# 16 End Stages of Nuclear Burning

## 16.1 *Useful references*

- Prialnik, 2nd ed., Appendix B
- Choudhuri, Secs. 5.1–5.2
- Kippenhahn, Weiger, and Weiss, 2nd ed., Ch. 15
- Hansen, Kawaler, and Trimble, Sec. 3.5

## 16.2 *Introduction*

Nuclear burning continues until fuel is exhausted. For the Sun, the p-p chain can continue for about $10^{10}$ yr. Once H is exhausted at the core, thermal pressure is lost: the upper layers of the star are no longer supported, and the core compresses. By the virial theorem, half of that gravitational binding energy goes into heating the gas.

When we hit $T_C \sim 10^8$ K at the core, the triple-$\alpha$ process kicks in and begins converting $^4$He into $^{12}$C. After that, ever-larger nuclei continue to fuse until either (1) we get up to $^{56}$Fe or (2) something besides nuclear burning can provide a (non-thermal) pressure source to maintain hydrostatic equilibrium. For Sunlike stars, **degeneracy pressure** provides that support.

## 16.3 *Degeneracy Pressure*

Degeneracy pressure results from the Pauli exclusion principle, which states that only one fermion is allowed to occupy any particular quantum state. In effect, fermions begin to repel each other in order to keep their quantum wavefunctions from overlapping.

Recall that in the very first lecture (Sec. 1, also in Sec. 16.2) we discussed an order-of-magnitude criterion for a classical ideal gas, namely

(380) $n \ll \lambda_D^{-3}.$

For ionized H gas, we found that this was equivalent to requiring

(381) $\rho \ll 10^3 \text{ g cm}^{-3} \left( \frac{T}{10^7\text{K}} \right)^{3/2}.$

We will now improve on this using kinetic theory.

We previously defined the density of states in phase space (Eq. 41) to be

(382) $\frac{dN}{d^3r d^3p} = f(\vec{r}, \vec{p}).$

We'll make the simplifying assumption that $f$ is both homogeneous (i.e., there is no $\vec{r}$ dependence) and isotropic (thus replacing $\vec{p}$ with $p$). For fermions, we

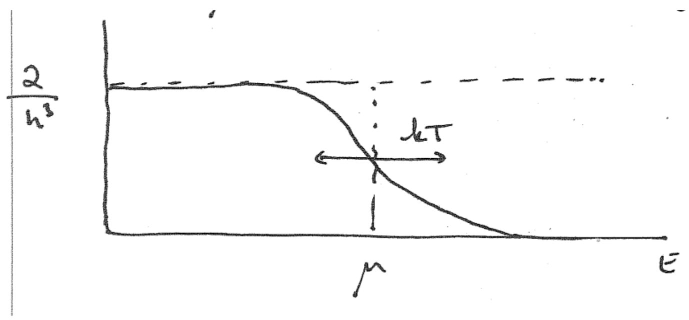


Figure 33: Phase space density of fermions as a function of energy $E$. The indicated value $\mu$ is the Fermi Energy. In the fully degenerate limit, the phase-space density approaches a step function.

have seen that this takes the form (Eq. 101)

$$f(p) = \frac{2}{h^3}\frac{1}{e^{[E-\mu]/k_BT+1}} = \frac{2}{h^3}n_{occ}. \quad (383)$$

Here, $\mu$ is not the mean molecular weight but rather $E_f$, the **Fermi energy** of the distribution. This quantity is derived by maximizing the number of microstates given $E$, $N$, and no more than two particles per state (see Fig. 33). So, much above $E_f$ almost no states are occupied; much below it almost all states are full.

The quantities derived from $f(p)$ above are key for us. In particular, we have the number density

$$n = \int f(p) d^3p \quad (384)$$

and also the gas pressure

$$P = \frac{1}{3}\int vpf(p)d^3p. \quad (385)$$

Note that we can relate $n$, $T$, and $\mu$ – so typically we will solve for $\mu$ in terms of these other variables.

**Non-degenerate (classical) case:**

In the classical limit, particles are widely spaced, $n << \lambda_D^3$, and $n_{occ} << 1$. This means that $\exp[(E-\mu)/kT] >> 1$. Furthermore, if we are fully classical then

$$E = \frac{p^2}{2m}. \quad (386)$$

This means that we have

(387)
$$f \approx \frac{2}{h^3} e^{-E/kT} e^{\mu/kT}$$

(388)
$$\approx \frac{2}{h^3} e^{-p^2/2mkT} e^{\mu/kT}.$$

If we integrate Eq. 388 over momentum to find $n$, we can solve for $\mu$ and find that

(389) $$f(p) \frac{n}{(2\pi mkT)^{3/2})} e^{-p^2/2mkT}$$

which we should recognize as being directly related to the Maxwell-Boltzmann distribution for an ideal gas (Eq. 95).

