MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 2
Due: Friday, February 22, 2018, in class
This problem set is worth **120 points**.

1. **The radial velocity equation [20 pts].** In this problem you will derive the equation for the radial (line-of-sight) velocity variations of a star in Keplerian motion. This equation is used to model spectroscopic binaries.

   (a) Suppose two stars with masses M_1 and M_2 are in a Keplerian orbit with period P and eccentricity e. Show that the radial velocity of star 1 can be written

   $$v_{\mathrm{rad},1}(t) = \left(\frac{2\pi}{P}\right)\frac{a_1 \sin I}{\sqrt{1-e^2}}\left\{\cos\left[\phi(t)+\omega\right] + e\cos\omega\right\}. \tag{1}$$

   Here, a_1 is the semimajor axis of star 1's orbit, I is the inclination, $\phi(t)$ is the "true anomaly" (the polar angle in the orbital plane, measured from pericenter), and ω is the argument of pericenter; see Figure 1 for a description of the angles I and ω. One way to proceed is as follows:

   - Show that the radial coordinate of the relative orbit can be written

     $$Z = r(\phi)\sin(\phi+\omega)\sin I, \tag{2}$$

     with reference to the coordinate system depicted in Figure 1.
   - Take the time derivative of z and use the conservation of angular momentum,

     $$L = \mu r^2 \dot{\phi} = \sqrt{GM\mu^2 a(1-e^2)}, \tag{3}$$

     as well as the polar equation for the ellipse,

     $$r = \frac{a(1-e^2)}{1+e\cos\phi}, \tag{4}$$

     to obtain the desired form for v_{rad} of the effective one-body orbit.
   - Scale the result for v_{rad} appropriately to obtain the radial velocity of star 1.

   (b) Show that the radial velocity may also be written

   $$v_{\mathrm{rad},1}(t) = \left(\frac{2\pi G}{P}\right)^{1/3}\frac{M_2 \sin I}{(M_1+M_2)^{2/3}\sqrt{1-e^2}}\left\{\cos\left[\phi(t)+\omega\right] + e\cos\omega\right\}. \tag{5}$$

   (c) To calculate $v_{\mathrm{rad},1}(t)$, you will need to compute $\phi(t)$. Last week you derived parametric equations for $r(t)$ but not for $\phi(t)$. Show that $\phi(t)$ can be obtained by

   $$\tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}, \tag{6}$$

   where u is the the same parameter as in problem 5(b) of problem set 1. (This parameter is known as the "eccentric anomaly.")

   One way to proceed is

   - Use the polar equation for an ellipse to solve for $\cos\phi$ in terms of r, a, and e.
   - Use the previous result $r = a(1-e\cos u)$ to eliminate r in favor of u.
   - Write expressions for $(1+\cos\phi)$ and $(1-\cos\phi)$ in terms of u and e.
   - Use trigonometric identities to derive the desired result.

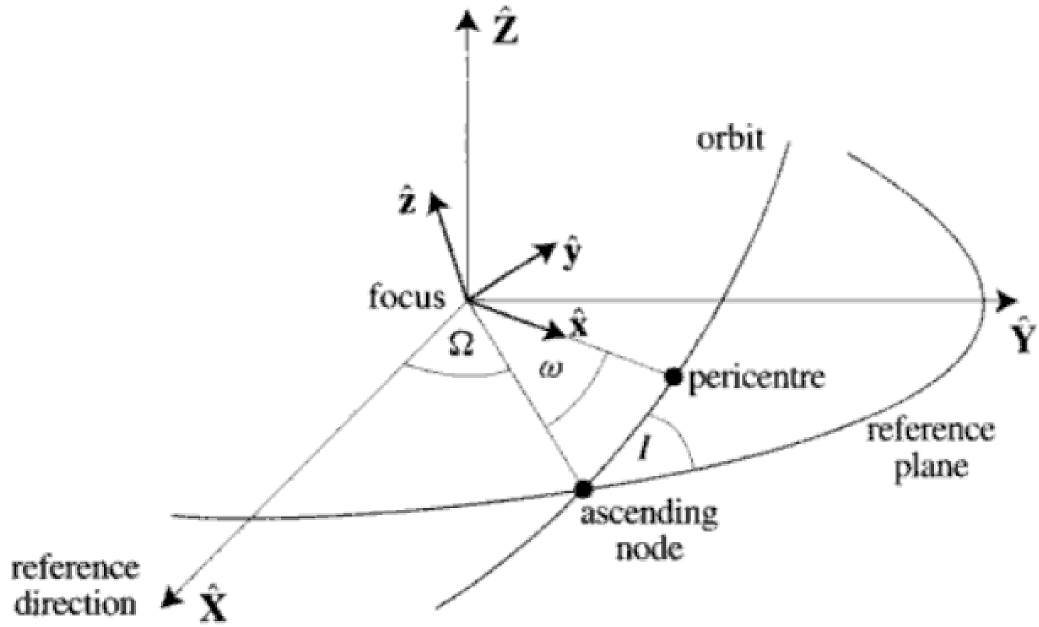

Figure 1: Definition of the orbital elements Ω, I, and ω (problem 1). For observational astronomy, the z-axis is chosen to be the line of sight.

