**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

**Problem Set 3**

**Due**: Friday, March 1, 2019, in class
This problem set is worth **110 points**

1. **Apparent intensity and angular resolution [10 pts].**

   Photons are produced in a spherical cloud of radius $R$ at a uniform rate $\Gamma$ photons cm$^{-3}$ s$^{-1}$. The cloud is a distance $d$ away. Neglect absorption of the photons, i.e., assume the medium is optically thin. A detector on Earth has an effective area $\Delta A$, and an angular acceptance of half-angle $\Delta\theta$ (i.e. the detector is sensitive to incoming photons that are arriving within a cone of half-angle $\Delta\theta$).

   (a) Suppose the angular size of the source is much larger than $\Delta\theta$ (the source is completely *resolved*). What is the observed photon intensity toward the center of the cloud, in photons s$^{-1}$ cm$^{-2}$ sr$^{-1}$? You should find that the answer is independent of $d$ as well as the properties of the detector ($\Delta\theta$ and $\Delta A$).

   (b) Now suppose the angular size of the source is much smaller than $\Delta\theta$ (the source is *unresolved*). What is the observed average intensity when the source is in the beam of the detector? Does it depend on the properties of the detector?

2. **Angular diameters and effective temperatures [10 pts].**

   (a) Show that if you can measure the bolometric flux $F$ and the angular diameter $\phi$ of a star, then you can determine the effective temperature $T_{\rm eff}$ even if you do not know the distance to the star. Note, "bolometric" means "integrated over all frequencies."

   (b) In one recent application of this technique, astronomers used optical interferometry to measure the angular diameters of both stars in the binary system $\beta$ CrB. The results were $0.699 \pm 0.017$ mas for star A, and $0.415 \pm 0.017$ mas for star B, where "mas" means milli-arcseconds. The bolometric apparent magnitudes of stars A and B are $3.87 \pm 0.05$ and $5.83 \pm 0.10$, respectively. The bolometric absolute magnitude of the Sun is 4.75, and the effective temperature of the Sun is 5777 K. Use this information to calculate the effective temperatures of stars A and B. You need not calculate the uncertainties.

   (In case you are curious to learn more, the reference is Bruntt et al. 2010, *Astron. & Astrophys.*, 512, 55.)

3. **Saha equation and pure hydrogen [20 pts].** Consider a gas of pure hydrogen at fixed density and temperature. The ionization energy of hydrogen is $\chi_0 = 13.6$ eV. You may assume that all the hydrogen atoms (whether neutral or ionized) are in their ground energy state.

   (a) Write down the Saha equation relating the number densities of neutral and ionized hydrogen ($n_0$ and $n_1$, respectively). Make reasonable approximations to use numerical values for the partition functions.

   (b) To find the individual densities, further constraints are required. Reasonable constraints are charge neutrality ($n_e = n_1$) and conservation of nucleon number ($n_1 + n_0 = n$), where the total hydrogen number density $n$ is a constant if the density $\rho$ is fixed. Rewrite the Saha equation in terms of the hydrogen ionization fraction $x = n_1/n$, eliminating $n_1$, $n_0$, and $n_e$. Does this equation have the expected limiting behavior for $T \to 0$ and $T \to \infty$?

   (c) Use the relation $\rho = m_{\rm H} n$ (where $m_{\rm H} = 1$ gm/$N_{\rm A}$, where $N_{\rm A} = 6.023 \times 10^{23}$ is Avogadro's number) to replace $n$ with $\rho$. Find an expression for the half-ionized ($x = 0.5$) path in the $\rho$-$T$ plane. Plot this path on a log-log plot for densities in the interesting range from $10^{-10}$–$10^{-2}$ g cm$^{-3}$

