Here η is

$$\eta = \beta - \frac{2}{3}\mu \tag{671}$$

which relates β, the bulk velocity coefficient, with μ, the shear velocity coefficient.

Conservation of Energy

The $n = 2$ moment expresses conservation of energy in the fluid, and equivalently determines the pressure P as well. Its derivation is truly marvelous but these notes are too narrow to contain it (well, almost). Nonetheless for completeness the final result is:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + e\right) + \vec{\nabla} \cdot \left(\rho \vec{u} u^2/2 + \vec{u}\mu\right) = \frac{\rho \vec{u} \cdot \vec{F}}{m} + \vec{\nabla} \cdot \left(\vec{u} \cdot \vec{\vec{\Pi}}\right) - \vec{\nabla} \cdot \vec{q} \tag{672}$$

where $\vec{\vec{\Pi}}$ is still the viscous diffusion tensor, e is the internal energy

$$e = \frac{P}{\gamma - 1}, \tag{673}$$

q is the heat flow

$$q_i = \Sigma_i \int v_i v_j^2 f d^3 v, \tag{674}$$

and μ (a different μ than immediately above) is the enthalpy (total heat content)

$$\mu = e + P = \frac{\gamma P}{\gamma - 1}. \tag{675}$$

24.3 *Shocks: Rankine-Hugoniot Equations*

Shocks are a frequent topic of study in astrophysical (and other) fluid studies. A **shock** occurs whenever a propogating wave is sufficiently intense that nonlinear wave theory no longer applies. In this case, the increased pressure at the traveling wave front builds up and ultimately leads to a sharp discontinuity in fluid velocity, ρ, and P.

In a coordinate system moving with a shock, under the right conditions the three moment equations are simplified greatly. The resulting, simplified statements of conservation of number (mass), momentum, and energy are the **Rankine-Hugoniot jump conditions**:

$$\rho \vec{u} = \text{const.}, \tag{676}$$

$$P + \rho u^2 = \text{const.}, \tag{677}$$

and

$$(678)\quad \frac{P}{\rho}\frac{\gamma}{\gamma-1}+\frac{1}{2}\rho u^2 = \text{const.}.$$

Using these three conservation equations, if we know the pre-shock conditions (in region 1) then we can calculate the post-shock (region 2) conditions.

Velocity and Density

A shock wave zooms through; in the shock's frame of reference, the unshocked material is moving at speed u_1. What will be the speed of the shocked medium: i.e., what is u_2?

From the energy equation,

$$(679)\quad \frac{1}{2}\left(u_1^2-u_2^2\right)=\frac{\gamma}{\gamma-1}\left(\frac{P_2}{\rho_2}\frac{P_1}{\rho_1}\right).$$

Invoking the continuity equation and rearranging gives

$$(680)\quad \frac{\rho_1 u_1}{2}\left(u_1^2-u_2^2\right)=\frac{\gamma}{\gamma-1}(P_2u_2-P_1u_1).$$

Applying the momentum equation and dividing out a factor of (u_1-u_2) yields

$$(681)\quad \frac{\rho_1 u_1}{2}(u_1+u_2)=\frac{\gamma}{\gamma-1}(-P_1+\rho_1u_1u_2).$$

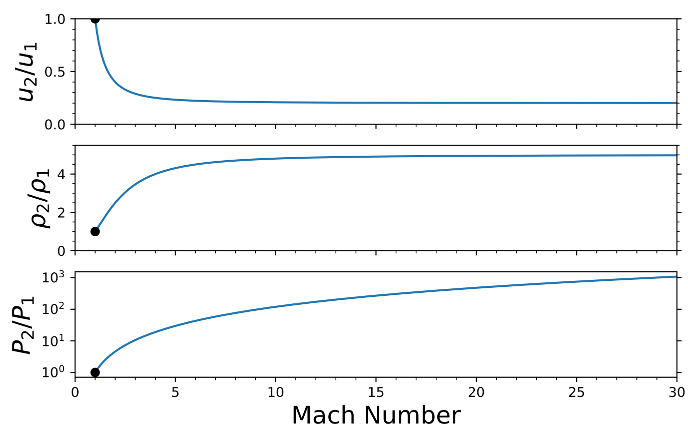

Figure 58: Post-shock conditions vs. shock speed: velocity (top; Eq. 684), density (middle; Eq. 686), and pressure (bottom; Eqs. 691 and 692). The black dot indicates $M=1$; below this speed, there is no shock.

