8 Stellar Classification, Spectra, and Some Thermodynamics

Questions you should be able to answer after this lecture:

- How are stars classified?

- What is the difference between Thermal equilibrium and Thermodynamic equilibrium?

- What are the different temperatures that must be equal in Thermodynamic equilibrium?

- When is Local Thermodynamic Equilibrium valid for a region?

Classification is a key step toward understanding any new class of objects. When modern astronomy began, classification of the stars was a key goal — also an elusive one, until the physical processes became better understood.

8.1 *Classification*

One of the first successful frameworks used **photometry** (broadband, $\Delta \nu / \nu \approx 20\%$, measurements of stellar flux density) at different colors. Assuming again that stellar spectra are approximately blackbodies, the Planck function shows that we should see the hotter stars have bluer colors and be intrinsically brighter. This led to the **Hertzsprung-Russell diagram** (HR diagram), which plots absolute magnitude against color – we'll see the HR diagram again when we discuss stellar evolution.

It's fair to say that spectroscopy is one of our key tools for learning about astronomical objects, including stars. Fig. 12 shows a sequence of stars arranged from hot to cool: one can easily see the Wien peak shift with temperature, although none of the stars are perfect blackbodies. Other features come and go, determined (as we will see) mainly by stellar temperature but also surface gravity (or equivalently, surface pressure).

Through decades of refinement, spectra are now classified using **Morgan-Keenan spectral types**. These include a letter to indicate the approximate

Table 1: Stellar spectral types.

SpT	T_{eff}	Spectral features
O	$> 3 \times 10^4$	Ionized He or Si; no H (or only very weak)
B	$10^4 - 3 \times 10^4$	H Balmer lines, neutral He lines
A	$7500 - 10^4$	Strong H lines
F	$6000 - 7500$	H Balmer, first metal lines appear (Ca)
G	$5200 - 6000$	Fading H lines, increasing metal lines
K	$3700 - 5200$	Strong Ca and other metals, hydride molecules appear
M	$2400 - 3700$	Molecular bands rapidly strengthen: hydrides, TiO, H_2O
L	$1400 - 2400$	A melange of atomic and molecular bands; dust appears
T	$\sim 400 - 1400$	CH_4 strengthens, dust clears
Y	$\lesssim 400$	NH_3 strengthens

temperature, an arabic numeral to refine the temperature, and a roman numeral to indicate the star's luminosity. The order of letters seems disjointed because stars were classified before the underlying physical causes were well-understood. The temperature sequence is OBAFGKMLTY, where the last three typically apply to brown dwarfs (intermediate in mass between planets and stars) and the rest apply to stars. Table 8.1 briefly describes each of the alphabetic spectral types. Additional resolution is added to the system through the use of numbers 0–9, so that F9–G0–G1 is a sequence of steadily decreasing

Table 2: Stellar luminosity classes.

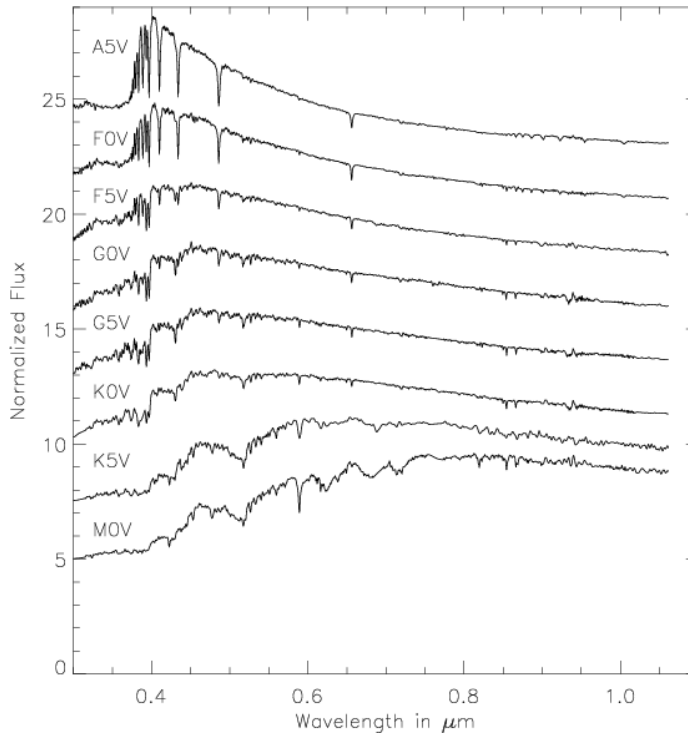| Lum	name	examples
VI	subdwarf	Kapteyn's Star (M1VI)
V	dwarf	Sun (G2V), Vega (A0V)
IV	subgiant	Procyon (F5IV)
III	giant	Arcturus (K1III)
II	bright giant	
I	supergiant	Rigel (B8Ia), Betelgeuse (M1Ia)
0	hypergiant	η Carinae, Pistol Star

