some initial conditions in (perhaps unstable) equilibrium, assume a small perturbation to those conditions, and observe the results: if the perturbation grows with time then an instability is indicated.

One such scenario is the **Rayleigh-Taylor Instability**: given two media with densities ρ_u and ρ_ℓ and a local acceleration (i.e. gravity) field $\vec{g}$ perpendicular to the interface between the media, the media will be unstable if the denser material is "on top."

We assume initial conditions $u = 0$, $\rho = \rho_0(z)$, and $P = P_0$. We then examine the situation if a small density perturbation ρ_1 is applied. This perturbation may also change the velocity and pressure, so the new conditions are

$$\rho = \rho_0 + \rho_1$$
$$\vec{u} = u_1$$
$$P = P_0 + P_1.$$

The standard approach is to model ρ_1 as the complex function

(705) $\rho_1 = \rho_1(z)e^{i(kx-\omega t)}$.

The utility of this approach is that ω will determine when our situation is stable or not. Specifically, if $\omega^2 > 1$ then ω is real and the perturbation (i.e., its real part) will merely oscillate with time; but if $\omega^2 < 1$ then our perturbation will grow exponentially with time, indicating an unstable system.

We then proceed to apply each of the fluid equations of Sec. 24.2, beginning

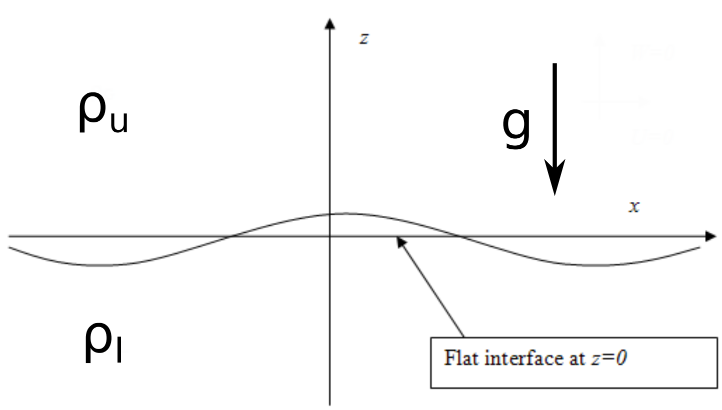

Figure 60: Initial conditions for considering the Rayleigh-Taylor instability, just after an initial perturbation has been applied.

with the continuity equation (Eq. 666):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0.$$

Applying Eq. 705, we then obtain

$$-i\omega\rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{u}_1) + \vec{\nabla} \cdot (\rho_1 \vec{u}_1) = 0. \tag{706}$$

Since ρ_1 and $\vec{u}_1$ are both small their product is negligible, and the last term can be dropped. Furthermore, if the fluids are incompressible then

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{707}$$

We then have

$$-i\omega\rho_1 + \vec{u}_1 \cdot \vec{\nabla}\rho_0 = 0$$
$$-i\omega\rho_1 + \frac{\partial \rho_0}{\partial z} u_1^z = 0$$

and so the amplitude of the density perturbations is

$$\rho_1 = \frac{(\partial\rho_0/\partial z)u_1^z}{i\omega}. \tag{708}$$

The next step is to determine the pressure perturbation, P_1, that results from the applied density perturbation. For this we begin with the next moment equation, of momentum conservation. A simplified statement of momentum conservation (Eq. 667) is

$$\frac{\partial}{\partial t}(\rho \vec{u}) = -\vec{\nabla} P + \rho \vec{g}.$$

Expanding this using our perturbed quantities gives

$$\frac{\partial}{\partial t}(\rho_0 \vec{u}_1) = -\vec{\nabla} P_1 - \rho_1 g \hat{z}.$$

The gradient of the pressure perturbation (which has a form analogous to Eq. 705), is

$$\vec{\nabla} P_1 = ikP_1 \hat{x} + \frac{\partial P_1}{\partial z}\hat{z}. \tag{709}$$

We thus have two equations for each of the two component directions:

$$-i\omega\rho_0 u_1^x = -ikP_1 \qquad (\hat{x} \text{ direction}) \tag{710}$$

and

$$-i\omega\rho_0 u_1^z = -\rho_1 g - \frac{\partial P_1}{\partial z} \quad (\hat{z} \text{ direction}). \tag{711}$$

From the first of these (Eq. 710), we have

(712) $P_1 = \frac{\omega}{k}\rho_0 u_1^x$

which is usually recast using the definition of incompressibility,

(713) $\vec{\nabla} \cdot u = iku_1^x + \frac{\partial u_1^z}{\partial z} = 0$

to give

(714) $P_1 = \frac{i\omega}{k^2}\rho_0 \frac{\partial u_z^z}{\partial z}.$

For the final piece of the puzzle, to determine ω^2 and so determine whether our stratified fluid is stable or not, we begin with Eq. 711:

(715)
$$-i\omega\rho_0 u_1^z = -\rho g - \frac{\partial P_1}{\partial z}$$
(716)
$$= -\rho_1 g - \frac{\partial}{\partial z}\left(\frac{i\omega}{k^2}\rho_0 \frac{\partial u_1^z}{\partial z}\right)$$
(717)
$$= -\rho_1 g - \frac{i\omega}{k^2}\left(\frac{\partial}{\partial z}\left(\rho_0 \frac{\partial u_1^z}{\partial z}\right)\right).$$

The density gradient is zero in all directions but $\hat{z}$, and even then it is zero everywhere except at the boundary $z = 0$. Thus we have

(718) $\vec{\nabla}\rho_0 = \frac{\partial \rho_0}{\partial z}\hat{z} = (\rho_u - \rho_\ell)\,\delta(z)\hat{z}.$

With this and Eq. 708, we then have

(719) $\frac{\partial}{\partial z}\left(\rho_0 \frac{\partial u_1^z}{\partial z}\right) - k^2\rho_0 u_1^z = \frac{gk^2}{\omega^2}u_1^z\,(\rho_u - \rho_\ell)\,\delta(z).$

Because of the delta function, two different expressions will result depending on whether or not $z = 0$. If not, then $\delta(z) = 0$ and so Eq. 719 becomes

(720) $\frac{\partial^2}{u_1^z}\partial z^2 = k^2 u_1^z$

whose solution has the usual form:

(721) $u_1^z = u_{10}^z e^{-k|z|}.$

On the other hand, if $z = 0$ then we take Eq. 719 and integrate up across the boundary layer; schematically, we are calculating $\int_{0^-}^{0^+} dz$. Since $\rho_0 u_1^z$ is continuous at $z = 0$ (by Eq. 721),

$$\int_{0^-}^{0^+} k^2 \rho u_1^z dz = 0. \tag{722}$$

Again making use of Eq. 721, integrating the second term in Eq. 719 will give

$$\int_{0^-}^{0^+} \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial u_1^z}{\partial z}\right) dz = -k u_{10}^z \left(\rho_u + \rho_\ell\right). \tag{723}$$

Finally, integrating over the delta function in the right-hand side of Eq. 719 yields

$$\int_{0^-}^{0^+} \frac{gk^2}{\omega^2} u_1^z \left(\rho_u - \rho_\ell\right) \delta(z) = \frac{gk^2}{\omega^2} u_{10}^z \left(\rho_u - \rho_\ell\right). \tag{724}$$

The key result of all this that we have now shown that

$$\boxed{\omega^2 = -gk \frac{\rho_u - \rho_\ell}{\rho_u + \rho_\ell}.} \tag{725}$$

The implication is that if the denser material is "on top," $\rho_u > \rho_\ell$, $\omega^2 < 0$, and the situation is unstable. If the the denser material starts out underneath, then the situation is stable. This is why oil always floats on water (even if you try to pour a layer of water onto a pre-existing layer of oil). It is also responsible for the fascinating surface shape of the interface between supernova remnants and the ISM, as seen (e.g.) in the Crab Nebula, and it is of fundamental importance for inertially-confined fusion experiments.

In the case of instability, the density perturbation grows exponentially with time so long as the perturbations are small. One sometimes defines the Atwood Number

$$A \equiv \frac{\rho_u - \rho_\ell}{\rho_u + \rho_\ell} \tag{726}$$

in which case the characteristic growth timescale is

$$\tau_{\rm RT} = (Agk)^{-1/2}. \tag{727}$$

Since the wavenumber $k = 2\pi/\lambda$, this means

$$\tau_{\rm RT} = \left(\frac{\lambda}{2\pi A g}\right)^{1/2} \tag{728}$$

So the shortest-wavelength perturbations grow most rapidly.

Once the perturbation amplitude is comparable to its wavelength, this linear regime begins to break down. We will then have alternating rising and sinking plumes, moving at different relative velocities. In the presence of a velocity shear and different densities, we have the **Kelvin-Helmholtz-Rayleigh-Taylor instability.** It turns out that in the presence of velocity shear ω is always complex, and so the fluid will always be unstable. The Kelvin-Helmholtz instability is responsible for some cloud patterns on Earth, and it sculpts the shapes of outflow jets from compact, accreting sources. The combined KHRT instability is responsible for the characteristic "mushroom clouds" that form above large (or even nuclear) explosions.