We now bring in the **Mach number**, defined as the velocity relative to the local soundspeed:

(682) $$M = \frac{u}{c_s} = u\sqrt{\frac{\rho}{\gamma P}}.$$

This, plus another round of algebra, gives

(683) $$\frac{u_1 + u_2}{2} = \frac{\gamma}{\gamma - 1}\left(u_2 - \frac{u_1}{\gamma M_1^2}\right).$$

Finally one can factor out the terms containing u_1 and u_2, divide, and find that

(684) $$\frac{u_2}{u_1} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{1}{M_1^2}.$$

Since *gamma* ≈ 1.5 and shocks are supersonic $M > 1$, this means $u_2 < u_1$. The velocity after the shock has passed will always be less than the speed of the shock front. In the limit of a very fast-moving shock,

(685) $$\lim_{M\to\infty} \frac{u_2}{u_1} = \frac{\gamma - 1}{\gamma + 1}.$$

From Eq. 684 and the continuity equation, the density relation is then also quickly derived:

(686) $$\rho_2 = \rho_1 \frac{u_1}{u_2}$$

So as the shock speed increases the density will asymptotically approach

(687) $$\lim_{M\to\infty} \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}.$$

Thus the density is always greater after the shock has passed through, but never more than a $\sim$few times greater.

Pressure

To determine the new pressure, we start with the combined continuity and momentum equations

(688) $$P_2 = P_1 + \rho_1 u_1 (u_1 - u_2)$$

(689) $$= P_1 + \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right).$$

Using Eq. 682 to substitute for P_1 and Eq. 684 for u_2/u_1, after some algebraic gymnastics we have

(690)
$$P_2 = \rho_1 u_1^2 \left(\frac{1}{\gamma M_1^2} + \frac{2(M_1^2 - 1)}{M_1^2(\gamma+1)} \right)$$
(691)
$$= \rho_1 u_1^2 \left(\frac{1}{\gamma M_1^2} + \frac{2}{\gamma+1} - \frac{2}{M_1^2(\gamma+1)} \right).$$

The ratio of pressures requires a bit more work; defining $r = \rho_2/\rho_1$, it is

(692) $$\frac{P_2}{P_1} = \frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1) - (\gamma-1)r}$$

The pressure will also be greater after the shock has passed through; unlike for density, a shock can increase the pressure to arbitrarily large values.

24.4 *Supernova Blast Waves*

A common shock is the sudden, cataclysmic injection of energy into the interstellar medium by a supernova. The outer layers of the dying star are ejected from the remnant core at extremely high velocities, where they interact with an ISM that is essentially at rest. The evolution of the SN blast wave can be considered in two distinct phases: the initial (energy-conserving) **Sedov-Taylor** expansion phase, and the later (momentum-conserving) **snowplow phase**.

Sedov-Taylor Expansion Phase

Soon after the supernova goes off, the ejecta's energy content (thermal plus kinetic) is much greater than what is being radiated away. Thus we can approximate the expansion as adiabatic. (Note that the Sedov phase really only begins when the mass swept up in the blast wave shell becomes comparable to the inidial ejecta mass; this takes roughly 70–100 yr.)