Figure 12: Optical-wavelength spectra of main-sequence stars across a range of spectral types.

T_{eff}. Finally, the roman numerals described in Table 2 indicate the luminosity class, which typically correlates with the stellar radius (and inversely with the surface gravity).

8.2 *Thermodynamic Equilibrium*

Our goal is to quantitatively explain the trends observed in Fig. 12. To do that, we need the tools provided to us by thermodynamics and statistical mechanics. We claimed earlier that $S_\nu = B_\nu(T)$, the source function is equal to the blackbody function, for a source in thermodynamic equilibrium. So, what are the conditions of thermodynamic equilibrium, and in what typical astronomical sources are these conditions satisfied?

There are two main conditions for thermodynamic equilibrium.

1. **Thermal equilibrium**: There is no heat transfer in a source: classically, it is at a constant, uniform temperature. However, as we will describe further in Section 14, for a star we generally take this just to mean that the temperature can vary spatially (but not in time), and that local energy losses (say, due to energy transport) are exactly balanced by gains (say, due to nuclear fusion).

2. Every temperature in the source is the same: the source is also in a radiation, ionization, and excitation equilibrium.

So, how do we define all of these different temperatures a source or system can have?

First, there is the **kinetic temperature** T_{kin}. This temperature describes the random motion of particles in a system. For a system in thermodynamic equilibrium, the distribution of speeds of particles (atoms or molecules) in this system is given by the **Maxwell-Boltzmann distribution**:

$$(95) \quad dN_v = 4\pi n \left(\frac{m}{2\pi k_B T_{kin}} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2 k_B T_{kin}} \right) dv$$

Here, dN_v is the number of particles with mass m and number density n between speeds v and $v + dv$.

Second, there is the **excitation temperature** T_{ex}. This temperature describes the distribution of internal energies in the particles in a system. This internal energy can be the energy of different electronic states of an atom, or the energy of rotation or vibration in a molecule. For a system in thermodynamic equilibrium, the fraction of atoms (or molecules) occupying a particular energy state is given by the **Boltzmann distribution** (not to be confused with Eq. 95!):

$$(96) \quad \frac{N_1}{N_2} = \frac{g_1}{g_2} \exp\left(-\frac{E_1 - E_2}{k_B T_{ex}} \right)$$

Here, N_1 is the number of atoms or molecules in a state with an energy E_1 above the ground state, and N_2 is the number of atoms or molecules in a state

with an energy E_2 above the ground state. The statistical weight of each state is given by g, which accounts for multiple configurations that might all have the same energy (i.e., the statistical degeneracy).

Next is T_{rad}, the **radiation temperature** in the system. This temperature is defined by an equation we have seen before: the Planck distribution, or the Blackbody law of Equation 49.

Finally, there is the **ionization temperature** T_i. This temperature describes the degree to which electrons are bound to the particles in a system. The fraction of the atoms in a gas which are ionized is given by the **Saha equation**, derived below and given as Eq. 124.

8.3 *Local Thermodynamic Equilibrium*

How typical is it for astronomical sources (like stars or planets or gas clouds) to be in thermodynamic equilibrium? In general, it is rare! Most sources are going to have significant temperature variations (for example, from the interior to the exterior of a star or planet). However, the situation is not hopeless, as in most sources, these changes are slow and smooth enough that over a small region, the two conditions we described are sufficiently satisfied. Such a situation is referred to as Local Thermodynamic Equilibrium or LTE.

