9 Stellar Atmospheres

Having developed the machinery to understand the spectral lines we see in stellar spectra, we're now going to continue peeling our onion by examining its thin, outermost layer – the stellar atmosphere. Our goal is to understand how specific intensity I_ν varies as a function of increasing depth in a stellar atmosphere, and also how it changes depending on the angle relative to the radial direction.

9.1 *The Plane-parallel Approximation*

Fig. 14 gives a general overview of the geometry in what follows. The star is spherical (or close enough as makes no odds), but when we zoom in on a small enough patch the geometry becomes essentially plane-parallel. In that geometry, S_ν and I_ν depend on both altitude z as well as the angle θ from the normal direction. We assume that the radiation has no intrinsic dependence on either t or ϕ – i.e., the radiation is in steady state and is isotropic.

We need to develop a few new conventions before we can proceed. This is because in our definition of optical depth, $d\tau_\nu = \alpha_\nu ds$, the path length ds travels along the path. This Lagrangian description can be a bit annoying, so it's common to formulate our radiative transfer in a path-independent, Eulerian, prescription.

Let's call our previously-defined optical depth (Eq. 71) τ'_ν. We'll then create a slightly altered definition of optical depth – a vertical,ingoing optical depth (this is the convention). The new definition is almost identical to the old one:

(130) $d\tau_\nu = -\alpha_\nu dz$

But now our optical depth, is vertical and oriented to measure inward, toward the star's interior. In particular since $dz = ds\cos\theta$, relative to our old optical depth we now have

(131) $d\tau_\nu = -d\tau'_\nu \cos\theta$

Our radiative transfer equation, Eq. 78, now becomes

(132) $-\cos\theta \dfrac{dI_\nu}{d\tau_\nu} = S_\nu(\tau_\nu, \theta) - I_\nu(\tau_\nu, \theta)$

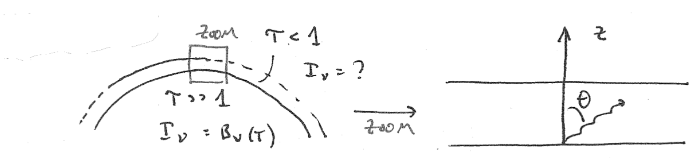

Figure 14: Schematic view of a stellar atmosphere, and at right a zoomed-in view showing the nearly plane-parallel nature on small scales.

It's conventional to also define $\mu = \cos\theta$, so our **radiative transfer equation** for stellar atmospheres now becomes

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu, \mu) \quad (133)$$

We can solve this in an analogous manner to how we treated Eq. 85, multiplying all terms by $e^{-\tau_\nu{}'/\mu}$, then rearranging to see that

$$\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu/\mu}\right) = -\frac{S_\nu}{\mu} e^{\tau_\nu/\mu} \quad (134)$$

And our solution looks remarkably similar to Eq. 86, except that we now explicitly account for the viewing angle μ:

$$I_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} \frac{S_\nu(\tau_\nu', \mu)}{\mu} e^{(\tau_\nu' - \tau_\nu)/\mu} d\tau_\nu' \quad (135)$$

So we now have at least a formal solution that could explain how I_ν varies as a function of the vertical optical depth τ_ν as well as the normal angle θ. It's already apparent that I_ν at a given depth is determined by the contributions from S_ν at all deeper levels, but these S_ν themselves depend on I_ν there. So we'd like to develop a more intuitive understanding than Eq. 135 provides.

Our goal will be to make a self-consistent model for S_ν and I_ν (or, as we'll see, I_ν and T). We'll again assume local thermodynamic equilibrium (LTE), so that

$$S_\nu = B_\nu(T) = B_\nu\left[T(\tau_\nu)\right] \quad (136)$$

(since T increases with depth into the star).

First, let's assume a simple form for S_ν so we can solve Eq. 135. We already tried a zeroth-order model for S_ν (i.e. a constant; see Eq. 79), so let's add a first-order perturbation, assuming that

$$S_\nu = a_\nu + b_\nu \tau_\nu \quad (137)$$

Where a_ν and b_ν are independent of τ_ν – for example, two blackbodies of different temperatures. When we plug this form into the formal solution of Eq. 135 and turn the crank, we find that the **emergent intensity** from the top of the star's atmosphere ($\tau_\nu = 0$) is

$$I_\nu(\tau_\nu = 0, \mu) = a_\nu + b_\nu \mu \quad (138)$$

Fig. 15 explains graphically what this solution means: namely, that the *angular* dependence of a star's emergent radiation encodes the *depth* dependence of its atmosphere's source function. If the depth dependence is small, so will the angular dependence be – and the reverse will also hold. So if $b_\nu \approx 0$, I_ν will be nearly isotropic with θ.