A common approach is to assume (reasonably) that the shock wave's expansion will depend on the shock front radius r at time t, the ISM's initial density ρ_1, and the energy injected E. One defines a characteristic, dimensionless quantity

(693) $$\xi \equiv r t^l E^m \rho_1^n.$$

For ξ to be dimensionless and not retain units of length (L), mass (M), or time (T) it must be true that

(694) $$[\xi] = 1 = L T^l \left(\frac{ML^2}{T^2} \right)^m \left(\frac{M}{L^3} \right)^n$$

which implies

$$l = -2/5$$
$$m = -1/5$$
$$n = +1/5.$$

This dimensional argument immediately implies that the shock front radius scales as

$$r_{\rm sh} = \xi_0 \left(\frac{Et^2}{\rho_1}\right)^{1/5}. \tag{695}$$

For reasonable assumptions, ξ_0 is of order unity (typically with $\lesssim 20\%$). This also gives an expression for the speed of the expanding shock front,

(696)
$$u_{\rm sh} = \frac{2}{5}\xi_0 \left(\frac{E}{\rho_1 t^3}\right)^{1/5}$$
(697)
$$= \frac{2}{5} r_{\rm sh}/t.$$

We can compare these predictions to observations, e.g. of the relatively young Crab Nebula. For $E = 10^{51}$ erg, $\rho_1 = 10^{-24}$ g cm^{-3}, and $t = 1000$ yr our predictions come moderately close to reality:

(698)
$$r_{\rm pred} \approx 5 \text{ pc} \sim r_{\rm obs} \approx 3 \text{ pc}$$
(699)
$$u_{\rm pred} \approx 2000 \text{ km s}^{-1} \sim u_{\rm obs} \approx 900 \text{ km s}^{-1}.$$

With the size and speed of the shock wave in hand, we can use the Rankine-Hugoniot conditions to determine the density, bulk velocity, and pressure inside the shock front (just after the blast wave has passed through). Using the nomenclature of Fig. 59, for a very strong (highly supersonic) blast wave, *im-*

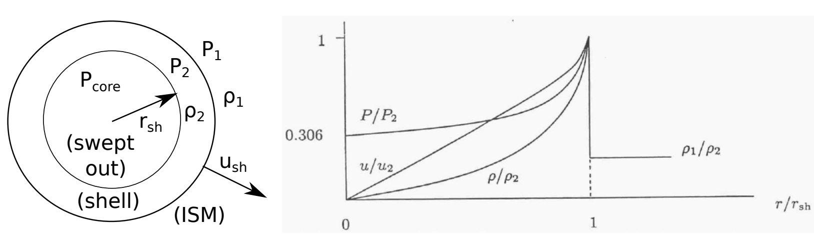

Figure 59: *Left:* Schematic diagram of a supernova shock wave with finite shell width. *Right:* Sedov solution for $\gamma = 5/3$ (Fig. 17.3 of Shu, Vol. II).

mediately inside the blast front Eqs. 684, 686, and 691 will give

$$u_{2,Lab} = -u_{2,Shock} + u_{\rm sh} = -\left(\frac{\gamma-1}{\gamma+1}\right) u_{\rm sh} + u_{\rm sh} = \frac{2}{\gamma+1} u_{\rm sh}$$
$$\rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1$$
$$P_2 = \frac{2}{\gamma+1}\rho_1 u_{\rm sh}^2.$$

By combining these with the fluid equations (Eqs. 666, 667, and 672) and setting $u = (2/5)r_{\rm sh}/t$ (Eq. 697), one obtains a set of analytically-tractable relations for the pressure, density, and velocity as a function of radius. These relations are plotted in the right-hand panel of Fig. 59.

Snowplow Phase

Long after the supernova goes off, the ejecta has lost enough energy and is expanding slowly enough that energy losses via radiation become significant. This happens with the total radiated energy is comparable to the initial input energy E, which typically takes $\sim 10^5$ yr. At this point the energy of the shock wave is no longer conserved, but its momentum should still be conserved. In this case, the shock front acts like a snowplow coasting into a snow bank; the series of collisions is inelastic and the wave continues to slow down at a new, different rate.

