and forth between the two objects a time or two, but before too long one object or another will end its stellar life, as either a white dwarf, neutron star, or black hole.

23.5 *Accretion Disks*

Once our binary contains a compact stellar remnant, if the binary separation and mass ratio are right then one last phase of mass transfer can occur. When overflow occurs in a system with a compact object (WD, NS, or BH; call it m_2), the material has a long way to fall. It is pushed over the brink by the unbalanced pressure at L_1, and falls down toward m_2 with a velocity $v \approx c_s \sim 10s$ km s^{-1} — much smaller than the orbital speeds of ~ 100 km s^{-1}. When m_2 had a large radius this material would easily hit its target, but in this later phase of evolution the target is far smaller.

Now, the overflowing gas heads down, down toward m_2 — but all the while, the $\vec{v} \times \vec{\Omega}$ Coriolis force is steadily acting on the material, causing it to veer away from a direct path. The combined potential leads to the matter entering into an orbit around m_2, with the material's trajectory passing through its former position and smashing into the material that was coming along behind it.

Shock heating sets in where the infalling stream impacts the growing disk of matter, converting bulk kinetic energy into heat. Radiation can try to cool the hot, shocked material but it can't transport much angular momentum: so the accreted material ends up in a circular **accretion disk.**

Further evolution of the disk is set by its ability to transport mass inward through the disk while simultaneously moving angular momentum outward – these parameters are set in turn by the viscosity of the disk. Each concentric annulus of material in the disk wants to travel at a slightly different Keplerian speed. Very close to m_2 at the center of our accretion disk, orbits are determined solely by m_2 and so travel at the Keplerian angular velocity

$$\Omega_K(r) = \frac{v}{r} = \sqrt{\frac{GM}{r^3}}. \tag{595}$$

Meanwhile, the angular momentum per unit mass is

$$\ell(r) = rv = r^2\Omega_K = \sqrt{GMr}. \tag{596}$$

So as we go outward through successive annuli of the disk, Ω decreases but ℓ increases. These rings, rotating at different speeds, are coupled by viscosity – this effectively acts like friction. So each interior ring tries to speed up the rotation of its exterior neighbor, sending angular momentum outward and pushing out that exterior neighbor. At the same time, the ring interacts viscously with the next ring inward, trying to slow it down and so causing it to fall inward. The net effect is that the disk will spread toward smaller and larger radii, transporting angular momentum outward. Energy is dissipated by the viscous interactions (plus emitted radiation), so material falls steadily inward.

23.6 *Alpha-Disk model*

Our modest goal here is to find a steady-state model for an accretion disk with fixed mass transfer rate $\dot{M}$. When disk material spills into the compact object's potential well it has near-zero velocity but a long way to fall. Thus the ultimate power source of an accretion disk comes from the conversion of gravitational potential energy. Dropping in some small amount of mass m will liberate

$$\Delta E = \frac{GMm}{R} \tag{597}$$

and so the overall luminosity of the accretion disk should scale as

$$L_{\text{acc}} \approx \frac{GM\dot{M}}{R}. \tag{598}$$

The Stress Tensor

We will shortly introduce the so-called "α-disk" model that is often used to provide a phenomenological description of accretion disk physics. As background to this discussion, we first describe two useful foundational concepts. The first is the **viscous stress tensor** T_{ij}. (Some students have already encountered a variant of this tensor, the stress-energy tensor, in a general relativity class. For our discussion here, we only need the 3-D purely spatial stress tensor.) The quantity T_{ij} represents a flux of momentum:

$$T_{ij} = \text{Flux of momentum } p^i \text{ in the } j\text{-th direction} . \tag{599}$$

Imagine you have a box with sides parallel to the x, y, and z axes:

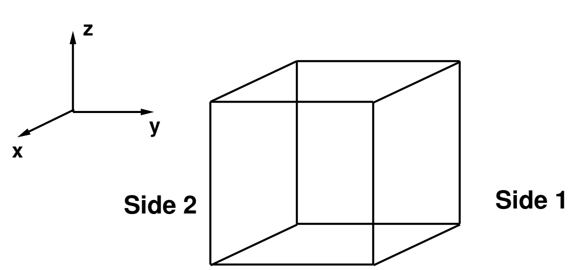

Figure 55: Fiducial box for computing fluxes.