4. **Saha equation and pure helium [30 pts].**

   Consider a gas of pure helium at fixed density and temperature. The ionization energies for helium are $\chi_0 = 24.6$ eV (from neutral to singly ionized) and $\chi_1 = 54.4$ eV (from singly to doubly ionized). You may assume that all the helium atoms (whether neutral, singly ionized, or doubly ionized) are in their ground energy state. Let $n_e$, $n_0$, $n_1$, and $n_2$ be the number densities of, respectively, free electrons, neutral atoms, singly-ionized atoms, and doubly-ionized atoms. The total number density of neutral atoms and ions is denoted by $n$. Define $x_e$ as the ratio $n_e/n$, and let $x_i$ be $n_i/n$ where $i = 0, 1, 2$. You should assume that the gas is electrically neutral. The degeneracy factors you need for the atoms and ions are 2 for He, 4 for $He^+$, and 2 for $He^{2+}$.

   (a) Construct the ratios $n_1/n_0$ and $n_2/n_1$ using the Saha equation. In doing so, take care in establishing the zero points of energy for the various constituents.

   (b) Apply charge neutrality and nucleon number conservation ($n = n_0 + n_1 + n_2$) and recast the above Saha equations so that only $x_1$ and $x_2$ appear as unknowns. The resulting two equations have $T$ and $n$ [or, equivalently, $\rho = n m_{\rm He} = n(4\,{\rm gm}/N_{\rm A})$] as parameters.

   (c) Simultaneously solve the two Saha equations for $x_1$ and $x_2$ for temperatures in the range $4 \times 10^4 \le T \le 2 \times 10^5$ K. Do this for a fixed density with the three values $\rho = 10^{-4}$, $10^{-6}$, or $10^{-8}$ g cm$^{-3}$. You may find it more convenient to use the logarithm of your equations. Choose a dense grid in temperature because you will soon plot the results. Once you have found $x_1$ and $x_2$, also find $x_e$ and $x_0$ for the same range of temperature. Note that this is a numerical exercise; you will want to use a tool like Mathematica or Matlab for this.

   (d) Plot all your $x$s as a function of temperature for your chosen value of $\rho$. (Plot $x_0$, $x_1$, and $x_2$ on the same graph.) Identify the transition temperatures (half-ionization) for the two ionization stages.

5. **Limb darkening [20 pts].**

   In this problem you will derive a relation between the measured limb darkening of a star, and the source function of its photosphere. Let the intensity of the stellar disk be $I_\nu(r)$, where $r$ is the distance from the center of the stellar disk in units of the stellar radius (i.e. $r = 0$ at the center, and $r = 1$ at the limb).

   (a) Instead of $r$ it is traditional to express $I_\nu$ as a function of $\mu \equiv \sqrt{1 - r^2}$. Show that $\mu = \cos\theta$, where $\theta$ is the angle between the line of sight and the normal to the stellar surface.

   (b) We want an expression for the intensity at the stellar surface in terms of the source function. Start from the the radiative transfer equation for a plane-parallel atmosphere. Show that for an upward-propagating ray coming from far below to the top surface, the formal solution is

   $$I_\nu(\mu) = \int_0^\infty d\tau_\nu \, \frac{S_\nu(\tau_\nu)}{\mu} e^{-\tau_\nu/\mu}, \tag{1}$$

   where $\tau_\nu$ is the vertical optical depth.

   (c) Suppose the (unknown) source function can be represented by a polynomial,

   $$S_\nu(\tau_\nu) = a_0 + a_1\tau_\nu + a_2\tau_\nu^2 + \cdots + a_n\tau_\nu^n. \tag{2}$$

   Show that under this assumption the emergent intensity is given by

   $$I_\nu(\mu) = a_0 + a_1\mu + 2a_2\mu^2 + \cdots + (n!)a_n\mu^n, \tag{3}$$

   using the definite integral $\int_0^\infty x^n \exp(-x)dx = n!$. In this way the measured limb-darkening law can be used to determine the source function, and therefore the temperature stratification for an LTE atmosphere.