(d) HD 80606b is a giant planet with an orbital eccentricity exceeding 0.9, presenting an extreme example of the "eccentric exoplanet" problem: the observation that exoplanets often have eccentric orbits, despite the 20th century expectation that more circular orbits would be common. The planet and star have masses $4.2\ M_{\rm Jup}$ and $1.05\ M_\odot$ respectively. The orbital parameters are $P = 111.44$ days, $e = 0.933$, $I = 89.32$ degrees, and $\omega = 300.83$ degrees.

Write a computer program to plot the radial velocity of the star as a function of time. The radial velocity should be in units of m s^{-1}, and the time should be in units of days relative to pericenter. Plot at least two full orbits.

2. **Visual-spectroscopic binary [30 pts].** Sirius, the brightest star in the night sky, is a binary system consisting of a normal A-type star (referred to as Sirius A) and a white dwarf (Sirius B). It is both a visual binary and a spectroscopic binary. Moreover, it is close enough to Earth for its parallax to be measured accurately.

   A plot of the Sirius orbit is shown in Figure 2. In this plot, the more massive A star is taken to be a fixed reference point, and the position of the white dwarf relative to the A star is plotted as a function of time.

   (a) Estimate the orbital period P.

   (b) The plotted orbital shape is elliptical. However, even a circular orbit will appear elliptical if the orbital plane is tipped at an angle with respect to our line of sight. Present an argument, based on the plot, that the orbit is truly eccentric.

   (c) Measure the apparent semimajor axis of the binary in arcsec.

   A careful analysis of the trajectory (which you need not perform) would reveal e, ω, Ω, and i, from which you could deduce that the actual semimajor axis is obtained by *dividing* the apparent semimajor axis by 0.95.

   (d) The parallax of the center of mass of the binary is $\pi = 0.379''$. Use the implied distance to find the physical size of the semimajor axis in AU, and the sum of the masses, $M_A + M_B$.

   (e) If the orbits of both stars had been plotted around the center of mass of the binary, the orbit of the white dwarf would be 2.4 times larger than that of Sirius A. What is the mass ratio M_A/M_B? What are the values of M_A and M_B?

   (f) The apparent bolometric magnitudes of Sirius A and B are -2.1 and $+8.3$, respectively. Compute the luminosity of each star in terms of the solar luminosity, $L_\odot$. The absolute bolometric magnitude of the Sun is $+4.75$.

   (g) Use the Stefan-Boltzmann law and the known surface temperatures of the Sun and Sirius A & B (5777 K, 11200 K, and 28500 K, respectively) to compute the radii of Sirius A & B in units of the Sun's radius, $R_\odot$.

   (h) Compute the density (in g cm^{-3}) of Sirius A and B. Compare to the Earth's mean density of 5.5 g cm^{-3}.