**Degenerate cases:**

In the fully degenerate limit, particles are packed as tightly together as their fermionic wavefunctions will allow. This means that

(390)
$$f(p) = 2/h^3 \qquad (E \leq \mu)$$

(391)
$$= 0 \qquad (E > \mu)$$

To calculate the number density from $f(p)$, we again calculate

(392)
$$n = \int f(p) d^3p$$

(393)
$$= 4\pi \int_0^{p_F} \frac{2p^2}{h^3} dp$$

(394)
$$= \frac{8\pi}{h^3} \frac{p_F^3}{3}$$

where $p_F$ is the Fermi momentum, at which $E = \mu$. Given the number density $n$, this means we can also recast things as

(395) $$p_F = \left( \frac{3nh^3}{8\pi} \right)^{1/3}$$

which will hold regardless of the particular relation between energy and momentum (i.e., whether we are fully relativistic or totally non-relativistic).

Let's then use Eq. 395 to calculate the pressure from a fully degenerate gas. Again, from Eq. 385 we have

$$P = \frac{1}{3}\int_0^{p_F} vp\frac{2}{h^3}4\pi p^2 dp. \tag{396}$$

We'll consider two limits:

1. $v = p/m$ (fully non-relativistic), and

2. $v \approx c$ (ultra-relativistic)

In the **non-relativistic degenerate** case, we calculate Eq. 385 as

$$P = \frac{8\pi}{3}\frac{1}{m}\frac{1}{h^3}\int_0^{p_F} p^4 dp \tag{397}$$

$$= \frac{8\pi}{15}\frac{p_F^5}{mh^3}. \tag{398}$$

If we then plug in Eq. 395, we see that in this limit the gas pressure is

$$P = \frac{8\pi}{15}\frac{1}{mh^3}\left(\frac{3h^3}{8\pi}\right)^{5/3} n^{5/3}. \tag{399}$$

Note this expression for pressure contains the term $1/m$, so the smallest-mass particles dominate the pressure. Thus the electrons are what really matter. If we want to cast $P$ in terms of the mass density, we use Eq. 174 and the mean molecular weight of electrons,

$$n_e = \frac{\rho_{tot}}{\mu_e m_p}, \tag{400}$$

to write

$$P = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20 m_e m_p^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3} \tag{401}$$

$$= K_{\rm NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}. \tag{402}$$

By comparison back to Eq. 310, we see that a non-relativistic gas is a polytrope (Sec. 13) with index $\gamma = 5/3$.

In the **relativistic degenerate** case, the particle velocities are independent

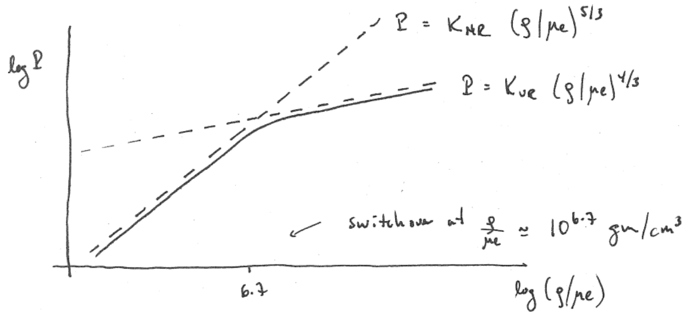


Figure 34: Pressure $P$ vs. density $\rho$ in the non-relativistic (NR) and ultra-relativistic (UR) limits. A switchover occurs at high densities above $\rho/\mu_e \approx 10^{6.7}$ g cm$^{-3}$.

of $p$, so we have one less $p$ in our integral for pressure:

(403)
$$P = \frac{1}{3}\int_0^{P_f} cp\frac{2}{h^3}4\pi p^2 dp$$

(404)
$$= \left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8m_p^{4/3}}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

(405)
$$= K_{\rm UR}\left(\frac{\rho}{\mu_e}\right)^{4/3}.$$

So the ultra-relativistic degenerate gas is also a polytrope, but now with a slightly shallower index $\gamma = 4/3$.