This describes the phenomenon of **limb darkening**, wherein the center of a stellar disk appears brighter than the edge. This is commonly seen in photographs of the Sun – it often looks to the eye like merely shadow effects of a 3D sphere, but in fact this represents temperature stratification.

Another interesting consequence involves the fact that an observer can only tpically observe down to $\tau_\nu \approx 1$. Because of the depth and angular dependencies we have just identified, this means that the surface where $\tau_\nu = 1$ (or any other constant value) occurs higher in the stellar atmosphere at the limb then at the disk center. Fig. 16 shows this effect. Since (as previously mentioned) temperature drops with decreasing pressure for most of a star's observable atmosphere, this means that we observe a cooler blackbody at the limb than at the center – and so the center appears brighter. (This is just a different way of thinking about the same limb-darkening effect mentioned above.) For the same reason, spectral lines look dark because at these lines α_ν is largest and so τ_ν occurs higher in the atmosphere, where temperatures are lower.

9.2 *Gray Atmosphere*

Now let's try to build a more self-consistent atmospheric model. To keep things tractable, we'll compensate for adding extra complications by simplifying another aspect: we'll assume a **gray atmosphere** in which the absorption coefficient (and derived quantities are independent of frequency). So we will use α instead of α_ν, and τ in place of τ_ν.

In this case, the equation of radiative transfer still has the same form as in

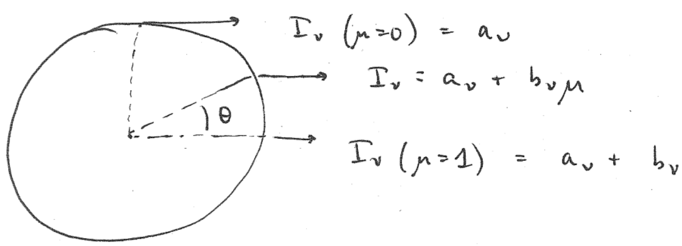

Figure 15: Emergent intensity as a function of θ assuming the linear model for S_ν given by Eq. 137.

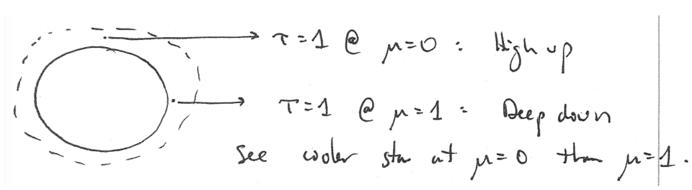

Figure 16: Depth dependence on the depth to which an observer can see into a stellar atmosphere: we see deeper at the center than at the limb.

Eq. 133 above:

$$(139)\quad \mu \frac{dI}{d\tau} = I(\tau, \mu) - S(\tau, \mu)$$

With the difference that by ignoring frequency effects, we are now equivalently solving for the **bolometric quantities**

$$(140)\quad I = \int_{\nu} I_{\nu}\, d\nu$$

and

$$(141)\quad S = \int_{\nu} S_{\nu}\, d\nu$$

Up until now, we've always assumed LTE with a Planck blackbody source function whose temperature varies with depth. But we've only used ad hoc models for this source — now, let's introduce some physically meaningful constraints. Specifically, let's require that flux is conserved as it propagates through the atmosphere. This is equivalent to saying there is no energy generation in the atmosphere: we just input a bunch of energy at the base and let it transport through and escape from the top.

This requirement of **flux conservation** means that $\frac{dF}{dt} = 0$, where

$$(142)\quad F = \int I \cos\theta d\Omega = \int \mu I d\Omega$$

(by definition; see Eq. 37).

To apply this reasonable physical constraint, let's integrate Eq. 139 over all solid angles:

$$(143)\quad \int \mu \frac{\partial I}{\partial \tau} d\Omega = \int (I - S) d\Omega$$

which implies that

$$(144)\quad \frac{dF}{d\tau} = 4\pi \langle I \rangle - 4\pi S$$

which equals zero due to flux conservation. Note that S is isotropic, while in general I may not be (i.e. more radiation comes out of a star than goes into it from space). The perhaps-surprising implication is that in our gray atmosphere,

$$(145)\quad S = \langle I \rangle$$

at all altitudes.