At this late stage, the supernova blast wave with radius r has swept up a spherical region of mass, which is carried along with the shock: this is just

(700) $$M_{\rm shell} = \frac{4}{3}\pi r^3 \rho_1.$$

In the momentum-conserving phase, we should have

(701) $$p_{\rm shell} = M_{\rm shell}(t) u_{\rm sh}(t) = \text{ const.}$$

This means

(702) $$r^3 \dot{r} = \text{ const,}$$

which implies

(703)
$$r_{\rm sh} \propto t^{1/4} \quad \text{and}$$
(704)
$$u_{\rm sh} \propto t^{-3/4}.$$

24.5 *Rayleigh-Taylor Instability*

Another common use of fluid dynamics is to determine when a given system becomes unstable. The approach used here is perturbation theory: assume some initial conditions in (perhaps unstable) equilibrium, assume a small perturbation to those conditions, and observe the results: if the perturbation grows with time then an instability is indicated.

One such scenario is the **Rayleigh-Taylor Instability**: given two media with densities ρ_u and ρ_ℓ and a local acceleration (i.e. gravity) field $\vec{g}$ perpendicular to the interface between the media, the media will be unstable if the denser material is "on top."

We assume initial conditions $u = 0$, $\rho = \rho_0(z)$, and $P = P_0$. We then examine the situation if a small density perturbation ρ_1 is applied. This perturbation may also change the velocity and pressure, so the new conditions are

$$\rho = \rho_0 + \rho_1$$
$$\vec{u} = u_1$$
$$P = P_0 + P_1.$$

The standard approach is to model ρ_1 as the complex function

$$\rho_1 = \rho_1(z)e^{i(kx-\omega t)}. \tag{705}$$

The utility of this approach is that ω will determine when our situation is stable or not. Specifically, if $\omega^2 > 1$ then ω is real and the perturbation (i.e., its real part) will merely oscillate with time; but if $\omega^2 < 1$ then our perturbation will grow exponentially with time, indicating an unstable system.

We then proceed to apply each of the fluid equations of Sec. **??**, beginning with the continuity equation (Eq. 666):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0.$$

Applying Eq. 705, we then obtain

$$-i\omega\rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{u}_1) + \vec{\nabla} \cdot (\rho_1 \vec{u}_1) = 0. \tag{706}$$

Since ρ_1 and $\vec{u}_1$ are both small their product is negligible, and the last term can be dropped. Furthermore, if the fluids are incompressible then

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{707}$$

We then have

$$-i\omega\rho_1 + \vec{u}_1 \cdot \vec{\nabla}\rho_0 = 0$$
$$-i\omega\rho_1 + \frac{\partial \rho_0}{\partial z} u_1^z = 0$$

and so the amplitude of the density perturbations is

$$\rho_1 = \frac{(\partial \rho_0 / \partial z) u_z^z}{i\omega}. \tag{708}$$

The next step is to determine the pressure perturbation, P_1, that results from the applied density perturbation. For this we begin with the next moment equation, of momentum conservation. A simplified Eq. 667 is

$$\frac{\partial}{\partial t}(\rho\vec{u}) = -\vec{\nabla}P + \rho\vec{g}.$$

Expanding this using our perturbed quantities gives

$$\frac{\partial}{\partial t}(\rho_0\vec{u}_1) = -\vec{\nabla}P_1 - \rho_1 g\hat{z}.$$

The gradient of the pressure perturbation (which has a form analogous to Eq. 705, is

$$\vec{\nabla}P_1 = ikP_1\hat{x} + \frac{\partial P_1}{\partial z}\hat{z}. \tag{709}$$

We thus have two equations for each of the two component directions:

$$-i\omega\rho_0 u_1^x = -ikP_1 \qquad (\hat{x}\text{ direction}) \tag{710}$$

and

$$-i\omega\rho_0 u_1^z = -\rho_1 g - \frac{\partial P_1}{\partial z} \quad (\hat{z}\text{ direction}). \tag{711}$$

