MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 1
Due: Friday, February 15, 2019, in class
This problem set is worth **106 points**.

1. **Gravity and density [12 pts].** The gravitational constant is $G = 6.673 \times 10^{-11}$ kg^{-1} m^3 s^{-2}. A more interesting way to express G is

$$\frac{1}{G} = 1.16\ (\mathrm{g\ cm^{-3}})\ \mathrm{hr}^2.$$

This suggests that gravitational timescales depend on density, and that for "terrestrial" densities ($\rho \sim 1 - 10$ g cm^{-3}) the timescales are measured in hours. You will show this in two specific examples.

   (a) A test particle is in a circular orbit just above the surface of a spherical planet. Calculate the orbital period, in terms of the planet's mean density ρ. What is the period for an Earth-like planet ($\rho = 5.5$ g cm^{-3})?

   (b) Consider a uniform-density sphere of non-interacting particles, initially at rest. How long does it take to gravitationally collapse? Give your answer in terms of the initial mean density ρ, and evaluate it for $\rho = 5.5$ g cm^{-3}. *Hint*: A sneaky way to perform this calculation to relate the given situation to the $e \to 1$ limiting case of the Kepler problem.

2. **Eclipsing binaries [10 pts].** Assume that two stars are in circular orbits about a mutual center of mass and are separated by a distance a. Assume also that the binary inclination angle is i (defined as the angle between the line-of-sight and the orbital angular momentum vector, with $0° \leq i \leq 90°$) and that the two stellar radii are R_1 and R_2. Find an expression for the smallest inclination angle that will just barely produce an eclipse.

3. **The Kepler acceleration [18 pts].** Kepler's first law says the orbit of a planet is an ellipse with the Sun at one focus:

$$r(\phi) = \frac{a(1-e^2)}{1 + e\cos\phi},$$

where r is measured from the focus, ϕ is measured from the distance of closest approach (*pericenter*), and e is the eccentricity.

Kepler's second law says that equal areas are swept out in equal times: $\frac{1}{2}r^2\dot{\phi} =$ constant.

These two laws give a complete *kinematic* description of a planetary orbit: the position and velocity of the planet as a function of time. Newton showed that these laws *imply* that the planet experiences an inward radial acceleration varying inversely with distance. In this problem you will do the same. Importantly, in what follows do not use Newtonian physics; simply use Kepler's two laws, kinematics, and calculus.

   (a) Write the x and y coordinates of the planet as a function of a, e, and ϕ.

   (b) Differentiate x and y with respect to time, and use Kepler's second law to show that the velocity vector can be written

$$\vec{v} = K_1\left[(-\sin\phi)\hat{x} + (e + \cos\phi)\hat{y}\right],$$

   where K_1 is a constant.

   (c) Show that the acceleration vector can be written

$$\vec{a} = -K_2\frac{\hat{r}}{r^2},$$

   where K_2 is another constant.

4. **Kepler problem: parametric solution [16 pts].** In the Kepler problem there is no closed-form solution for $r(t)$. In this problem you will derive a parametric solution.

   (a) Start with the energy equation

   $$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GM\mu}{r},$$

   where E is the energy, μ is the reduced mass, r is the distance between the two bodies, L is the angular momentum in the center-of-mass frame, and M is the total mass. Show that the time can be written

   $$t - t_0 = \sqrt{\frac{a}{GM}} \int_{r(t_0)}^{r(t)} \frac{r\,\mathrm{d}r}{\sqrt{a^2e^2 - (r-a)^2}}.$$

   (b) Use the substitution $r - a = -ae\cos u$, and Kepler's third law, to derive the parametric solution

   $$\begin{aligned} t - t_0 &= \left(\frac{P}{2\pi}\right)(u - e\sin u) \\ r &= a(1 - e\cos u), \end{aligned}$$

   where t_0 is the time of closest approach (pericenter).

5. **Orbital precession [20 pts].** Although Keplerian orbits are closed ellipses, any departure from a pure Keplerian potential will generally produce non-closing orbits. In particular, the first-order effect of small departures is to cause an elliptical orbit to *precess*: the direction of pericenter rotates at a steady rate within the plane of the orbit.

