13 Polytropes

Much of the challenge in making self-consistent stellar models comes from the connection between T and L. The set of so-called **polytrope models** derives from assuming that we can just ignore the thermal and luminosity equations of stellar structure. This assumption is usually wrong, but it is accurate in some cases, useful in others, and historically was essential for making early progress toward understanding stellar interiors. A polytrope model assumes that for some proportionality constant K and index γ (or equivalently, n),

(308)
$$P = K\rho^{\gamma}$$

(309)
$$= K\rho^{1+1/n}$$

We have already discussed at least two types of stars for which a polytrope is an accurate model. For fully convective stars, energy transport is dominated by bulk motions which are essentially adiabatic (since $\tau_{dyn}\langle\tau_{\gamma,\mathrm{diff}}$); thus $\gamma = \gamma_{ad} = 5/3$. It turns out that the same index also holds for degenerate objects (white dwarfs and neutron stars); in the non-relativistic limit these also have $\gamma = 5/3$, even though heat transport is dominated by conduction not convection. When degenerate interiors become fully relativistic, γ approaches $4/3$ and (as we saw previously) the stars can come perilously close to global instability.

The key equation in polytrope models is that of hydrostatic equilibrium (Eq. 192),

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

which when rearranged yields

(310) $$\frac{r^2}{\rho}\frac{dP}{dr} = -GM.$$

Taking the derivative of each side, we have

(311) $$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -GdM,$$

and substituting in the mass-radius equation (Eq. 227) for dM gives

(312) $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho.$$

It is then customary to define the density in terms of a dimensionless density function $\phi(r)$, such that

(313) $$\rho(r) = \rho_c\phi(r)^n$$

and n is the polytrope index of Eq. 309. Note that $\phi(r=0)=1$, so ρ_c is the density at the center of the star, while $\phi(r=R)=0$ defines the stellar surface. Combining Eq. 313 with Eq. 309 above gives

(314) $P(r) = K\rho_c^{1+1/n}\phi(r)^{n+1}$.

Plugging this back into Eq. 312 and rearranging yields the formidable-looking

(315) $$\lambda^2 \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\phi^n$$

where we have defined

(316) $$\lambda = \left(\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\right)^{1/2}.$$

When one also then defines

(317) $r = \lambda\xi$,

then we finally obtain the famous **Lane-Emden Equation**

(318) $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n.$$

The solutions to the Lane-Emden equation are the set of functions $\phi(\xi)$, each of which corresponds to a different index n and each of which completely specifies a star's density profile in the polytrope model via Eq. 313. The solution for a given n is conventionally denoted $\phi_n(\xi)$. Each solution also determines the temperature profile $T(r)$ (as you will see in Problem Set 5).

What are the relevant boundary conditions for $\phi(\xi)$, and what are the possible values of this dimensionless ξ anyway? Well, just as with $\phi(r)$ we must also have that $\phi(\xi=0)=1$, and analogously we will have $\phi(\xi=\xi_{\rm surf})=0$. As for $\xi_{\rm surf}$ (the value of ξ at the stellar surface), its value will depend on the particular form of the solution, $\phi(\xi)$. A final, useful boundary condition is that we have no cusp in the central density profile – i.e., the density will be a smooth function from $r=+\epsilon$ to $-\epsilon$. So our boundary conditions are thus

(319)
$$\phi(\xi=0)=1$$

(320)
$$\phi(\xi=\xi_{\rm surf})=0$$

(321)
$$\frac{d\phi}{d\xi}|_{\xi=0}=0$$

Just three analytic forms of $\phi(\xi)$ exist, corresponding to $n=0,\ 1,\ 5$. So-

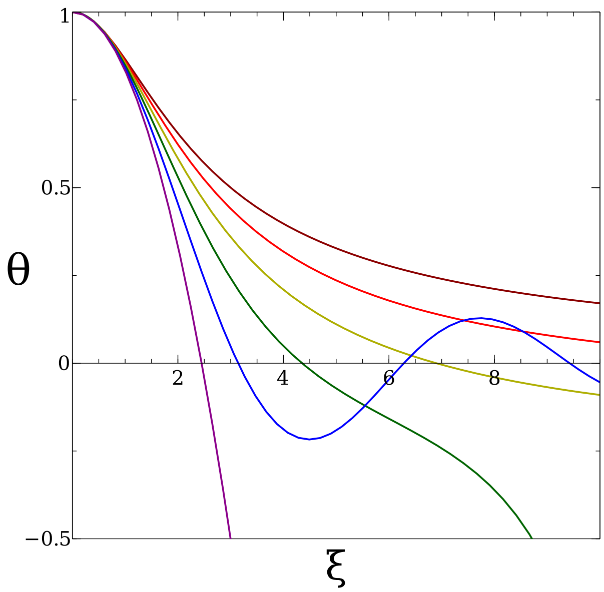

Figure 26: Solutions to the Lane-Emden Equation, here denoted by θ instead of ϕ, for $n = 0$ (most concentrated) to 5 (least concentrated). The applicability of each curve to stellar interiors ends at the curve's first zero-crossing. Figure from Wikipedia, used under a Creative Commons CCO 1.0 license.

lutions give finite stellar mass only for $n \leq 5$. Textbooks on stellar interiors give examples of these various solutions. One example is $n = 1$, for which the solution is

$$\phi_1(\xi) = a_0 \frac{\sin \xi}{\xi} + a_1 \frac{\cos \xi}{\xi} \tag{322}$$

where a_0 and a_1 are determined by the boundary conditions. A quick comparison to those conditions, above, shows that the solution is

$$\phi_1(\xi) = \frac{\sin \xi}{\xi} \tag{323}$$

which is the well-known sinc function. For a reasonable stellar model in which ρ only decreases with increasing r, this also tells us that for $n = 1$, $\xi_{\mathrm{surf}} = \pi$.

The point of this dense thicket of ϕ's and ξ's is that once n is specified, you only have to solve the Lane-Emden equation once. (And this has already been done – Fig. 26 shows the solutions for $n = 0$ to 5.) Merely by scaling K and ρ_c one then obtains an entire family of stellar structure models for each ϕ_n – each model in the family has its own central density and total mass, even though the structure of all models in the family (i.e., for each n) are homologously related.