MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 9

Due: Wednesday, May 1, 2019, in class
This problem set is worth **95 points**

1. **Numerical models of neutron stars (50 pts)**

   In this problem you will model a neutron star using a more realistic equation of state. To prepare for this problem, please peruse the classic paper by Arnett & Bowers (1977). (There are more recent papers on this subject, but they are a lot more complicated; Arnett & Bowers does a good job covering the key points and presenting a representive sample of equations of state.)

   (a) Show that a power law
   $$P = K_{\mathrm{AB}}\rho^{\gamma_{\mathrm{AB}}}$$
   with $\gamma_{\mathrm{AB}} = 2$, $K_{\mathrm{AB}} = 10^5\,\mathrm{ba}\,(\mathrm{gm-cm}^{-3})^{-\gamma_{\mathrm{AB}}}$ is a good "average" of the models A through G in Figure 4 of Arnett & Bowers. (A "barye" [ba] is the cgs unit of pressure; 1 ba = 1 dyne cm^{-2} = 0.1 Pa.)

   You will assume that this expression is valid down to arbitrarily low densities, including $\log_{10}\rho \lesssim 14.6$ (where ρ is in g cm^{-3}), the smallest value plotted in Figure 4.

   (b) Consider a range of central densities $14 < \log_{10}\rho_c < 16.5$ (where ρ_c is in g cm^{-3}), uniformly spaced in $\log_{10}\rho_c$. For each of these, integrate the Tolman-Oppenheimer-Volkoff equation to find $\rho(r)$. Although it is not necessary, you may find it useful to start by nondimensionalizing the equations
   $$\frac{d\rho}{dr} = \frac{1}{K_{\mathrm{AB}}\gamma_{\mathrm{AB}}\rho^{\gamma_{\mathrm{AB}}-1}}\left(\frac{dP}{dr}\right)_{\mathrm{TOV}} \qquad (1)$$
   $$\frac{dM}{dr} = 4\pi r^2\rho. \qquad (2)$$
   In other words, introduce some fiducial density scale ρ_0 and normalize ρ to that; deduce related length, mass, and pressure scales, and scale those variables accordingly.

   Hint 1: Some numerical integrators become quite sad if you begin integrating at $r = 0$ — because the enclosed mass there is zero, you find lots of $0/0$ type singularities. Start at some very small radius ϵ, and assume that ρ varies very slowly in that region to put $M(\epsilon) \simeq (4\pi/3)\rho_c\epsilon^3$.

   Hint 2: If you elect to nondimensionalize, note that $(G\rho)^{-1/2}$ is a time and so $(G\rho_0)^{-1/2}$ can be regarded as a fiducial timescale. Likewise, in any relativistic calculation c is a relevant fiducial velocity.

   (c) Plot the total mass as a function of $\log_{10}\rho_c$ for your models. What is the maximum mass of a neutron star for this equation of state?

   (d) Plot the radius as a function of $\log_{10}\rho_c$ (for masses corresponding to stable neutron stars).

   (e) Plot the mass-radius relationship for your models (for masses corresponding to stable neutron stars).

   (f) Finally, investigate a "maximally stiff" equation of state, for which (at high densities) the sound speed is equal to the speed of light. Please repeat steps 1b and 1c using an equation of state
   $$P = \rho c^2 \qquad \text{for} \qquad \rho > 10^{14.6}\ \mathrm{g\ cm}^{-3} \qquad (3)$$
   $$P = K\rho^{5/3} \qquad \text{for} \qquad \rho < 10^{14.0}\ \mathrm{g\ cm}^{-3}\,, \qquad (4)$$
   where $K \approx 5.5 \times 10^9$ (cgs) is the appropriate constant for a non-relativistic Fermi gas of neutrons. For densities between $10^{14.0}$ and $10^{14.6}$ g cm^{-3}, calculate $\log_{10} P$ as a function of $\log_{10}\rho$ via linear interpolation between the two expressions given above.