The net rate of change of (for example) p^x associated with flow through the sides normal to the y axis is

$$\left.\frac{dp^x}{dt}\right|_{\hat{y}} = \int_{\text{side 1}} T_{xy}\,dx\,dz - \int_{\text{side 2}} T_{xy}\,dx\,dz\ . \tag{600}$$

Similar equations describe the rates of change of components associated with momentum flow through the sides normal to the other axes.

Viscosity

The second concept is the notion of **viscosity**. In a fluid, viscosity is the quantity which transports momentum component i in some direction that is not i — a *non-normal* stress. (Momentum component i transported in direction i is probably much more familiar: it is pressure.) It is related to the density of the fluid and the velocity *gradient*. In Cartesian coordinates, the coefficient of dynamic viscosity ν is defined as

$$T_{ij} = \rho\nu\frac{\partial v^i}{\partial x^j}\ . \tag{601}$$

When we convert to cylindrical coordinates (which we'll want to use to model an accretion disk), one particularly important component is

$$T_{\phi r} = \rho\nu r\frac{d\Omega}{dr}\ . \tag{602}$$

This tells us how axial motions in a disk are coupled to one another in the radial direction.

In fluids we typically encounter in our daily lives, the value of ν is of order the mean free path of the molecules in the fluid, λ, times their typical speed, $\bar{u}$. This intuition fails for accretion disks, which means that their viscosity (or their effective viscosity) must arise from some different physical mechanism. Determining this mechanism and thereby understanding the viscosity of astrophysical accretion disks is an important problem in modern astrophysics research.

Overview

Turn now to accretion disks. In steady state, accretion disks are a stable assembly of fluid in which various energy sources and various forces are in balance. This is very similar to the root underlying physics of stars; as such, it is not too surprising that the equations which govern the structure of accretion disks bear a more than passing resemblance to the equations of stellar structure. They are slightly more complicated, however, because of the different symmetry of the two systems: Stars are spherically symmetric, whereas accretion disks are cylindrically symmetric. A stellar model tells us how key quantities (temperature, pressure, density) vary with spherical radius r. An accretion disk model tells us how key quantities (temperature, pressure, density) vary with cylindrical radius r and with height z above or below the disk's midplane, $z = 0$. (If there is any ambiguity, we will sometimes write the radius r_c to emphasize the cylindrical symmetry.)

Beyond the difference in symmetry, the basic physics describing stars and accretion disks are more-or-less identical. In particular we have

1. **Force balance.** In stars, we balance gravity with hydrodynamic pressure. Thanks to spherical symmetry, we only need to do this in the radial direction. In an accretion disk, we balance gravity, pressure, centrifugal forces, and viscous coupling of adjacent fluid elements. The single force balance equation we found for stars splits into 3 separate equations (one for each component) in an accretion disk.
2. **Mass conservation.** The donor star pumps mass into the disk; it is transported inward and eventually falls onto the object that is accreting. In steady state, the mass in a fluid element does not change; mass flows in and out, and the sum is constant.
3. **Power generation.** In a star, we generate power by nuclear fusion. In an accretion disk, we generate power by fluid elements rubbing against one another.
4. **Radiation transport.** That power is generated throughout the disk, and has to flow out to the surface before it is radiated away. To understand this process, we need to know about the opacity of the material in the disk, and how the radiation gets out.
5. **Equation of state.** Just as in stars, we need to relate pressure to density.

We now go through these, though not quite in this order. Our final disk model wil display a well-ordered hierarchy of velocities:

$$v_z \ll v_r \ll c_s \lesssim v_\phi. \tag{603}$$

I.e., the accreting material rapidly swirls around the disk, at Keplerian velocities typically faster than the soundspeed. Much slower than those speeds will be a steady inward radial drift; even slower will be vertical settling toward the disk midplane.

The Alpha-Disk Model

1. **Radial force balance.**

   We assume here that the orbital speed is much larger than the sound speed of the gas. If this is the case, then we can neglect gas forces in favor of the centrifugal force: Considering a fluid element of mass Δm, and a central body of mass M onto which the accretion flows, we have

$$\frac{GM\Delta m}{r^2} = \Delta m \Omega^2 r \longrightarrow \boxed{\Omega(r) = \Omega_K = \sqrt{\frac{GM}{r^3}}} \tag{604}$$

This has a built in consistency check: After we build our model, we compare dP/dr with $GM\rho/r^2$. If the pressure gradient is not small in this comparison, then we should not have neglected it, and we need to revisit this.