   (d) Show that for a gray LTE atmosphere, the predicted limb darkening law for the wavelength-integrated intensity at the stellar surface is

   $$\frac{I(\theta)}{I(0)} = \frac{2}{5} + \frac{3}{5}\cos\theta.$$

6. **Radiative transfer in spherical coordinates [20 pts].**

After this week's classes you should be familiar with the radiative diffusion equation for a plane-parallel atmosphere, an appropriate model for a thin photosphere. In this problem you will repeat those steps for a spherical atmosphere, as appropriate for the bulk of a star. We will assume the star is spherically symmetric and that consequently $I_\nu = I_\nu(r, \theta)$, where $r$ is the radial coordinate and $\theta$ is the angle of a ray relative to the local radius vector (and *not* the polar angle referring to the position with respect to the stellar center). See Fig. 1.

(a) Use the chain rule,

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}, \tag{4}$$

to show that the radiative transfer equation (RTE) can be written

$$\cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial I_\nu}{\partial \theta} + \rho\kappa_\nu I_\nu - j_\nu = 0. \tag{5}$$

In this expression, $\kappa_\nu$ is the *opacity*, measured in units of cm$^2$ g$^{-1}$; and $j_\nu$ is the *emission coefficient*, measured in units of erg cm$^{-3}$ s$^{-1}$ sr$^{-1}$ Hz$^{-1}$ [both as defined by Rybicki & Lightman (p. 9-10)].

(b) Integrate the RTE over all solid angles to show

$$\frac{dF_\nu}{dr} + \frac{2}{r}F_\nu + c\rho\kappa_\nu u_\nu - \rho\epsilon_\nu = 0, \tag{6}$$

where $\epsilon_\nu$ is the (angle-averaged) *emissivity* as defined on p. 9 of Rybicki & Lightman.

(c) Multiply the RTE by $\cos\theta$ and integrate over all solid angles to show

$$c\frac{dp_\nu}{dr} + \rho\kappa_\nu F_\nu = 0, \tag{7}$$

where you have assumed $j_\nu$ to be isotropic, and $I_\nu$ to be nearly isotropic. Here, $p_\nu$ is the *specific radiation pressure* given by

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega. \tag{8}$$

(d) Use the preceding equation, as well as the blackbody formula for radiation pressure, the relation $F = L/4\pi r^2$ and the definition of the Rosseland mean opacity $\kappa_R$ to show

$$\frac{dT}{dr} = -\frac{3\rho\kappa_R L}{64\pi\sigma r^2 T^3}. \tag{9}$$

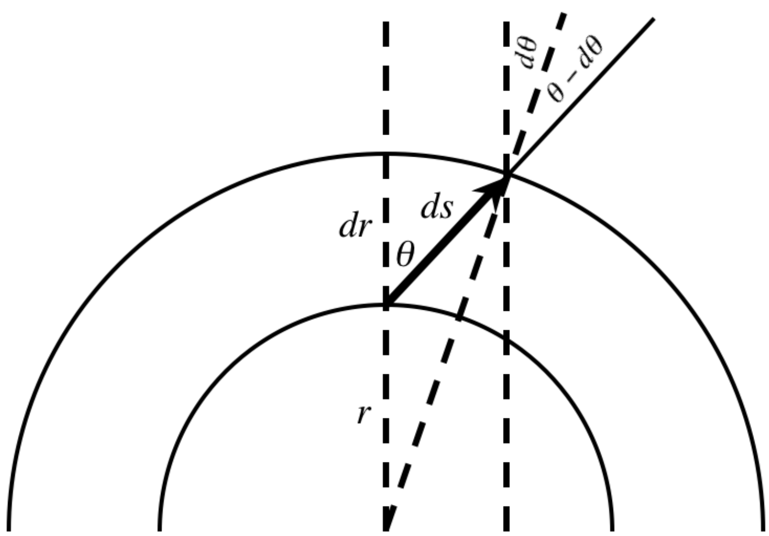


Figure 1: Geometry relevant to Prob. 6. A photon propagates a distance $ds$ along a direction $\theta$ from the local radius vector. As a result its radial coordinate increases by $dr$ and the angle to the local radius vector decreases by $d\theta$.