When does LTE hold? First, for particles to have a Maxwell-Boltzmann distribution of velocities, and so to have a single kinetic temperature, the particles must have a sufficient opportunity to 'talk' to each other through collisions. Frequent collisions are also required for particles to have a uniform distribution of their internal energy states. The frequency of collisions is inversely proportional to the mean free path of the gas: the typical distance a particle travels before undergoing a collision. In general, for a region to be in LTE, the mean free path should be small compared to the distance over which the temperature varies appreciably. As LTE further requires that the radiation temperature is equal to the kinetic and excitation temperature, the matter and radiation must also be in equilibrium. For this to happen, not only must the mean free path for particles to undergo collisions with each other be small, but the mean free path for photons to undergo collisions with matter must be small as well. We have actually already introduced the mean free path for photons: it is equal to α^{-1}, where α was given in Equation 70 as the extinction coefficient, with units of fractional depletion of intensity per distance traveled. As intensity is depleted by being absorbed by matter, the inverse of the extinction coefficient describes the typical distance a photon will travel before interacting with matter.

Very qualitatively then, our two conditions for LTE are that the mean free path for particle-particle and particle-photon interactions must be less than the distance over which there is a significant temperature variation.

8.4 *Stellar Lines and Atomic Populations*

When we study stellar spectra, we examine how the strengths of various features change. Fig. 12 suggests that this is a continuous process as a function of T_{eff}. For example, we never see lines of both He I (i.e., neutral He) and Ca II

(i.e., singly-ionized Ca, i.e. Ca^+) at the same time – these lines appear at completely different temperatures. What we want is a quantitative understanding of spectra.

When do we expect substantial excitation of these various atoms? Let's consider the electronic lines of atomic hydrogen. The H atom's energy levels are given by:

$$(97) \quad E_n = -\frac{13.6 \text{ eV}}{n^2}$$

which gives rise to the Rydberg formula (Eq. 89) for the locations of individual lines.

To see conditions we need to excite these H atoms, we might make use of the relative probability of 2 atomic states with different energies (given by the Boltzmann distribution, Eq. 96). Statistical mechanics tells us that the statistical weight of each level in a hydrogen atom is

$$(98) \quad g_n = 2n^2$$

So for transitions between the ground state (-13.6 eV, $n = 1$) and the first excited state (-3.4 eV, $n = 2$) the relative fraction is given by

$$(99) \quad \frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left[-\left(E_1 - E_2\right)/k_B T\right]$$

When the levels are approximately equal, we then have

$$(100) \quad 1 = \frac{2}{8} \exp\left[10.2 \text{ eV}/k_B T\right]$$

The calculation above would thus imply that to get appreciable levels of excited hydrogren, we would need $T \approx 90,000$ K — much hotter than the observed temperatures of stars. In fact, H is totally ionized (not just mildly excited) even at much lower temperatures. Meanwhile, even A and F stars (with $T_{\text{eff}} \leq 10,000$ K) show prominent $n = 2$ hydrogen lines. We got the energetics right, but missed some other important thermodynamic quantities.

8.5 *The Saha Equation*

Let's investigate our hydrogen atom in further detail. From statistical mechanics, the distribution function of particles leads to the phase space density (see Eqns. 47 and 48):

$$(101) \quad f(\vec{r}, \vec{p}) = \frac{g}{h^3} \frac{1}{e^{[E-\mu]/k_B T} \pm 1}$$

where μ is the chemical potential and g is still the degeneracy factor:

(102)
$$g = 2s + 1 \text{(for fermions)}$$

(103)
$$g = 2 \text{(for photons)}$$

and where the $\pm$ operator is positive for Fermi-Dirac statistics and negative for Bose-Einstein statistics.

Again, we'll transform this six-dimensional density into a number density by integrating over momentum (see Eq. 43):

(104) $n = 4\pi \int\limits_0^\infty f(p)p^2 dp$

But integrating Eq. 101 is going to be a bear of a job, so we'll make two additional approximations. First, we'll assume for now that all particles are non-relativistic – so their energy is given classically by

(105) $E = \dfrac{p^2}{2m} + mc^2$

And we'll also assume that we're dealing with large energies, such that $E - \mu >> k_B T$. In practice, this second point means we can neglect the ± 1 in the denominator of Eq. 101. Both these assumptions are reasonable for the gas in most stars. We'll come back later to some especially interesting astrophysical cases, when these assumptions no longer hold.

We can now make the attempt to calculate n from Eq. 104.

