macroscopic scale, it means that the rate at which energy is produced in the center of the star is exactly equal to the star's luminosity: the rate at which that energy exits the surface.

How likely is it that a star satisfies this requirement? While a star may spend most of its life near Thermal Equilibrium while it is on the main sequence, most of the evolutionary stages it goes through do not satisfy Eq. 236: for example, pre-main sequence evolution (protostars) and post-main sequence evolution (red giants). How can we describe conservation of energy for an object that is not in Thermal equilibrium?

Following standard texts (e.g., Prialnik), we can make use of u, the internal energy density in a shell in our star. We can change u either by doing work on the shell, or by having it absorb or emit heat. We have already described how the heat in the shell can change with L_r and ϵ_m. Similarly, the incremental work done on the shell can be defined as a function of pressure and the incremental change in volume:

$$dW = -PdV \tag{237}$$

$$= -P\left(\frac{dV}{dm}dm\right) \tag{238}$$

$$= -P\, d\left(\frac{1}{\rho}\right) dm \tag{239}$$

The change in internal energy per unit mass (du) is equal to the work done per unit mass ($\frac{dW}{dm}$), so finally we can rewrite Eq. 239 as:

$$du = -Pd\left(\frac{1}{\rho}\right) \tag{240}$$

Taking the time derivative of each side,

$$\frac{du}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) \tag{241}$$

Compression of the shell will decrease dV, and thus require energy to be added to the shell, while expansion increases dV and is a way to release energy in the shell.

Changes in the internal energy of the shell u with time can then be described in terms of the both the work done on the shell and the changes in heat:

$$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right) \tag{242}$$

The general form of Eq. 242 is the next equation of stellar structure, known either as the **Energy Equation** or the **Equation of Conservation of Energy**.

You may also sometimes see this equation written in various other forms, such as in terms of the temperature T and entropy S of the star. In this form, you then have

$$\frac{\partial L_r}{\partial m} = \epsilon_m - T\frac{dS}{dt} \tag{243}$$

Chemical Composition

An additional relationship that is useful for determining stellar evolution is the change in a star's composition.This relation will be less of an 'equation' for the purposes of this class, and more a rough depiction of how the composition of a star can vary with time.

We can define the composition of a star using a quantity called the mass fraction of a species:

$$X_i = \frac{\rho_i}{\rho}. \tag{244}$$

Here, ρ_i is the partial density of the i^{th} species.

Particles in a star are defined by two properties: their baryon number $\mathcal{A}$ (or the number of total protons and neutrons they contain) and their charge $\mathcal{Z}$. Using the new notation of baryon number, we can rewrite Equation 292 to be the corresponding partial number density of the i^{th} species:

$$n_i = \frac{\rho_i}{\mathcal{A}_i m_H}. \tag{245}$$

We can then slightly rewrite our expression for the composition as

$$X_i = n_i \frac{\mathcal{A}_i}{\rho} m_H. \tag{246}$$

Changes in composition must obey (at least) two conservation laws. Conservation of charge:

$$\mathcal{Z}_i + \mathcal{Z}_j = \mathcal{Z}_k + \mathcal{Z}_l. \tag{247}$$

and conservation of baryon number:

$$\mathcal{A}_i + \mathcal{A}_j = \mathcal{A}_k + \mathcal{A}_l. \tag{248}$$

If you also consider electrons, there must also be a conservation of lepton number.

Without attempting to go into a detailed formulation of an equation for the rate of change of X we can see that it must depend on the starting composition and the density, and (though it does not explicitly appear in these equations) the temperature, as this will also govern the rate of the nuclear reactions responsible for the composition changes (analagous to our description of molecular collisions in Equation 281 and shown in Figure 26, in which the velocity of particles is set by the gas temperature). This leads us to our last 'equation' of stellar structure, which for us will just be a placeholder function **f** representing that the change in composition is a function of these variables:

(249) $\dot{\mathbf{X}} = \mathbf{f}(\rho, T, \mathbf{X}).$

Technically, this **X** is a vector representing a series of equations for the change of each X_i.

The final fundamental relation we need in order to derive the structure of a star is an expression for the temperature gradient, which will be derived a bit later on.

11.3 *Pressure*

We have already introduced a relationship for the gas pressure, for an ideal gas, in Equation 291. However, now that we have begun talking more about the microscopic composition of the gas we can actually be more specific in our description of the pressure. Assuming the interior of a star to be largely ionized, the gas will be composed of ions (e.g., H^+) and electrons. Their main interactions ('collisions') that are responsible for pressure in the star will be just between like particles, which repel each other due to their electromagnetic interaction. As a result, we can actually separate the gas pressure into the contribution from the ion pressure and the electron pressure:

(250) $P_{gas} = P_e + P_{ion}$

For a pure hydrogen star, these pressures will be equivalent, however as the metallicity of a star increases, the electron pressure will be greater than the ion pressure, as the number of free electrons per nucleon will go up (for example, for helium, the number of ions is half the number of electrons).

Assuming that both the ions and electrons constitute an ideal gas, we can rewrite the ideal gas equation for each species:

(251) $P_e = n_e kT$

and

(252) $P_{ion} = n_{ion} kT$

However, this is not the full story: there is still another source of pressure in addition to the gas pressure that we have not been considering: the pressure from radiation.