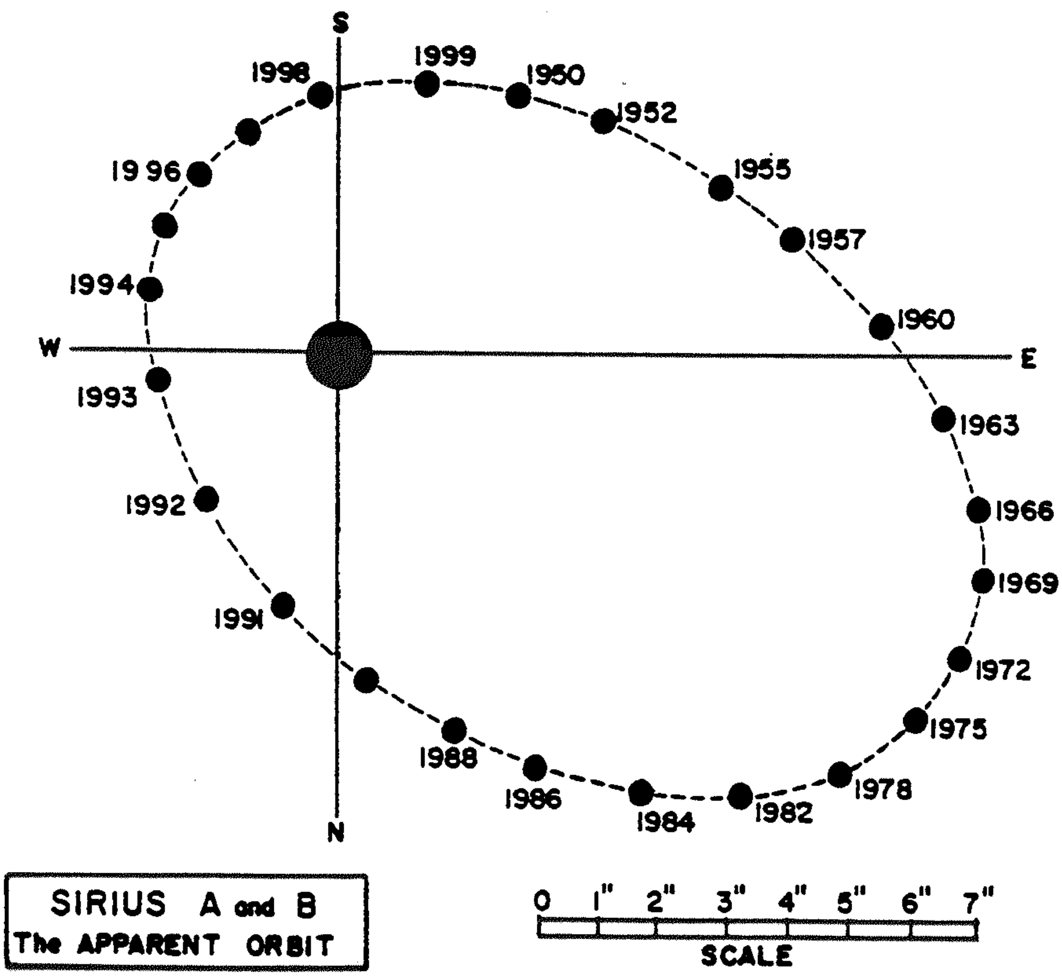

Figure 2: The relative orbit of Sirius A and B (problem 2).

3. **Transiting planet [12 pts].** A planet is observed to transit a star. The transits recur with period P. The duration of each transit is T, and during each transit the apparent brightness of the star drops by a fraction δ. By modeling the transit light curve, it is found that the orbital inclination is $i = 90°$ (i.e. the orbital axis is perpendicular to the line of sight). Doppler measurements of the star reveal a sinusoidal radial-velocity variation with an amplitude K.

   In what follows you may assume $M_p \ll M_\star$ and $R_p \ll R_\star \ll a$, where M_p is the planetary mass, $M_\star$ is the stellar mass, R_p is the planetary radius, $R_\star$ is the stellar radius, and a is the orbital semimajor axis.

   (a) Derive an approximate formula for the star's mean density, $\langle \rho_\star \rangle$, in terms of observable quantities.

   (b) Derive an approximate expression for the planet's surface gravity g_p, in terms of observable quantities. (The surface gravity is defined as GM_p/R_p^2, where M_p and R_p are the mass and radius of the planet, respectively.)

4. **Gravitational-wave chirp [30 pts]** Consider a binary consisting of two masses m_1 and m_2 in a circular orbit of radius R. Consider the orbit to be adequately described using Newtonian gravity.

   (a) Compute the rate at which energy is carried away from the system by gravitational waves. Express your answer in terms of the reduced mass μ, the total mass M, the orbital frequency Ω, and the orbital separation R.

   Due to this loss of energy, the radius of the orbit will gradually shrink, and the frequency of the binary will "chirp" to higher frequencies as time passes.

   (b) By asserting *global* conservation of energy in the following form,

$$\frac{d}{dt}\left(E_{\text{kinetic}} + E_{\text{potential}}\right) + \frac{dE_{\text{GW}}}{dt} = 0 \ ,$$

   derive an equation for dR/dt, the rate at which the orbital radius shrinks. *Hint*: You may find it useful to express all the terms in this equation as functions of R before evaluating terms and solving for dR/dt.

   (c) Find the rate of change of the orbital frequency Ω caused by gravitational-wave emission. You should find that the masses only appear in the combination $\mathcal{M} = \mu^{3/5} M^{2/5}$, perhaps raised to some power. This combination of masses is know as the "chirp mass," since it sets the rate at which the frequency "chirps."

   (d) Integrate the $d\Omega/dt$ you obtained in part (c) to obtain $\Omega(t)$, the time evolution of the binary's orbital frequency. Let T_c be the "coalescence time," the time at which the frequency formally[1] goes to infinity. Your answer should be a power law in $T_c - t$.