2. **Vertical force balance.**

   We cannot neglect pressure gradients for this force component — they are the whole effect. We look at the z component of the usual equation of hydrostatic balance:

$$\frac{\partial P}{\partial z} = -g_z \rho \tag{605}$$

$$= -\frac{GM}{r^2}\left(\frac{z}{r}\right)\rho\ . \tag{606}$$

Integrate this up using the fact that pressure is zero outside the disk, $z \geq H/2$ (defining H as the disk's height, and $z = 0$ as the disk's midplane):

$$\int_0^{H/2} \frac{\partial P}{\partial z} dz = P_{H/2} - P_m = -P_m\ . \tag{607}$$

Here P_m is pressure on the midplane. We pull various factors out of the integral on the right-hand side, and find

$$P_m \simeq \frac{GM}{r^3} \int_0^{H/2} z\,\rho\,dz \approx \frac{GM\rho H^2}{r^3}\ . \tag{608}$$

In the final approximate result, we neglected factors of order unity. Errors due to this neglect should be comparable to any errors made in neglecting how ρ varies with z.

In some applications, we need somewhat better approximations than this. Note that the r which appears here is strictly speaking the *spherical* radius, not the cylindrical one. One way to improve the calculation would be to replace r with $\sqrt{r_c^2 + z^2}$ in the equation for dP/dz. For the applications we will pursue in 8.901, we use $r \simeq r_c$. This is known as the "thin disk" approximation. It is used quite widely, but it is worth being aware of its limitations.

Another approximation we have made is that the disk has a well-defined upper edge at $H/2$, when in reality the vertical pressure will tend to have an exponential decrease – in this case, H becomes the disk's vertical scale height.

3. **Mass conservation.**

Mass flows from large radius to small radius. Consider a cross section of one annulus of the disk, as shown in Fig. **??**. Think about mass flowing into and out of the volume associated with this cross section in a time Δt:

$$\begin{aligned}\Delta M &= \text{Mass entering outer radius in } \Delta t - \text{Mass leaving inner radius in } \Delta t \\ &= \{[\text{Cross section at } r_o]\,[\text{Mass flux at } r_o] - [\text{Cross section at } r_i]\,[\text{Mass flux at } r_i]\}\,\Delta t \\ &= \{[2\pi r_o H(r_0)]\,[\rho(r_o) v_r(r_o)] - [2\pi r_i H(r_i)]\,[\rho(r_i) v_r(r_i)]\}\,\Delta t\end{aligned}$$

Now divide everything by the volume of this annulus, $2\pi r H(r)\Delta r$, and take the limit as $\Delta r \to 0$, $\Delta t \to 0$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2\pi r H(r)} \frac{d}{dr} (2\pi r H \rho v_r) \ . \tag{609}$$

Strictly speaking Eq. 609 should be a partial derivative rather than a total derivative on the right-hand side. In the thin disk model, we neglect the dependence of quantities on z. As such, taking $\partial/\partial r \to d/dr$ is a fine approximation as long as the thin-disk conditions are met. We'll similarly use $\partial \to d$ in the calculations that follow.

In steady state, $\partial \rho / \partial t = 0$. Imagining that everything depends only on r, we then have

$$2\pi r H \rho v_r = \text{constant} \ . \tag{610}$$

We can tell by inspection that this constant is just the rate at which mass enters one side of the volume and then leaves the other, so we have

$$\boxed{2\pi r H \rho v_r = \dot{M}} \tag{611}$$

Note that the above calculation is equivalent to starting from the

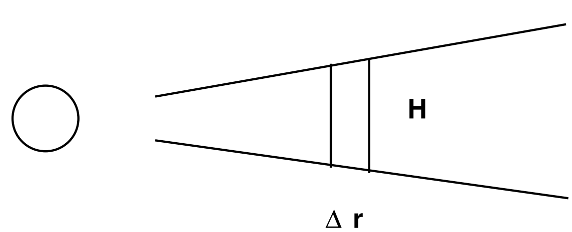

Figure 56: Cross section of one annulus of thickness Δr, height H.

mass continuity equation,

(612) $$\frac{\partial}{\partial r}(\rho v_r r) = 0.$$

This implies that $\rho v_r r$ is constant, which one then integrates:

(613) $$\int \rho v_r r \; d\phi \; dz = 2\pi r H \rho v_r = \dot{M}.$$

4. **Angular momentum.**

How to handle the transport of angular momentum in the disk is a little bit delicate. We begin by essentially repeating the mass transport analysis, but looking at how angular momentum flows radially through the disk:

$$\begin{aligned} \Delta L &= L \text{ entering outer radius} - L \text{ leaving inner radius} \\ &= \{[\text{Cross section at } r_o]\,[L \text{ flux at } r_o] - [\text{Cross section at } r_i]\,[L \text{ flux at } r_i]\}\,\Delta t \\ &= \left\{[2\pi r_o H(r_0)]\left[\rho(r_o) v_r(r_o)\Omega(r_o) r_o^2\right] - [2\pi r_i H(r_i)]\left[\rho(r_i) v_r(r_i)\Omega(r_i) r_i^2\right]\right\}\Delta t \end{aligned}$$

I've used the fact that the angular momentum of a mass element is $\Delta m r^2 \Omega$, so the radial flux associated with this angular momentum is $(\rho v_r) r^2 \Omega$. Divide by Δt, by the volume $2\pi r H \Delta r$, and take the limits. The result is

(614) $$\tau = \frac{1}{2\pi r H} \frac{d}{dr}\left(2\pi r H \rho v_r \Omega r^2\right) .$$

We have defined $\tau \equiv dL/dVdt$, the torque on the annulus per unit volume. We massage this one step further, using the result from our analysis of mass conservation to simplify:

(615) $$\boxed{\tau = \frac{1}{2\pi r H} \frac{d}{dr}\left(\dot{M} \Omega r^2\right)}$$

Now comes the tricky bit: What do we use for τ? Fundamentally, we know that τ arises from viscosity coupling adjacent annuli of the disk to one another: viscosity "wants" the disk to rotate as a solid body, so it tries to slows down annuli on the inside and speed up annuli on the outside. Our goal is to compute the torque associated with the ϕ component of momentum that flows in the r direction, as shown in Fig. 57.

We compute the angular momentum ΔL delivered to this annulus:

$$\begin{aligned}
\Delta L &= [r_o(\text{axial force at } r_o) + r_i(\text{axial force at } r_i)]\,\Delta t \\
&= \left[r_o \left(\int_{\text{outer face}} T_{\phi r}\, dA \right) - r_i \left(\int_{\text{inner face}} T_{\phi r}\, dA \right) \right] \Delta t \\
&= \left\{ T_{\phi r}(r_o) \left[2\pi r_o^2 H(r_o) \right] - T_{\phi r}(r_i) \left[2\pi r_i^2 H(r_i) \right] \right\} \Delta t \qquad (616)
\end{aligned}$$

Note on the first line that we *add* the two torques together: the torque associated with momentum flux at both the outer and the inner boundaries of the annulus contributes to the angular momentum in this volume. The math which follows determines the signs of these contributions; we find a relative minus sign because the normal associated with the inner face of the annulus points in, and that associated with the outer face points out.

Dividing and taking limits appropriately, we find

$$\tau = \frac{1}{rH} \frac{d}{dr} \left(T_{\phi r} r^2 H \right) \; . \qquad (617)$$

So far, our analysis has effectively just moved our ignorance from one place to another. This isn't a bad thing, since we've now moved our unknown into one quantity, the stress-tensor component $T_{\phi r}$.

To proceed, we need to figure out what to use for this quantity. If we could estimate the viscosity ν, we would use Eq. (602) to estimate $T_{\phi r}$. Estimating ν is rather tricky; however, we know that the resulting stress $T_{\phi r}$ must have the same dimensions as pressure. A very quick-and-dirty approximation is to imagine that P and $T_{\phi r}$ are proportional to one another:

$$T_{\phi r} = \alpha P \; . \qquad (618)$$

This approximation yields what is known as the **Shakura-Sunyaev α-disk model**. The key idea here is that, since the pressure P is *a* stress, it gives us a reasonable guess for typical values of *all* stresses in the disk. The parameter α parameterizes our ignorance, and is

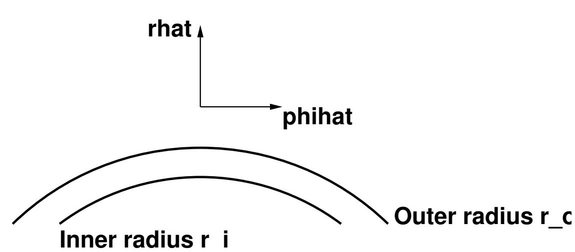

Figure 57: Top view of disk, looking down on our fiducial annulus.