Energy

To begin to define the properties of radiation from astronomical objects, we will start with the energy that we receive from an emitting source somewhere in space. Consider a source of radiation in the vacuum of space (for familiarity, you can think of the sun). At some point in space away from our source of radiation we want to understand the amount of energy dE that is received from this source. What is this energy proportional to?

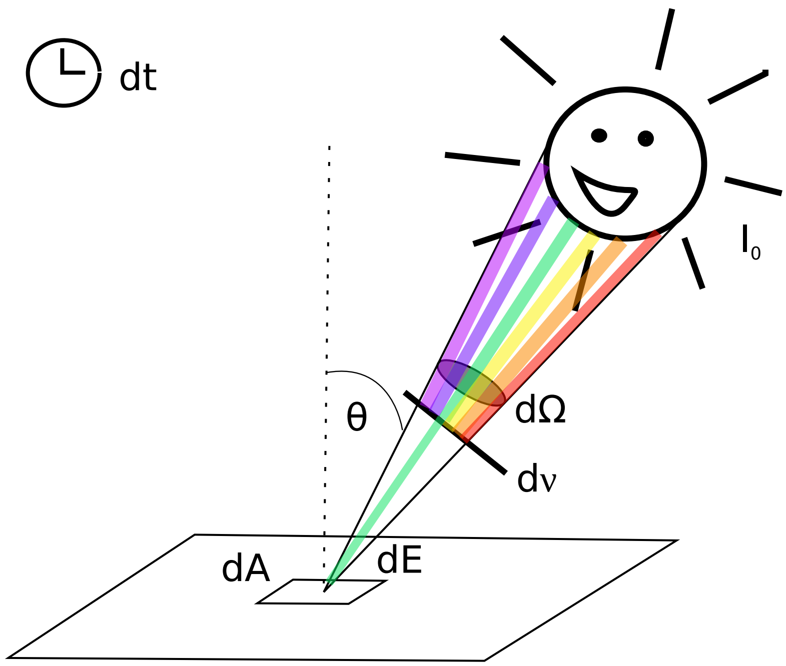

Figure 5: Description of the energy detected at a location in space for a period of time dt over an area dA arriving at an angle θ from an object with intensity I_0, an angular size $d\Omega$, through a frequency range $d\nu$ (in this case, only the green light).

As shown in Figure 5, our source of radiation has an intensity I_0 (we will get come back to this in a moment) over an apparent angular size (solid angle) of $d\Omega$. Though it may give off radiation over a wide range of frequencies, as is often the case in astronomy we only concern ourselves with the energy emitted in a specific frequency range ν+$d\nu$ (think of using a filter to restrict the colors of light you see, or even just looking at something with your eyeball, which only detects radiation in the visible range). At the location of detection, the radiation passes through some area dA in space (an area perhaps like a spot on the surface of earth) at an angle θ away from the normal to that surface. The last property of the radiation that we might want to consider is that we are detecting it over a given window of time (and many astronomical sources are time-variable). You might be wondering why the distance between our detector and the source is not being mentioned yet: we will get to this.

Considering these variables, the amount of energy that we detect will be proportional to the apparent angular size of our object, the range of frequencies over which we are sensitive, the time over which we collect the radiation, and the area over which we do this collection. The constant of proportionality is the **specific intensity** of our source: I_0. Technically, as this is the intensity just over a limited frequency range, we will write this as $I_{0,\nu}$.

In equation form, we can write all of this as:

$$dE_\nu = I_{0,\nu}\, cos\theta\; dA\; d\Omega\; d\nu\; dt \tag{36}$$

Here, the cos θ dA term accounts for the fact that the area that matters is actually the area "seen" from the emitting source. If the radiation is coming straight down toward our unit of area dA, it "sees" an area equal to that of the full dA (cos $\theta = 1$). However, if the radiation comes in at a different angle θ, then it "sees" our area dA as being tilted: as a result, the apparent area is smaller (cos $\theta < 1$). You can test this for yourself by thinking of the area dA as a sheet of paper, and observing how its apparent size changes as you tilt it toward or away from you.

