Lecture 3

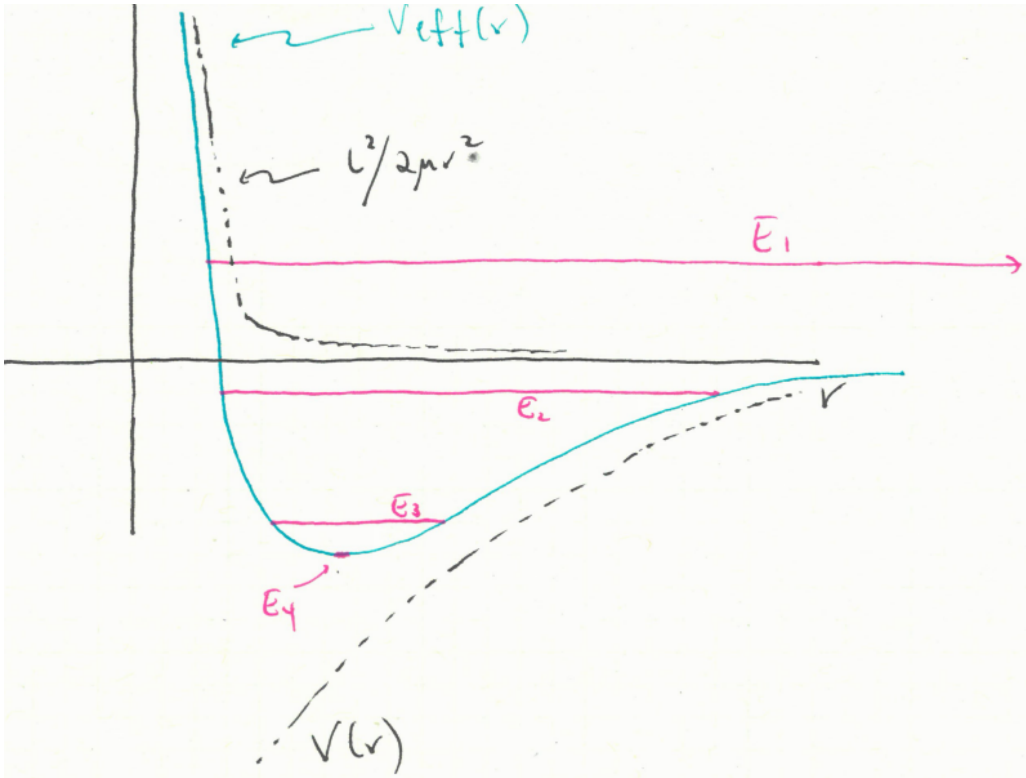
- So, we've reduced the **2-body problem** to a one-body problem. Next, we reduce the dimensionality:
  - We have a central force, so $\vec{F} \propto \vec{r}$
    Thus we have no torque, since $\vec{\tau} = \vec{r} \times \vec{F} = 0$
    Thus angular momentum is conserved in the 2-body system.

  - That angular momentum is always perpendicular to the orbital plane, since
    $\vec{L} \cdot \hat{r} = (\vec{r} \times \mu \dot{\vec{r}}) \cdot \hat{r}$
    $... = (\hat{r} \times \vec{r}) \cdot \mu \dot{\vec{r}} = 0$

  - Since the orbit is always in a single, constant plane we can just describe it using 2D polar coordinates, r and φ
    Thus we have $\vec{L} = \vec{r} \times \mu \vec{v}$ (by definition of L)
    $... = \mu r v_\varphi = \mu r^2 \dot{\varphi}$ = constant
    → **Equal area law (Kepler's 2nd) follows – true for <u>any central force</u> (not just $1/r^2$)**

  - Next, we go from 2D to 1D:
    - $E = \frac{1}{2} \mu \vec{v} \cdot \vec{v} + V(r)$
      $... = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\varphi}^2 + V(r)$
      $L = \mu r^2 \dot{\varphi}$ (from above), and so $\dot{\varphi}^2 = \frac{L^2}{\mu^2 r^4}$
      and so $E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2 \mu r^2} + V(r)$. We call those last two terms V_{eff}.
    - The formal solution to solve for the orbital motion is:
      $$dt = \frac{dr}{\sqrt{\frac{2}{\mu}[E - V_{eff}(r)]}}$$
    - For any given potential, one can integrate to get *t(r)* and then invert to find *r(t)*. Usually one gets nasty-looking Elliptic integrals for a polynomial potential.