   (e) Suppose that a signal from a neutron star binary with $m_1 = 1.3\, M_\odot$, $m_2 = 1.4\, M_\odot$ enters the band of the LIGO detector when the gravitational-wave frequency $f = 30$ Hz. Bearing in mind that the orbital frequency Ω is related to the gravitational-wave frequency f by[2]

$$f = 2\frac{\Omega}{2\pi} \ ,$$

   estimate how long the system will be in band before the neutron stars collide (i.e., before our formula predicts that Ω will diverge).

[1] In reality, various approximations we have introduced break down before we reach this time. T_c is nonetheless not a bad proxy for the time at which the members of the binary merge due to gravitational-wave emission.

[2] The $1/2\pi$ is the usual relation between angular frequency and frequency; the 2 is due to the radiation between quadrupolar in nature

5. **Habitable zone [12 pts]** The "habitable zone" of a star is defined as the range of orbital distances where a planet would have a surface temperature appropriate for water to exist as a liquid.

   A planet's surface temperature can be estimated as follows. Suppose it is located a distance d from a star of radius $R_\star$ and effective temperature $T_{\rm eff}$. Assume the planet has a Bond albedo A, meaning that it reflects a fraction A of the incident power, and absorbs the complementary fraction $1 - A$. Further assume the planet is in thermal equilibrium, meaning that the energy absorbed by the planet in a given time interval is equal to the energy that it re-radiates during the same time interval. Finally, assume that the planet radiates as a blackbody, and that its atmosphere efficiently redistributes the heat over the planet's surface, causing the energy to be re-radiated isotropically.

   (a) Find $T_{\rm eq}$, the planet's equilibrium temperature, in terms of $T_{\rm eff}$, d, $R_\star$, and A.

   A naive calculation of the habitable zone would require $T_{\rm eq}$ to be in the range 273-373 K. However, the greenhouse effect of the planet's atmosphere will likely raise the surface temperature. To account for this and other atmospheric effects, it has been suggested that the habitable zone should be defined by the criterion 175 K $<$ $T_{\rm eq}$ $<$ 270 K, where $T_{\rm eq}$ is calculated as above (Kaltenegger & Sasselov 2011, ApJ Letters, 736, 25).

   (b) Using this definition of the habitable zone, and assuming $A = 0.3$ (the Earth's approximate Bond albedo), determine the range of orbital distances and orbital periods corresponding to the habitable zone of
   - the Sun ($R = R_\odot$, $M = M_\odot$, $T_{\rm eff} = 5777$ K).
   - Vega ($R = 2.26\ R_\odot$, $M = 2.14\ M_\odot$, $T_{\rm eff} = 9602$ K).
   - Proxima Centauri ($R = 0.141\ R_\odot$, $M = 0.123\ M_\odot$, $T_{\rm eff} = 3042$ K).

6. **Blackbody radiation [16 pts].** The Planck radiation spectrum is given by

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \quad (\text{erg cm}^{-2}\ \text{s}^{-1}\ \text{Hz}^{-1}\ \text{steradian}^{-1}),$$

   per unit frequency.

   (a) **Wavelength spectrum.** Show by explicit calculation that the equivalent Planck radiation spectrum per unit *wavelength* is given by

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT) - 1} \quad (\text{erg cm}^{-2}\ \text{s}^{-1}\ \text{cm}^{-1}\ \text{steradian}^{-1}),$$

   starting from the expression for B_ν.

   (b) **Stefan-Boltzmann law.** Derive the Stefan-Boltzmann law ($F = \sigma T^4$) by integrating the Planck blackbody spectrum over all wavelengths or frequencies. (Note that there is an extra factor of π to convert from brightness per unit solid angle to total brightness, so that $F = \pi \int B_\nu d\nu = \pi \int B_\lambda d\lambda$.) You may use the fact that

$$\int_0^\infty \frac{u^3}{e^u - 1} du = \frac{\pi^4}{15}.$$

   Give an expression for the Stefan-Boltzmann constant σ in terms of fundamental physical constants, and check its numerical value and units, $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}.

   (c) **Wavelength of radiation peak.** Derive the Wien displacement law, which relates the wavelength of the radiation at the peak of the Planck function B_λ to the temperature: $T\lambda_{\rm max} = 0.29$ cm K. [When you differentiate to find the maximum of B_λ, you will obtain a nonlinear equation of the form $5(1-e^{-y})-y = 0$ which you can solve numerically.]

   (d) **Frequency of radiation peak.** Repeat the previous part, but this time find the relation between the *frequency* at the peak of the Planck function B_ν and the temperature: $\nu_{\rm max}/T = 5.9 \times 10^{10}$ Hz K^{-1}. For a given temperature T, does the photon energy corresponding to $\nu_{\rm max}$ agree with that for $\lambda_{\rm max}$ in the previous part? Should they agree? Explain.