(106)
$$n = 4\pi \int\limits_0^\infty f(p)p^2 dp$$

(107)
$$= \frac{4\pi g}{h^3} \int\limits_0^\infty p^2 dp \exp\left(\frac{\mu}{k_B T}\right) \exp\left(-\frac{mc^2}{k_B T}\right) \exp\left(-\frac{p^2}{2m k_B T}\right)$$

(108)
$$= \frac{4\pi g}{h^3} \exp\left[\left(\mu - mc^2\right)/k_B T\right] \int\limits_0^\infty p^2 \exp\left(-\frac{p^2}{2m k_B T}\right)$$

(109)
$$= \frac{g}{h^3} \left(2\pi m k_B T\right)^{3/2} \exp\left[\left(\mu - mc^2\right)/k_B T\right]$$

So now we have a relation between the number density and other relevant

quantities. We can rearrange this expression to get

(110) $\exp\left(\dfrac{\mu - mc^2}{k_B T}\right) = \dfrac{1}{g}\dfrac{n}{n_Q}$

where n_Q is the "quantum density"

(111) $n_Q \equiv (2\pi m k_B T / h^2)^{3/2}$

When $n = n_Q$, then the spacing between particles $n^{-1/3}$ is roughly equal to the thermal de Broglie wavelength — the particles' wave functions start to overlap, quantum effects ramp up, and degeneracy effects become increasingly important.

Ideally we want to get rid of the pesky μ and set things in terms of other quantities. Recall from thermodynamics that the **chemical potential** μ is just the energy absorbed or released during reactions. At constant volume V and entropy S, μ is determined by the change in internal energy U:

(112) $\mu \equiv \left(\dfrac{\partial U}{\partial n}\right)\bigg|_{V,S}$

The implication is that in equilibrium, all chemical potentials in a reaction sum to zero. So given a notional reaction
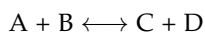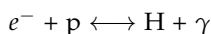
A + B $\longleftrightarrow$ C + D

we must have both

(113) $A + B = C + D$

and also

(114) $\mu_A + \mu_B = \mu_C + \mu_D$

Just as energies flow to equalize temperature and reach thermal equilibrium, numbers of different particle species flow to reach chemical equilibrium. For the H system under consideration, the reaction to ionize our hydrogren is

$e^- + p \longleftrightarrow H + \gamma$

In chemical equilibrium, we will then have

(115) $\mu_e + \mu_p = \mu_H$

since the chemical potential of a photon is zero. Why is this useful? Because we can rearrange Eq. 110 to find an expression for μ, and then use this in what follows (we're getting close). We find

(116) $\mu_i = m_i c^2 + k_B T \ln\left(\dfrac{n_i}{g_i n_{Q_i}}\right)$

and of course mass-energy must also be conserved in the reaction:

(117) $m_H c^2 = m_p c^2 + m_e c^2 + \epsilon_n$

where $\epsilon_1 = -13.6$ eV for full ionization. The statistical weights are a tad tricker, but for our fermions we have $g_p = g_e = 2$ while for ionizing atomic H we have $g_H = n^2 g_p g_e = 4$.

Requiring that the chemical potentials must balance, following Eq. 115 we then have:

(118)
$$m_p c^2 + m_e c^2 + k_B T \left[\ln \left(\frac{n_p}{2 n_{Q_p}} \right) + \ln \left(\frac{n_e}{2 n_{Q_e}} \right) \right] = m_H c^2 + k_B T \ln \left(\frac{n_H}{4 n_{Q_H}} \right)$$

Bringing in the results of Eq. 117, we then have

(119) $\ln \left(\dfrac{n_p n_e}{4 n_{Q_p} n_{Q_e}} \right) = \dfrac{-13.6 \text{ eV}}{k_B T} + \ln \left(\dfrac{n_H}{4 n_{Q_H}} \right)$

Rearranging terms, we then have

(120) $\dfrac{n_p n_e}{n_H} \dfrac{n_{Q_H}}{n_{Q_p} n_{Q_e}} = e^{-(13.6 \text{ eV}/k_B T)}$

We can simplify this one more step by recalling from Eq. 111 that $n_{Q_p} \approx n_{Q_H}$. This means that we have finally reached our goal:

(121) $\dfrac{n_p n_e}{n_H} = n_{Q_e} e^{-(13.6 \text{ eV}/k_B T)}$

which is famous as the **Saha equation** for hydrogen ionization. This tells us how the relative number densities of p, e^-, and H atoms will depend on the temperature of the system of particles.