Intensity

Looking at Equation 36, we can figure out the units that the specific intensity must have: energy per time per frequency per area per solid angle. In SI units, this would be W Hz^{-1} m^{-2} sr^{-1}. Specific intensity is also sometimes referred to as **surface brightness**, as this quantity refers to the brightness over a fixed angular size on the source (in O/IR astronomy, surface brightness is measured in magnitudes per square arcsec). Technically, the specific intensity is a 7-dimensional quantity: it depends on position (3 space coordinates), direction (two more coordinates), frequency (or wavelength), and time. As we'll see below, we can equivalently parameterize the radiation with three coordinates of position, three of momentum (for direction, and energy/frequency), and time.

Flux

The **flux density** from a source is defined as the total energy of radiation received from all directions at a point in space, per unit area, per unit time, per frequency. Given this definition, we can modify equation **??** to give the flux density at a frequency ν:

$$F_\nu = \int_\Omega \frac{dE_\nu}{dA\; dt\; d\nu} = \int_\Omega I_\nu cos\theta d\Omega \tag{37}$$

The total flux at all frequencies (the **bolometric flux**) is then:

$$F = \int_\nu F_\nu\, d\nu \tag{38}$$

As expected the SI units of flux are W m^{-2}; e.g., the aforementioned Solar Constant (the flux incident on the Earth from the Sun) is roughly 1400 W m^{-2}.

The last, related property that one should consider (particularly for spatially well-defined objects like stars) is the **Luminosity**. The luminosity of a source is the total energy emitted per unit time. The SI unit of luminosity is just Watts. Luminosity can be determined from the flux of an object by integrating over its entire surface:

$$L = \int F \, dA \tag{39}$$

As with flux, there is also an equivalent luminosity density, L_ν, defined analogously to Eq. 38.

Having defined these quantities, we now ask how the flux you detect from a source varies as you increase the distance to the source. Looking at Figure 6, we take the example of our happy sun, and imagine two spherical shells or bubbles around the sun: one at a distance R_1, and one at a distance R_2. The amount of energy passing through each of these shells per unit time is the same: in each case, it is equal to the luminosity of the sun, $L_\odot$. However, as $R_2 > R_1$, the surface area of the second shell is greater than the first shell. Thus, the energy is spread thinner over this larger area, and the flux (which by definition is the energy per unit area) must be smaller for the second shell. Comparing the equations for surface area, we see that flux decreases proportional to $1/d^2$.

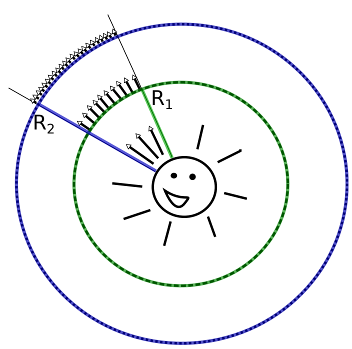

Figure 6: A depiction of the flux detected from our sun as a function of distance from the sun. Imagining shells that fully enclose the sun, we know that the energy passing through each shell per unit time must be the same (equal to the total luminosity of the sun). As a result, the flux must be less in the larger outer shell: reduced proportional to $1/d^2$

6.2 *Conservation of Specific Intensity*

We have showed that the flux obeys an inverse square law with distance from a source. How does the specific intensity change with distance? The specific intensity can be described as the flux divided by the angular size of the source, or $I_\nu \propto F_\nu/\Delta\Omega$. We have just shown that the flux decreases with distance, proportional to $1/\mathrm{d}^2$. What about the angular source size? It happens that the source size also decreases with distance, proportional to $1/\mathrm{d}^2$. As a result, the specific intensity (just another name for surface brightness) is independent of distance.

Let's now consider in a bit more detail this idea that I_ν is conserved in empty space – this is a key property of radiative transfer. This means that in the absence of any material (the least interesting case!) we have $dI_\nu/ds = 0$, where s measures the path length along the traveling ray. And we also know from electrodynamics that a monochromatic plane wave in free space has a single, constant frequency ν.

