7 Radiative Transfer

Radiation through empty space is what makes astronomy possible, but it isn't so interesting to study on its own. **Radiative transfer**, the effect on radiation of its passage through matter, is where things really get going.

7.1 *The Equation of Radiative Transfer*

We can use the fact that the specific intensity does not change with distance to begin deriving the radiative transfer equation. For light traveling in a vacuum along a path length s, we say that the intensity is a constant. As a result,

$$\frac{dI_\nu}{ds} = 0 \;(for\ radiation\ traveling\ through\ a\ vacuum) \tag{66}$$

This case is illustrated in the first panel of Figure 9. However, space (particularly objects in space, like the atmospheres of stars) is not a vacuum everywhere. What about the case when there is some junk between our detector and the source of radiation? This possibility is shown in the second panel of Figure 9. One quickly sees that the intensity you detect will be less than it was at the source. You can define an **extinction coefficient** α_ν for the space junk, with units of extinction (or fractional depletion of intensity) per distance (path length) traveled, or m^{-1} in SI units. For our purposes right now, we will assume that this extinction is uniform and frequency-independent (but in real life of course, it never is).

We also define

$$\alpha_\nu = n\sigma_\nu \tag{67}$$

$$= \rho\kappa_\nu \tag{68}$$

Where n is the number density of absorbing particles and σ_ν is their frequency-dependent cross-section, while ρ is the standard mass density and κ_ν is the frequency-dependent **opacity**. Now, our equation of radiative transfer has

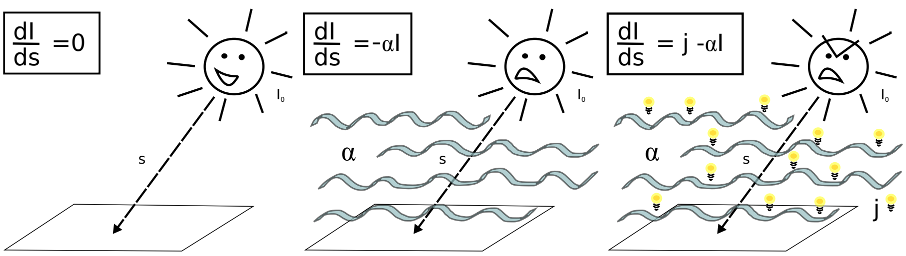

Figure 9: The radiative transfer equation, for the progressively more complicated situations of: (left) radiation traveling through a vacuum; (center) radiation traveling through a purely absorbing medium; (right) radiation traveling through an absorbing and emitting medium.

been modified to be:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \ (\textit{when there is absorbing material between us and our source}) \tag{69}$$

As is often the case when simplifying differential equations, we then find it convenient to try to get rid of some of these pesky units by defining a new unitless constant: τ_ν, or **optical depth**. If α_ν is the fractional depletion of intensity per path length, τ_ν is just the fractional depletion. We then can define

$$d\tau_\nu = \alpha_\nu ds \tag{70}$$

and re-write our equation of radiative transfer as:

$$\frac{dI_\nu}{d\tau} = -I_\nu \tag{71}$$

Remembering our basic calculus, we see that this has a solution of the type

$$I_\nu(s) = I_\nu(0)\exp\left(-\int_0^s d\tau_\nu\right) \tag{72}$$

$$= I_\nu(0)e^{-\tau_\nu} \ (\textit{for an optically thin source}) \tag{73}$$

So, at an optical depth of unity (the point at which something begins to be considered optically thick), your initial source intensity I_0 has decreased by a factor of e.

However, radiation traveling through a medium does not always result in a net decrease. It is also possible for the radiation from our original source to pass through a medium or substance that is not just absorbing the incident radiation but is also emitting radiation of its own, adding to the initial radiation field. To account for this, we define another coefficient: j_ν. This **emissivity coefficient** has units of energy per time per volume per frequency per solid angle. Note that these units (in SI: W m^{-3} Hz^{-1} sr^{-1}) are slightly different than the units of specific intensity. Including this coefficient in our radiative transfer equation we have:

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \tag{74}$$

or, putting it in terms of the dimensionless optical depth τ, we have:

$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu \tag{75}$$

After defining the so-called **source function**

$$(76) \quad S_\nu = \frac{j_\nu}{\alpha_\nu}$$

we arrive at the final form of the **radiative transfer equation**:

$$(77) \quad \frac{dI_\nu}{d\tau_\nu} = S_\nu \, - \, I_\nu$$

7.2 *Solutions to the Radiative Transfer Equation*

What is the solution of this equation? For now, we will again take the simplest case, and assume that the medium through which the radiation is passing is uniform (i.e., S_ν is constant). Given an initial specific intensity of $I_\nu(s=0) = I_{\nu,0}$, we obtain

$$(78) \quad I_\nu = I_{\nu,0}e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \;\; (for\, constant\, source\, function)$$

What happens to this equation when τ is small? In this case, we haven't traveled very far through the medium and so should expect that absorption or emission haven't had a strong effect. And indeed, in the limit that $tau_\nu = 0$ we see that $I_\nu = I_{\nu,0}$.