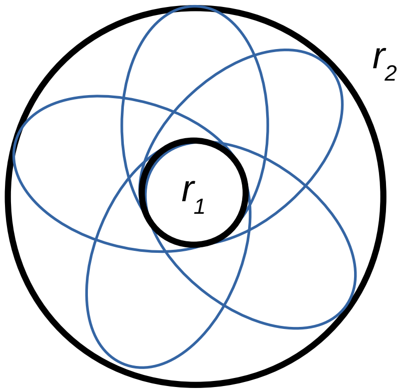
- Get more insight from graphical analysis.



- Plot V_{eff}, and then the total system E on the same graph. Given L & E:
  - Must have $V_{eff} < E$ (otherwise $v^2 < 0$)
  - Motion shows a turning point whenever $V_{eff} = E$.

- For different energies plotted:
  - E_1: unbound orbit. Hyperbolic – interstellar comets!
  - E_2, E_3: bound, eccentric orbits (outer (apastron) and inner (periastron) points)
  - E_4: circular orbit (single radius).
  - $E<E_4$: not allowed!

- Let's look at this motion in the plane (for the bound case):



We see that the possible paths will fill in the regions between an inner and an outer radius (r_1 and r_2). But there's no guarantee that the orbits actually repeat periodically.

- We get **periodic orbits, and closed ellipses, for two special cases**:
  - $V(r) \propto \frac{1}{r}$ (Keplerian motion)
  - $V(r) \propto r^2$ (simple harmonic oscillator)
  - "Bertrand's Theorem" says that these are the only two closed-orbit forms.

- These closed-form cases are also special because they have **an "extra" conserved quantity.**
  - Consider gravity: $V(r) = -\frac{G\mu M}{r}$ where $M = m_1 + m_2$
  - Define the "Laplace-Runge-Lenz" (LRL) vector,
    $\vec{A} \equiv \vec{p} \times \vec{L} - G M \mu^2 \hat{r}$ ← this is **conserved**! Describes shape & orientation of orbit
  - $\frac{d\vec{A}}{dt} = \frac{d\vec{p}}{dt} \times \vec{L} + \vec{p} \times \frac{d\vec{L}}{dt} - G M \mu^2 \frac{d\hat{r}}{dt}$ (second term goes to zero; L conserved!)
    $\frac{d\vec{p}}{dt} = -G\frac{\mu M}{r^2}\hat{r}$
    $\frac{d\hat{r}}{dt} = \frac{d\varphi}{dt}\hat{\varphi}$
    $\vec{L} = \mu r^2 \dot{\varphi} \hat{z}$

  - So, $\frac{d\vec{A}}{dt} = \left(-\frac{G\mu M}{r^2}\hat{r}\right) \times \left(\mu r^2 \dot{\varphi} \hat{z}\right) - G M \mu^2 \dot{\varphi} \hat{\varphi}$, which gives
    $\frac{d\vec{A}}{dt} = +G M \mu^2 \dot{\varphi} \hat{\varphi} - G M \mu^2 \dot{\varphi} \hat{\varphi} = 0$ … A is a conserved quantity!
  - But, what does the LRL vector *mean*?
    → It describes the elliptical equations of motion!

- A points in the orbital plane. Define it to point along the x-axis of our polar system:
  $$\vec{r}\cdot\vec{A}\equiv r\,A\cos\varphi=\vec{r}\cdot(\vec{p}\times\vec{L})-G\,M\,\mu^2\,r$$
  $$\ldots=\vec{L}\cdot(\vec{r}\times\vec{p})-G\,M\,\mu^2\,r \quad \rightarrow$$
  $$r\,A\cos\varphi=L^2-G\,M\,\mu^2\,r$$

- We can solve this for r:
  $$r(\varphi)=\frac{L^2/G\,M\,\mu^2}{1+(A/G\,M\,\mu^2)\cos\varphi}$$
  and this is just the **equation of an ellipse** that we saw in Lecture 2, with
  $$e=\frac{A}{G\,M\,\mu^2} \quad \text{and}$$
  $$L=\sqrt{G\,M\,\mu^2\,a(1-e^2)}$$

- We defined A to point along the x-axis (φ = 0). This is the same direction where r is minimized – so A (the LRL) points toward the closest approach in the orbit ("pericenter").