## 16.4 *Implications of Degeneracy Pressure*

So our discussion of polytropes in Sec. 13 was fruitless; it now turns out that they give an exact description of the behavior of a degenerate gas. The polytrope indices in the two cases, above, 5/3 vs. 4/3, seem close enough together that there might not be much difference. But comparison to Eqs. 296 and 297 show that the slightly lower index of 4/3 makes all the difference: a fully relativistic and degenerate gas will tend toward instability and collapse.

In the equations of state Eqs. 402 and 405 above, the degeneracy pressure will dominate over the gas pressure so long as

(406)

$$P_{deg} \gg P_{gas}$$

(407)

$$K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3} \gg \frac{\rho k T}{\mu_e m_p}.$$

And assuming a fully ionized medium (so $\mu_e = 1/2$), we then require

(408) $$\frac{\rho}{\mu_e} \gg 750 \text{ g cm}^{-3}\left(\frac{T}{10^7\ K}\right)^{3/2}$$

which is quite similar to our earlier estimate of $n \ll \lambda_D^{-3}$ (Eq. 381, and Sec. 16.2). As the density of a degenerate gas is increased, Fig. 34 demonstrates that the equation of state will switch over from non-relativistic (Eq. 402) to ultra-relativistic (Eq. 405) above densities $\rho/\mu_e \approx 10^{6.7}$ g cm$^{-3}$ or (equivalently) when

(409) $$p_F \approx m_e c = \left(\frac{3nh^3}{8\pi}\right)^{1/3}.$$

## 16.5 *Comparing Equations of State*

As we start moving into stellar evolution, we will encounter wildly different regimes of pressure, density, and temperature. Which equation of state dominates in each regime? We've seen several examples so far:

| Type | EOS | | Ideal gas | Temp. dependence |
|---|---|---|---|---|
| NR degeneracy pressure | $K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{2/3}$ |
| Rel degeneracy pressure | $K_{UR}\left(\frac{\rho}{\mu_e}\right)^{4/3}$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{1/3}$ |
| Radiation pressure | $\frac{4\sigma}{3c}T^4$ | = | $\frac{\rho}{\mu_e}\frac{kT}{m_p}$ | $T \propto \rho^{1/3}$ |

Note that the temperature for radiation pressure and ultra-relativistic degeneracy pressure have the same dependence on temperature; however, the coefficient is larger for the radiation pressure case.

One additional case we haven't yet discussed is: when does treatment as a gas break down? This turns out to happen when Coulomb interactions become increasingly important. Or equivalently, when

(410)

$$E_C \approx E_{Th}$$

(411)

$$\frac{e^2}{a} \approx n^{1/3} e^2 = kT$$

which implies that again, $T \propto \rho^{1/3}$ — but with a smaller coefficient that for

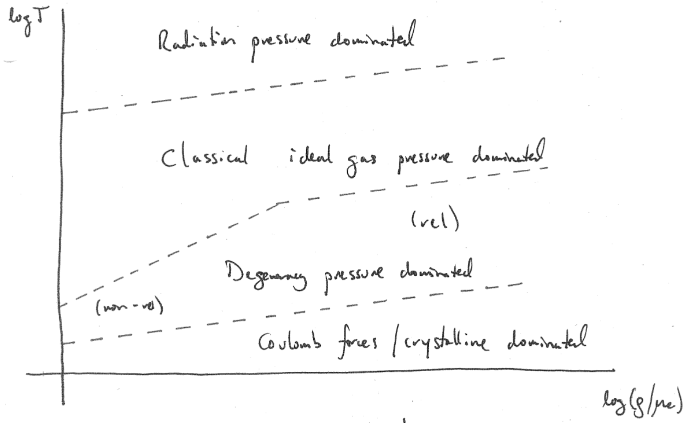


Figure 35: Different regimes in stellar interiors.

the ultra-relativistic degenerate gas. Fig. 35 summarizes all these different regimes.

Note that degeneracy pressure (like any good polytropic equation of state) is independent of temperature. So it halts stellar contraction even with no power generation. If nuclear power is somehow generated in a degenerate medium, there are interesting consequences:

- **Non-degenerate:** When extra energy is produced, the star expands and cools thanks to the virial theorem. Thus energy production will decreaes: negative feedback.
- **Degenerate star:** Extra energy production leads to no expansion of the star. The only place the energy can go is into heating the gas, so its temperature goes up – and thus energy production will increase as well. Positive feedback!

The positive feedback in the degenerate case can accelerate so rapidly that an entire star can become unbound. In other cases, the star will merely be heated up so much that the degenerate state is destroyed; then negative feedback via the virial theorem can once again come into play.