   (a) Consider a potential

   $$V(r) = -\frac{GM_1M_2}{r}\left(1 + \frac{r_0^2}{r^2}\right),$$

   where r_0 is a constant with the dimensions of length, representing a small perturbation to the Keplerian potential. Calculate the time derivative of the Laplace-Runge-Lenz vector,

   $$\vec{A} \equiv \vec{p} \times \vec{L} - GM_{\mathrm{tot}}\mu^2\hat{r},$$

   using a polar coordinate system in which the angle ϕ is measured with respect to the initial direction of $\vec{A}$. (*Hint*: Your answer should be a vector in the $\hat{\phi}$ direction.)

Problem continues on the next page

(b) Show that the effect of the perturbation is to cause $\vec{A}$ to rotate at a constant angular rate

$$\omega_{\rm prec} = \frac{2\pi}{P_{\rm prec}} = \left(\frac{2\pi}{P}\right)\frac{3r_0^2}{[a(1-e^2)]^2},$$

where a is the orbital radius, P is the orbital period of the unperturbed orbit, and $P_{\rm prec}$ is the (much longer) precession period. You may wish to follow these steps:

- Show that $|\vec{A}|$ is approximately constant by averaging $d/dt(|\vec{A}|^2)$ over an orbital period. Changes in $|\vec{A}|$ do not accumulate, but changes in the direction of $\vec{A}$ do accumulate.
- Make use of the fact that if any vector $\vec{V}$ is precessing, its instantaneous precession rate can be written

$$\vec{\omega}_{\rm prec} = \hat{V} \times \frac{d\hat{V}}{dt} = \frac{\vec{V} \times \frac{d}{dt}\vec{V}}{V^2}.$$

- In evaluating $\vec{\omega}_{\rm prec}$, you are calculating a leading-order correction to $\vec{A}(t)$ in terms of the small parameter r_0/r. Use the Keplerian result derived in class for the zeroth-order term: $\vec{A} = eGM\mu^2\hat{x}$ (where $\hat{x}$ is the direction to the pericenter).
- Average the instantaneous precession rate over an entire orbit, assuming a small but nonzero eccentricity. You should get an answer proportional to $(1-e^2)^{-2}$. Then let $e \to 0$.

6. **Tidal evolution of the Earth-Moon system [30 pts].** In this problem, you will compute the evolution of the Earth-Moon system by considering the tidal coupling between the Moon's orbit and the Earth's rotation. Angular momentum may be exchanged between these two components but must be conserved overall. Energy may be lost from the system via the heat generated by tidal friction. You should neglect any effects due to the rotation of the Moon.

(a) Write down expressions for the total energy E and and total angular momentum J of the Earth-Moon system. Some useful symbols will be the Earth's angular rotation frequency ω; the Moon's (Keplerian) orbital frequency Ω; the masses of the Earth and Moon, M_e and M_m; the Earth's moment of inertia I; and the mean separation of the Earth and Moon, a.

(b) Use the equation for J to eliminate ω from the energy equation.

(c) Show that the energy equation can be cast into the dimensionless form

$$\epsilon = -\frac{1}{s} + \alpha(j - s^{1/2})^2 ,$$

where ϵ is the total energy in units of $(GM_eM_m/2a_0)$, j is the total angular momentum in units of $(\mu a_0^2\Omega_0)$, $s = a/a_0$ is the dimensionless separation, μ is the reduced mass, and the subscript "0" refers to values in the present epoch of the Earth-Moon system's history.

(d) Find numerical values for α and j. Look up the masses of the Earth and Moon, look up the radius R_e of the Earth, and take the Earth's moment of inertia to be $(2/5)M_eR_e^2$. Take $a_0 = 3.84 \times 10^5$ km.

(e) Graph the dimensionless energy to estimate the two values of s for which ϵ is an extremum.

(f) Find the same two values of s quantitative by differentiating the energy equation and solving the resulting nonlinear equation numerically. Show that $\omega = \Omega$ at this orbital separations. Find the corresponding orbital period of the Moon and rotation period of the Earth.

(g) Find the difference in energy ΔE between the current epoch and the time in the future when the Earth's rotation and Moon's orbit will be synchronous.

(h) Estimate the rate of energy dissipation due to tidal friction by assuming that, twice per day, the top 1-m layer of the oceans is lifted by 1-m and then lowered. Further assume that some fraction η of this mechanical energy is dissipated as heat.

(i) Set $\eta = 0.01$ and estimate the time (from the current epoch) when this equilibrium configuration will be reached.