We mentioned above that I_ν can be parameterized with three coordinates of position, three of momentum (for direction, and energy/frequency), and time. So $I_\nu = I_\nu(\vec{r}, \vec{p}, t)$. For now we'll neglect the dependence on t, assuming a constant radiation field – so our radiation field fills a particular six-dimensional phase space of $\vec{r}$ and $\vec{p}$. This means that the particle distribution N is proportional to the phase space density f:

(40) $$dN = f(\vec{r}, \vec{p}) d^3r d^3p$$

By **Liouville's Theorem**, given a system of particles interacting with conservative forces, the phase space density $f(\vec{r}, \vec{p})$ is conserved along the flow of particles; Fig. 7 shows a toy example in 2D (since 6D monitors aren't yet mainstream).

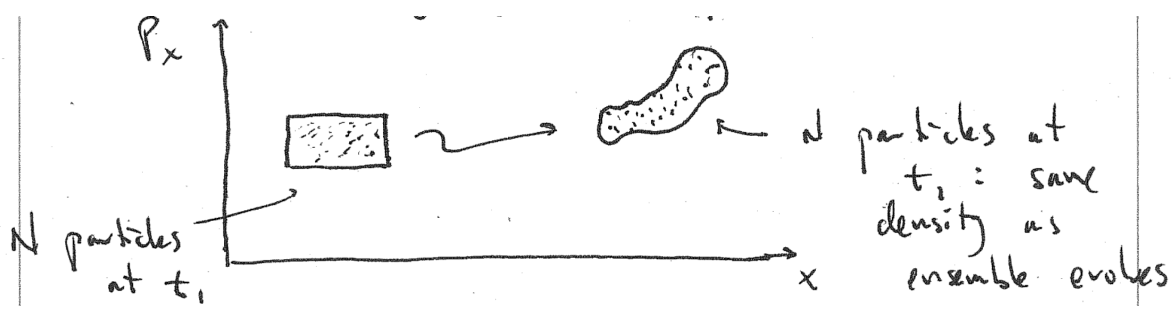

Figure 7: Toy example of Liouville's Theorem as applied to a 2D phase space of (x, p_x). As the system evolves from t_1 at left to t_2 at right, the density in phase space remains constant.

In our case, the particles relevant to Liouville are the photons in our radiation field. Fig. 8 shows the relevant geometry. This converts Eq. 40 into

(41) $$dN = f(\vec{r}, \vec{p}) c dt dA \cos\theta d^3p$$

As noted previously, $\vec{p}$ encodes the radiation field's direction and energy

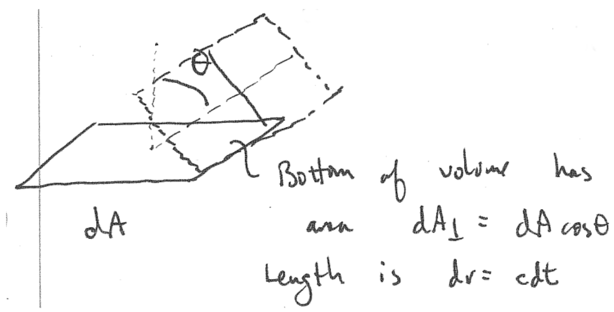

Figure 8: Geometry of the incident radiation field on a small patch of area dA.

(equivalent to frequency, and to linear momentum p) of the radiation field. So we can expand d^3p around the propagation axis, giving $d^3p = p^2 dp d\Omega$. This means we then have

$$(42)\quad dN = f(\vec{r}, \vec{p}) c dt dA \cos\theta p^2 dp d\Omega$$

Finally recalling that $dE = dN(h\nu)$ and $p = h\nu/c$, we have

$$(43)\quad dE = (h\nu) f(\vec{r}, \vec{p}) c dt dA \cos\theta \left(\frac{h\nu}{c}\right)^2 \left(\frac{h d\nu}{c}\right) d\Omega$$

We can combine this with Eq. 36 above, to show that specific intensity is directly proportional to the phase space density:

$$(44)\quad I_\nu = \frac{h^4 \nu^3}{c^2} f(\vec{r}, \vec{p})$$

Therefore whenever phase space density is conserved, I_ν/ν^3 is conserved. And since ν is constant in free space, I_ν is conserved as well.