2. **Pulsar spin-down properties (15 pts)**

Consider a pulsar with spin period $P = 2\pi/\Omega$ that is losing energy and therefore spinning down.

(a) If the energy loss mechanism is magnetic dipole radiation, then

$$\frac{dE}{dt} = -\frac{B^2\Omega^4 R^6 \sin^2\alpha}{6c^3}, \tag{5}$$

where B is the polar magnetic field strength, R is the pulsar radius, and α is the angle between the magnetic and rotational poles. Show that this implies $\dot{\Omega} = -k\Omega^3$ where k is a constant. Also show that in this case, $B \propto \sqrt{P\dot{P}}$.

(b) For the more general case $\dot{\Omega} = -k\Omega^n$, where n is the *braking index*, show that $n = \ddot{\Omega}\Omega/\dot{\Omega}^2$.

(c) Show that if the braking index is n, the age of the pulsar may be estimated as

$$\tau \approx \frac{|P/\dot{P}|_{\text{final}}}{n-1}\left[1 - \frac{P_{\text{initial}}^{n-1}}{P_{\text{final}}^{n-1}}\right]. \tag{6}$$

3. **Blackbody radiation from a compact object (15 pts)**

Because general relativity is important for compact objects, even seemingly basic quantities such as luminosity, temperature and radius need to be defined carefully, as you will see in this problem.

Consider a spherical blackbody of constant temperature and mass M and an outer surface defined by the radial coordinate $r = R$. Two observers are measuring the blackbody radiation: an observer located at the surface of the sphere, and a very distant observer.

(a) If the observer at the surface of the sphere measures the luminosity of the blackbody to be L, show that the observer at infinity measures

$$L_\infty = L\left(1 - \frac{2GM}{Rc^2}\right).$$

An important bit of physics to use here is the gravitational redshift z_g associated with a photon travelling radially in the Schwarzschild spacetime. It is given by

$$1 + z_g = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2}. \tag{7}$$

This is derived on p. 387-392 of Choudhuri. A photon that is emitted with energy E at radius R will be measured to have energy $E/(1+z_g)$ very far away. This redshift also applies to clocks — a time interval dt at radius R is measured to be an interval $dt(1+z_g)$ by observers far away. In other words, clocks deep in a gravitational potential well run slow. (This is why GPS satellites need to correct for general relativity — clocks in high orbit run demonstrably faster than clocks on the surface of the Earth.)

(b) Suppose both observers use Wien's law,

$$\lambda_{\max} T = 0.28978 \text{ cm K},$$

to determine the blackbody's temperature. Here $\lambda_{\max}$ is the wavelength corresponding to the peak in the blackbody spectrum. Show that

$$T_\infty = T\sqrt{1 - \frac{2GM}{Rc^2}}.$$

(c) Suppose both observers use the Stefan-Boltzmann law to determine the radius of the spherical blackbody. Show that

$$R_\infty = \frac{R}{\sqrt{1 - 2GM/Rc^2}}$$

Thus, using the Stefan-Boltzmann law without accounting for general relativity will lead to an *overestimate* of the size of a compact blackbody.

4. **Supernova explosion in a binary system (15 pts)**

Two stars of mass m_1 and m_2 are in a circular orbit. Star 1 undergoes a supernova explosion in which mass Δm is blown away spherically symmetrically (in the frame of star 1) on a time scale that is very short compared to the orbital period. Show that the condition for the orbit to remain bound is

$$\Delta m < \frac{m_1 + m_2}{2}.$$

Hint: Compute the total energy of the binary after the explosion in the (new) center of mass frame of the binary. Assume that immediately after the explosion, the binary separation and the orbital velocities of both stars are unchanged. A useful relation is that the total kinetic energy is $(1/2)\mu v_{\rm rel}^2$, where μ is the reduced mass and $v_{\rm rel}$ is the relative velocity.