What happens to this equation when τ becomes large? In this case, we've traveled through a medium so optically thick that the radiation has "lost all memory" of its initial conditions. Thus $e^{-\tau_\nu}$ becomes negligible, and we arrive at the result

$$(79) \quad I_\nu = S_\nu \;\; (for\, an\, optically\, thick\, source)$$

So the only radiation that makes it out is from the emission of the medium itself. What is this source function anyway? For a source in thermodynamic equilibrium, any opaque (i.e., optically thick) medium is a "black body" and so it turns out that $S_\nu = B_\nu(T)$, the Planck blackbody function. For an optically-thick source (say, a star like our sun) we can use Eq. 79 to then say that $I_\nu = B_\nu$.

The equivalence that $I_\nu = S_\nu = B_\nu$ gives us the ability to define key properties of stars – like their flux and luminosity – as a function of their temperature. As described the preceding chapter, using Eq. 37 and 38 we can integrate the blackbody function to determine the flux of a star (or other blackbody) as a function of temperature, the Stefan-Boltzmann law:

$$(80) \quad F = \sigma T^4$$

Another classic result, the peak frequency (or wavelength) at which a star (or other blackbody) radiates, based on its temperature, can be found by differentiating the blackbody equation with respect to frequency (or wavelength). The result must be found numerically, and the peak wavelength can be expressed in Wien's Law as

$$\lambda_{peak} = \frac{2.898 \times 10^{-3} \text{ m K}}{T} \tag{81}$$

We can improve on Eq. 78 and build a formal, general solution to the radiative transfer equation as follows. Starting with Eq. 77, we have

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \tag{82}$$

$$\frac{dI_\nu}{d\tau_\nu} e^{\tau_\nu} = S_\nu e^{\tau_\nu} - I_\nu e^{\tau_\nu} \tag{83}$$

$$\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right) = S_\nu e^{\tau_\nu} \tag{84}$$

We can integrate this last line to obtain the formal solution:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu) e^{(\tau'_\nu - \tau_\nu)} d\tau'_\nu \tag{85}$$

As in our simpler approximations above, we see that the initial intensity $I_\nu(0)$ decays as the pathlength increases; at the same time we pick up an increasing contribution from the source function S_ν, integrated along the path. In practice S_ν can be fairly messy (i.e., when it isn't the Planck function), and it can even depend on I_ν. Nonetheless Eq. 85 lends itself well to numerical solution.

7.3 *Kirchhoff's Laws*

We need to discuss one additional detail before getting starts on stars and nebulae: **Kirchhoff's Law for Thermal Emission.** This states that a thermally emitting object in equilibrium with its surrounding radiation field has $S_\nu = B_\nu(T)$.

Note that the above statement does *not* require that our object's thermal radiation is necessarily blackbody radiation. Whether or not that is true depends on the interactions betweens photons and matter – which means it depends on the optical depth τ_ν.

Consider two lumps of matter, both at T. Object one is optically thick, i.e. $\tau_\nu >> 1$. In this case, Eq. 85 does indeed require that the emitted radiation has the form $I_\nu(\tau_\nu) = S_\nu = B_\nu(T)$ — i.e., blackbody radiation emerges from and optically thick object. This is mostly the case for a stellar spectrum, but not quite (as we'll see below).

First, let's consider the other scenario in which our second object is optically thin, i.e. $0 < \tau_\nu << 1$. If our initial specific intensity $I_\nu(0) = 0$, then we have

$$I_\nu(\tau_\nu) = 0 + S_\nu\left(1 - (1 - \tau_\nu)\right) \tag{86}$$

$$= \tau_\nu B_\nu(T) \tag{87}$$

Thus for an optically thin object, the emergent radiation will be blackbody radiation, scaled down by our low (but nonzero) τ_ν.

It's important to remember that τ_ν is frequency-dependent (hence the ν subscript!) due to its dependence on the extinction coefficient α_ν. So most astronomical objects represent a combination of the two cases discussed immediately above. At frequencies where atoms, molecules, etc. absorb light most strongly, α_ν will be higher than at other frequencies.