6.3 *Blackbody Radiation*

For radiation in thermal equilibrium, the usual statistical mechanics references show that the Bose-Einstein distribution function, applicable for photons, is:

$$(45)\quad n = \frac{1}{e^{h\nu/k_B T} - 1}$$

The phase space density is then

$$(46)\quad f(\vec{r}, \vec{p}) = \frac{2}{h^3} n$$

where the factor of two comes from two photon polarizations and h^3 is the elementary phase space volume. Combining Eqs. 44, 45, and 46 we find that in empty space

$$I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \equiv B_\nu(T) \tag{47}$$

Where we have now defined $B_\nu(T)$, the **Planck blackbody function**. The Planck function says that the specific intensity (i.e., the surface brightness) of an object with perfect emissivity depends only on its temperature, T.

Finally, let's define a few related quantities for good measure:

$$J_\nu = \text{specific mean intensity} \tag{48}$$
$$= \frac{1}{4\pi} \int I_\nu d\Omega \tag{49}$$
$$= B_\nu(T) \tag{50}$$

$$u_\nu = \text{specific energy intensity} \tag{51}$$
$$= \int \frac{I_\nu}{c} d\Omega \tag{52}$$
$$= \frac{4\pi}{c} B_\nu(T) \tag{53}$$

$$P_\nu = \text{specific radiation pressure} \tag{54}$$
$$= \int \frac{I_\nu}{c} \cos^2 \theta d\Omega \tag{55}$$
$$= \frac{4\pi}{3c} B_\nu(T) \tag{56}$$

The last quantity in each of the above is of course only valid in empty space, when $I_\nu = B_\nu$. Note also that the correlation $P_\nu = u_\nu/3$ is valid whenever I_ν is isotropic, regardless of whether we have a blackbody radiation.

6.4 *Radiation, Luminosity, and Temperature*

The Planck function is of tremendous relevance in radiative calculations. It's worth plotting $B_\nu(T)$ for a range of temperatures to see how the curve behaves. One interesting result is that the location of maximal specific intensity turns out to scale linearly with T. When we write the Planck function in terms of wavelength λ, where $\lambda B_\lambda = \nu B_\nu$, we find that the **Wien Peak** is approximately

$$\lambda_{\max} \approx 3000 \mu m\ K \tag{57}$$

So radiation from a human body peaks at roughly 10μm, while that from a 6000 K, roughly Sun-like star peaks at 0.5μm = 500 nm — right in the response range of the human eye.

Another important correlation is the link between an object's luminosity L and its temperature T. For any specific intensity I_ν, the bolometric flux F is given by Eqs. 37 and 38. When $I_\nu = B_\nu(T)$, the **Stefan-Boltzmann Law** directly follows:

$$F = \sigma_{SB} T^4 \tag{58}$$

where σ_{SB}, the Stefan-Boltzmann constant, is

$$\sigma_{SB} = \frac{2\pi^5 k_B^4}{15 c^2 h^3} \tag{59}$$

(or $\sim 6\ times 10^{-8}$ W s^{-1} m^{-2} K^{-4}).

Assuming isotropic emission, the luminosity of a sphere with radius R and temperature T is

$$L = 4\pi R^2 F = 4\pi\sigma_{SB} R^2 T^4 \tag{60}$$

.

If we assume that the Sun is a blackbody with $R_\odot = 7\times 10^8$ m and $T = 6000$ K, then we would calculate

$$L_{\odot,approx} = 4\times 3\times(6\times 10^{-8})\times(7\times 10^8)^2\times(6\times 10^3)^4 \tag{61}$$

$$= 72\times 10^{-8}\times(50\times 10^{16})\times(1000\times 10^{12}) \tag{62}$$

$$= 3600\times 10^{23} \tag{63}$$

which is surprisingly close to the IAU definition of $L_\odot = 3.828\times 10^{26}$ W m^{-2}.

Soon we will discuss the detailed structure of stars. Spectra show that they are not perfect blackbodies, but they are often pretty close. This leads to the common definition of an **effective temperature** linked to a star's size and luminosity by the Stefan-Boltzmann law. Rearranging Eq. 60, we find that

$$T_{\text{eff}} = \left(\frac{L}{4\pi\sigma_{SB} R^2}\right)^{1/4} \tag{64}$$