▪ One remaining law: K**epler's 3rd Law**
- Consider the area of a curve in polar coordinates.
  $$d\,Area=\frac{1}{2}r^2\,d\,\varphi \quad \text{, so}$$
  $$\frac{d\,Area}{d\,t}=\frac{1}{2}r^2\,\dot{\varphi}=\frac{1}{2}\frac{L}{\mu}=constant$$

- If we integrate over a full period, we get the area of an ellipse:
  $$A_{ellipse}=\int_0^P\frac{d\,Area}{dt}=\frac{1}{2}\int_0^P\frac{L}{\mu}dt=\frac{L\,P}{2\mu} \quad .$$
- And from geometry, $A_{ellipse}=\pi a\,b$ (where $b=a\sqrt{1-e^2}$ is the semi*minor* axis)
- So set these equal:
  $$\frac{L\,P}{2\mu}=\pi\,a^2\sqrt{1-e^2}$$. Plugging the previous expression for L in:
  $$\frac{1}{2}\frac{\sqrt{G\,M\,\mu^2\,a(1-e^2)}}{\mu}P=\pi a^2\sqrt{1-e^2}$$, which simplifies to
  $$\sqrt{\frac{G\,M}{a^3}}=\frac{2\,\pi}{P}=\Omega_{Kepler}$$

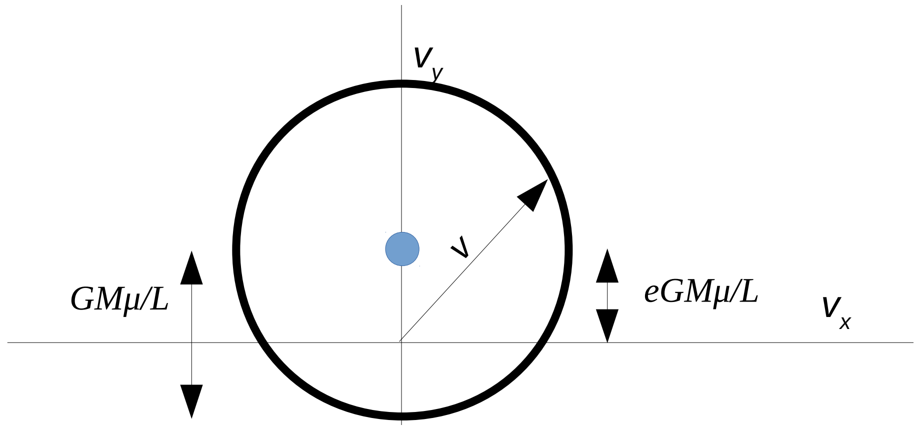
- Rearranging to the more familiar form, we find:
  $$P^2=\left(\frac{4\,\pi^2}{G\,M}\right)a^3$$. Or in Solar units, $$\left(\frac{P}{1\,yr}\right)^2=\left(\frac{M}{M_{sun}}\right)^{-1}\left(\frac{a}{1\,AU}\right)^3$$

- Other interesting bits and bobs:
  - A useful exercise for the reader is to show that $E=-\frac{GM\mu}{2a}$

    (use *rdot* = 0 at pericenter, $r = a\,(1-e)$, $\varphi=0$)

  - We have r(phi) --- **what about r(t) and phi(t) ?**
    - Unfortunately there's no general, closed-form solution – this is typically calculated iteratively using a numerical framework.
    - One can find parametric solutions (see Psets)

  - The position vector moves on an ellipse, but you can show that the velocity vector actually moves on a circle:



  - Really esoteric: all these conservation laws are tied to particular symmetries:
    - Energy conservation comes from time translation
    - Angular momentum conservation comes from SO(3) rotations
    - The RLR vector *A* is conserved because of rotations in *4D* (!!). *r* & *p* map onto the 3D surface of a 4D Euclidean sphere. Cool – but not too useful.