From the first of these (Eq. 710, we have

$$P_1 = \frac{\omega}{k}\rho_0 u_1^x \tag{712}$$

which is usually recast using the definition of incompressibility,

$$\vec{\nabla}\cdot u = iku_1^x + \frac{\partial u_1^z}{\partial z} = 0 \tag{713}$$

to give

$$P_1 = \frac{i\omega}{k^2}\rho_0\frac{\partial u_z^z}{\partial z}. \tag{714}$$

For the final piece of the puzzle, to determine ω^2 and so determine whether our stratified fluid is stable or not, we begin with Eq. 711:

(715)
$$-i\omega\rho_0 u_1^z = -\rho g - \frac{\partial P_1}{\partial z}$$

(716)
$$= -\rho_1 g - \frac{\partial}{\partial z}\left(\frac{i\omega}{k^2}\rho_0\frac{\partial u_1^z}{\partial z}\right)$$

(717)
$$= -\rho_1 g - \frac{i\omega}{k^2}\left(\frac{\partial}{\partial z}\left(\rho_0\frac{\partial u_1^z}{\partial z}\right)\right).$$

The density gradient is zero in all directions but $\hat{z}$, and even then it is zero everywhere except at the boundary $z = 0$. Thus we have

(718) $$\vec{\nabla}\rho_0 = \frac{\partial\rho_0}{\partial z}\hat{z} = (\rho_u - \rho_\ell)\,\delta(z)\hat{z}.$$

With this and Eq. 708, we then have

(719) $$\frac{\partial}{\partial z}\left(\rho_0\frac{\partial u_1^z}{\partial z}\right) - k^2\rho_0 u_1^z = \frac{gk^2}{\omega^2}u_1^z\,(\rho_u - \rho_\ell)\,\delta(z).$$

Because of the delta function, two different expressions will result depending on whether or not $z = 0$. If not, then $\delta(z) = 0$ and so Eq. 719

(720) $$\frac{\partial^2}{u_1^z}\partial z^2 = k^2 u_1^z$$

whose solution has the usual form:

(721) $$u_1^z = u_{10}^z e^{-k|z|}.$$

On the other hand, if $z = 0$ then we take Eq. 719 and integrate up across the boundary layer; schematically, we are calculating $\int_{0^-}^{0^+} dz$. Since $\rho_0 u_1^z$ is continuous at $z = 0$ (by Eq. 721),

(722) $$\int\limits_{0^-}^{0^+} k^2\rho u_1^z dz = 0.$$

Again making use of Eq. 721, integrating the second term in Eq. 719 will give

(723) $$\int\limits_{0^-}^{0^+}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial u_1^z}{\partial z}\right)dz = -ku_{10}^z\,(\rho_u + \rho_\ell)\,.$$

Finally, integrating over the delta function in the right-hand side of Eq. 719

yields

$$\int_{0^-}^{0^+} \frac{gk^2}{\omega^2} u_1^z \left(\rho_u - \rho_\ell\right) \delta(z) = \frac{gk^2}{\omega^2} u_{10}^z \left(\rho_u - \rho_\ell\right). \tag{724}$$

The key result of all this that we have now shown that

$$\boxed{\omega^2 = -gk\frac{\rho_u - \rho_\ell}{\rho_u + \rho_\ell}.} \tag{725}$$

The implication is that if the denser material is "on top," $\rho_u > \rho_\ell$, $\omega^2 < 0$, and so the density perturbation will grow exponentially with time. If the the denser material starts out underneath, then the situation is stable. This is why oil always floats on water (even if you try to pour a layer of water onto a pre-existing layer of oil). It is also responsible for the fascinating surface shape of the interface between supernova remnants and the ISM, as seen (e.g.) in the Crab Nebula.