Considering this pressure then at last gives us the total pressure in a star:

$$P = P_{ion} + P_e + P_{rad} \tag{253}$$

We can determine the radiation pressure using an expression for pressure that involves the momentum of particles:

$$P = \frac{1}{3}\int_0^\infty v\ p\ n(p)\ dp \tag{254}$$

Here v is the velocity of the particles responsible for the pressure, p is their typical momentum, and $n(p)$ is the number density of particles in the momentum range $(p, p + dp)$. We first substitute in values appropriate for photons ($v = c$, $p = \frac{h\nu}{c}$). What is $n(p)$? Well, we know that the Blackbody (Planck) function (Equation 49) has units of energy per volume per interval of frequency per steradian. So, we can turn this into number of particles per volume per interval of momentum by (1) dividing by the typical energy of a particle (for a photon, this is $h\nu$), then (3) multiplying by the solid angle 4π, and finally (4) using $p = \frac{E}{c}$ to convert from energy density to momentum density.

$$P_{rad} = \frac{1}{3}\,4\pi\int_0^\infty c\left(\frac{h\nu}{c}\right)\left(\frac{1}{h\nu}\right)\left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} d\nu \tag{255}$$

Putting this all together,

$$P_{rad} = \frac{1}{3}\left(\frac{4}{c}\right)\left[\pi\int_0^\infty\left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} d\nu\right] \tag{256}$$

Here, the quantity in brackets is the same integral that is performed in order to yield the Stefan-Boltzmann law (Equation 79). The result is then

$$P_{rad} = \frac{1}{3}\left(\frac{4}{c}\right)\sigma T^4 \tag{257}$$

The quantity $\frac{4\sigma}{c}$ is generally defined as a new constant, a.

We can also define the specific energy (the energy per unit mass) for radiation, using the relation

$$u_{rad} = 3\frac{P_{rad}}{\rho} \tag{258}$$

When solving problems using the Virial theorem, we have encountered a similar expression for the internal energy of an ideal gas:

$$KE_{gas} = \frac{3}{2}NkT \tag{259}$$

From the ideal gas law for the gas pressure (Equation 291), we can see that the specific internal energy $\frac{KE}{m}$ then can be rewritten in a similar form:

$$u_{gas} = \frac{P_{gas}}{\rho} \tag{260}$$

11.4 *The Equation of State*

In a star, an equation of state relates the pressure, density, and temperature of the gas. These quantities are generally dependent on the composition of the gas as well. An **equation of state** then has the general dependence $P = P(\rho, T, X)$. The simplest example of this is the ideal gas equation. Inside some stars radiation pressure will actually dominate over the gas pressure, so perhaps our simplest plausible (yet still general) equation of state would be

$$P = P_{gas} + P_{rad} \tag{261}$$

$$= nkT + \frac{4F}{3c} \tag{262}$$

$$= \frac{\rho kT}{\mu m_p} + \frac{4\sigma_{SB}}{3c}T^4 \tag{263}$$

where μ is now the **mean molecular weight per particle** – e.g., $\mu = 1/2$ for fully ionized H.

But a more general and generally applicable equation of state is often that of an adiabatic equation of state. As you might have encountered before in a physics class, an adiabatic process is one that occurs in a system without any exchange of hear with its environment. In such a thermally-isolated system, the change in internal energy is due only to the work done on or by a system. Unlike an isothermal process, an adiabatic process will by definition change the temperature of the system. As an aside, we have encountered both adiabatic and isothermal processes before, in our description of the early stages of star formation. The initial collapse of a star (on a free-fall time scale) is a roughly isothermal process: the optically thin cloud is able to essentially radiate all of the collapse energy into space unchecked, and the temperature does not substantially increase. However, once the initial collapse is halted when the star becomes optically thick, the star can only now radiate a small fraction of its collapse energy into space at a time. It then proceeds to contract nearly adiabatically.

Adiabatic processes follow an equation of state that is derived from the first law of thermodynamics: for a closed system, the internal energy is equal to the amount of heat supplied, minus the amount of work done.

As no heat is supplied, the change in the specific internal energy (energy per unit mass) u comes from the work done by the system. We basically already derived this in Equation 240:

$$du = -Pd\left(\frac{1}{\rho}\right) \tag{264}$$

As we have seen both for an ideal gas and from our expression for the radiation pressure, the specific internal energy is proportional to $\frac{P}{\rho}$:

$$u = \phi\frac{P}{\rho} \tag{265}$$

Where ϕ is an arbitrary constant of proportionality. If we take a function of that form and put it into Equation 264 we recover an expression for P in terms of ρ for an adiabatic process:

$$P \propto \rho^{\frac{\phi+1}{\phi}} \tag{266}$$

We can rewrite this in terms of an adiabatic constant K_a and an adiabatic exponent γ_a:

$$P = K_a\rho^{\gamma_a} \tag{267}$$

For an ideal gas, $\gamma_a = \frac{5}{3}$.

This adiabatic relation can also be written in terms of volume:

$$PV^{\gamma_a} = K_a \tag{268}$$

This can be compared to the corresponding relationship for an ideal gas, in which PV = constant.

11.5 *Summary*

In summary, we have a set of coupled stellar structure equations (Eq. 227, Eq. 231, Eq. 235, Eq. 242, and Eq. 267):

$$\frac{dr}{dm} = \frac{1}{4\pi r^2\rho} \tag{269}$$

(270) $$\ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}.$$

(271) $$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB}T^3r^2}$$

(272) $$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

(273) $$P = K_a\rho^{\gamma_a}$$

If we can solve these together in a self-consistent way, we have good hope of revealing the unplumbed depths of many stars. To do this we will also need appropriate boundary conditions. Most of these are relatively self-explanatory:

(274) $$M(0) = 0$$

(275) $$M(R) = M_{tot}$$

(276) $$L(0) = 0$$

(277) $$L(R) = 4\pi R^2\sigma_{SB}T_{\text{eff}}^4\rho(R) = 0$$

(278) $$P(R) \approx 0$$

(279) $$T(R) \approx T_{\text{eff}}$$

(280)

But to solve the equations of stellar structure even with all these constraints in hand is still a beast of a task. In practice one integrates numerically, given some basic models (or tabulations) of opacity and energy generation.