So in a simplistic model, assume we have a hot hydrogen gas cloud where α_ν is zero everywhere except at the locations of H lines. The location of these lines is given by the Rydberg formula,

$$\frac{1}{\lambda_{\mathrm{vac},1,2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \tag{88}$$

(where $n_1 = 1, 2, 3, 4, 5$, etc. for the Lyman, Balmer, Paschen, and Brackett series, respectively).

In a thin gas cloud of temperature T, thickness s, and which is "backlit" by a background of empty space (so $I_{\nu,0} \approx 0$), from Eq. 87 all we will see is $\tau_\nu B_\nu(T) = \alpha_\nu s B_\nu(T)$ — so an **emission-line spectrum** which is zero away from the lines and has strong emission at the locations of each line.

What about in a stellar atmosphere? A single stellar T (an **isothermal atmosphere**) will yield just a blackbody spectrum, regardless of the form of α_ν. The simplest atmosphere yielding an interesting spectrum is sketched in Fig. 10: an optically thick interior at temperature T_H and a cooler, optically thin outer layer at $T_C < T_H$.

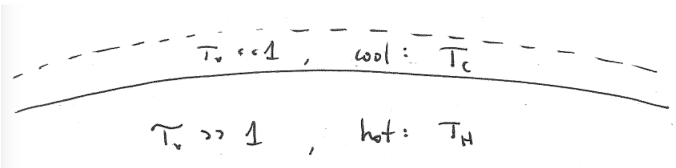

Figure 10: The simplest two-layer stellar atmospere: an optically thick interior at temperature T_H and a cooler, optically thin outer layer at $T_C < T_H$.

The hot region is optically thick, so we have $I_\nu = S_\nu = B_\nu(T_H)$ emitted from the lower layer – again, regardless of the form of α_ν. The effect of the upper, cooler layer which has small but nonzero τ_ν is to slightly diminish the contribution of the lower layer while adding a contribution from the cooler layer:

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) \tag{89}$$

$$= B_\nu(T_H)e^{-\tau_\nu} + B_\nu(T_C)\left(1 - e^{-\tau_\nu}\right) \tag{90}$$

$$\approx B_\nu(T_H)(1 - \tau_\nu) + B_\nu(T_C)\tau_\nu \tag{91}$$

$$\approx B_\nu(T_H) - \tau_\nu\left(B_\nu(T_H) - B_\nu(T_C)\right) \tag{92}$$

$$\approx B_\nu(T_H) - \alpha_\nu s\left(B_\nu(T_H) - B_\nu(T_C)\right) \tag{93}$$

So a stellar spectrum contains consists of two parts, roughly speaking. The first is $B_\nu(T_H)$, the contribution from the blackbody at the base of the atmosphere (the **spectral continuum**). Subtracted from this is a contribution wherever α_ν is strong – i.e., at the locations of strongly-absorbing lines. As we will see later, we can typically observe in a stellar atmosphere only down to $\tau_\nu \sim 1$. So at the line locations where (absorption is nonzero), we observe approximately $B_\nu(T_C)$. Thus in this toy model, the lines probe higher in the atmosphere (we can't observe as deeply into the star, because absorption is stronger at these frequencies – so we effectively observe the cooler, fainter upper layers). Meanwhile there is effectively no absorption in the atmosphere, so we see down to the hotter layer where emission is brighter. Fig. 11 shows a typical example.

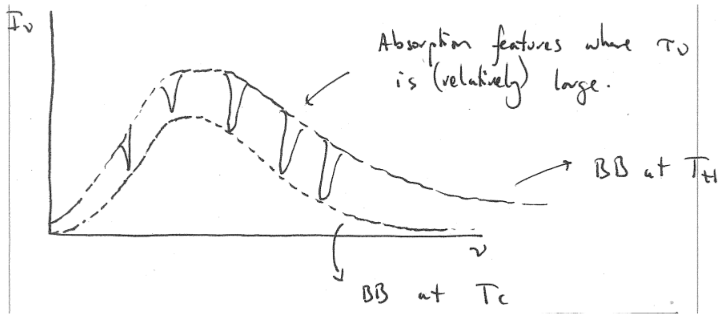

Figure 11: Toy stellar spectrum (solid line) for the toy stellar model graphed in Fig. 10.

Note that our assumption has been that temperature in the star decreases with increasing altitude. More commonly, stellar models will parameterize an atmosphere in terms of its **pressure-temperature profile**, with pressure P decreasing monotonically with increasing altitude. An interesting phenomenon occurs when T increases with decreasing P (increasing altitude): in this case we have a **thermal inversion**, all the arguments above are turned on their heads, and the lines previously seen in absorption now appear in emission over the same continuum. Thermal inversions are usually a second-order correction to atmospheric models, but they are ubiquitous in the atmospheres of the Sun, Solar System planets, and exoplanets.