It's traditional to refactor Eq. 121 by defining yet two more terms, the baryon number

(122) $n_B = n_H + n_p$

(which is conserved) and the ionization fraction

(123) $y = \dfrac{n_e}{n_B}$

which goes from zero (all neutral H) to unity (full ionization). When we divide both sides of Eq. 121 by n_B, we find the classical form of the **Saha equation**,

(124) $\dfrac{y^2}{1 - y} = \dfrac{n_{Q_e}}{n_B} e^{-(13.6 \text{ eV}/k_B T)}$

In a stellar photosphere, decent estimates are that $n_B \sim 10^{16}$ cm^{-3} and

$n_{Q_e} \approx 10^{21}$ cm^{-3} $(T/10^4$ K$)^{3/2}$. Eq. 124 is easily solved or plotted with numerical tools — the result, shown in Fig. 13, is a steep function of temperature that indicates ionization setting in at much lower temperatures than inferred in Eq. 100 alone. Instead, we see essentially no ionization in the 5800 K Solar photosphere, but we expect an ionization fraction of 5% at 9,000 K, rising to 50% at 12,000 K and 95% at 16,000 K. Although the Saha equation is a toy model with only two level populations, it still does an excellent job in predicting that H lines should be absent (as they are) from the hottest O and B stars.

In general, we also want to be able to properly treat the fact that there are an infinite number of energy levels (not just two) between the ground state and full ionization. This means that we need to account for the **partition function** $Z(\bar{T})$,

(125) $\quad Z(\bar{T}) \equiv \Sigma g_s e^{-E_s/k_B T}$

In principle one can calculate one's own partition functions, but in practice one often leaves that to the experts and borrows appropriately from the literature. So then the number density becomes

(126) $\quad n = \left(\dfrac{2\pi m k_B T}{h^2}\right)^{3/2} e^{\mu/k_B T} Z(\bar{T})$

Using this new form to repeat the analysis above, equality of chemical potentials will then yield

(127) $\quad \dfrac{n_B n_C}{n_A} = \left(\dfrac{2\pi k_B T}{h^2}\right)^{3/2} \left(\dfrac{m_B m_C^{3/2}}{m_A}\right)^{3/2} \left(\dfrac{\bar{Z}_B \bar{Z}_C}{\bar{Z}_A}\right)$

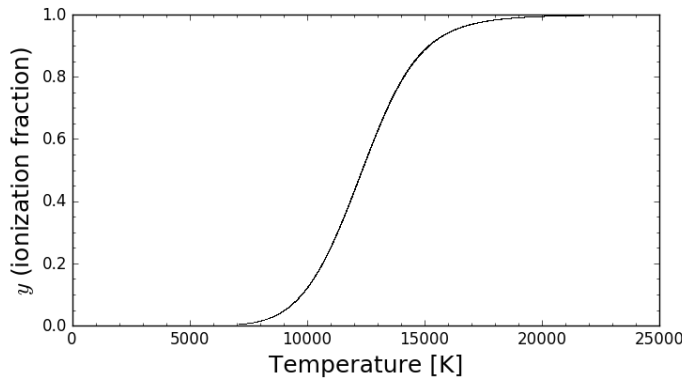If the partition function is dominated by a single state (as in our simple two-

Figure 13: Ionization fraction y as a function of temperature T as inferred from the two-level Saha equation (Eq. 124).

level example), we recover the earlier form:

(128) $\quad \dfrac{n_B n_C}{n_A} = \left(\dfrac{2\pi k_B T}{h^2} \right)^{3/2} \left(\dfrac{m_B m_C{}^{3/2}}{m_A} \right)^{3/2} \dfrac{g_B g_C}{g_A} e^{(E_A - E_B - E_C)/k_B T}$

Note that for simple level-change reactions within atomic H, most factors cancel and we recover the usual Boltzmann distribution:

(129) $\quad \dfrac{n_B}{n_A} = \dfrac{g_B}{g_A} e^{(E_A - E_B)/k_B T}$

As a few final remarks, note that the above analysis only applies for excitation caused by the thermal distribution of particles in our system. So this won't properly treat **photoionization** (i.e. ejection of an electron due to an incoming, highly-energetic photon). Also, everything here also requires mostly-classical conditions, i.e. $n << n_Q$ for all species involved. In Very dense plasmas, pressure begins to affect electron orbital shapes and subsequently affects both intermediate energy levels as well as ionization.