MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
Astrophysics I (8.901) — Prof. Crossfield — Spring 2019

Problem Set 10
Due: Wednesday, May 8, 2019, in class
This problem set is worth **102 points**

1. **Orbits of black holes (15 pts)**

   In the notes, we derived the following equation describing the orbit of a body around a non-rotating black hole:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\mathcal{E}^2}{c^2} - V_{\rm eff}(r) , \qquad \text{where}$$
$$V_{\rm eff}(r) = \left(1 - \frac{2GM}{rc^2}\right)\left(c^2 + \frac{\ell^2}{r^2}\right) .$$

   $\mathcal{E}$ is the orbital energy and ℓ the orbital angular momentum (each per unit mass of the orbiting body).

   (a) By requiring that r be constant along the orbit and that the orbit "sit" at the minimum of the effective potential, derive expressions for $\mathcal{E}$ and ℓ as functions of r for circular orbits.

   (b) As we move deeper into the strong field (i.e., to smaller values of r), the potential's shape changes. Compute the radius at which the minimum in $V_{\rm eff}$ goes away and stable circular orbits cease to exist. This radius you should find is known as the "innermost stable circular orbit," or ISCO. It is a starkly non-Newtonian characteristic of black hole orbits.

   (*Hint*: An easy mistake to make is to substitute the solutions for $\mathcal{E}$ and ℓ that you derived in the previous part too early in your calculation. Develop your criterion for whether a minimum exists or not assuming $\mathcal{E}$ and ℓ are constants; only then substitute the expressions appropriate for a circular orbit that you derived.)

2. **Orbital rearrangements due to mass transfer (20 pts)**

   Two stars of mass M_1 and M_2 are in a circular orbit. Star 2 transfers matter onto star 1 via Roche-lobe overflow. Assume that no mass is lost from the system, and also that the orbit remains circular during mass transfer. Use Newtonian physics for your analysis.

   (a) Using only conservation of mass and orbital angular momentum, find the fractional change in binary separation ($\Delta a/a$) in terms of the mass ratio $q = M_2/M_1$ and $\Delta M/M$. You may assume $\Delta M/M \ll 1$.

   Show that transfer from the lighter to the heavier star leads to a widening of the orbit ($\Delta a/a > 0$), while transfer from the heavier to the lighter star leads to a shrinking of the orbit ($\Delta a/a < 0$).

   (b) The Roche lobe radius[1] around star 2 is given very approximately by

$$R_{L2} \simeq \frac{a}{2}\left(\frac{M_2}{M_1 + M_2}\right)^{1/3} .$$

   Find the change in the Roche lobe radius, ΔR_{L2}, caused by the mass transfer. Express your answer in terms of $\Delta M/M$, q, and a.

   (c) Next we want to calculate the change in radius ΔR_2 of star 2 in response to the mass loss. For simplicity, assume that star 2 is fully convective, and thus well-described by an $n = 3/2$ polytrope, for which $R \propto M^{-1/3}$. From this relation, find $\Delta R_2/R_2$ in terms of $\Delta M/M$ and q. Then, using the fact that $R_2 = R_{L2}$ right at the onset of mass transfer, find ΔR_2 in terms of a, $\Delta M/M$, and q.

   (d) Argue that the mass transfer will be an unstable runaway process if $\Delta R_2 > \Delta R_{L2}$. Find the maximum value of the mass ratio q for stable mass transfer.

[1] The "Roche lobe radius" is the radius of a sphere whose volume is equal to the volume of a star that fills its Roche lobe — i.e., a star that is distorted so that it just touches the Lagrange point L_1. A star whose physical size is smaller than this radius will be distorted, but will not transfer mass to its companion; and vice versa if it is larger than this radius.

The Shakura-Sunyaev accretion disk model.

Problems 3–6 are based on the Shakura-Sunyaev (S-S) accretion disk model we discussed in lecture.

3. **Luminosity of the accretion disk (20 pts)**

   In the S-S accretion disk model, the dependence of the effective temperature $T_{\rm eff}$ on radial distance r is

$$T_{\rm eff} = \left[\frac{3GM\dot{M}}{8\pi\sigma r^3}\right]^{1/4} \left(1 - \sqrt{r_0/r}\right)^{1/4} \tag{1}$$

   where σ is the Stefan-Boltzmann constant.

   (a) Show that the total power radiated from the disk (including both sides!) is

$$L = \frac{1}{2}\frac{GM\dot{M}}{r_0}$$

   where r_0 is the radius of the inner edge of the accretion disk.

   (b) Denote by $L(> r)$ the power radiated from the portion of the disk with radius greater than r. Find an analytic expression for the ratio

$$\text{ratio} = \frac{L(> r)}{\frac{1}{2}\frac{GM\dot{M}}{r}}$$

   Sketch the ratio as a function of r. This result demonstrates that the gravitational potential energy that is released as the matter migrates inward does not emerge locally, but is instead redistributed by the viscous stresses.

4. **Temperature structure of the accretion disk (10 pts)**

   (a) Use Eq. (1) to find the location $r(T_{\rm max})$ where the temperature is a maximum. Express your answer in terms of r_0, the radius of the inner edge of the disk. If the central star is a non-rotating black hole, then $r_0 = 3R_s$ where $R_s = 2GM/c^2$ is the Schwarzschild radius. In this case, express your answer for the location of the maximum temperature in terms of R_s.

   (b) Compute $T_{\rm max}$ for the following two cases:

Accretor	Mass	$\dot{M}$	r_0	Source type
white dwarf	1.0 $M_\odot$	10^{17} g s^{-1}	9.0×10^8 cm	cataclysmic variable
neutron star	1.4 $M_\odot$	10^{18} g s^{-1}	1.2×10^6 cm	low-mass X-ray binary

5. **Inward radial speed of accreting material (17 pts)**

   We will now show that the radial inspiral speed v_r for material in the disk is always much smaller than the Keplerian orbital speed v_K. First, use the following expressions for $\rho(r)$ and $H(r)$ to compute an expression for v_r, the radial speed of inspiral for material in the disk:

$$H \simeq 1\times 10^8 \quad \alpha^{-1/10} \quad \dot{M}_{16}^{3/20} \quad r_{10}^{+9/8} \quad f^{3/5} \quad \text{cm}$$
$$\rho \simeq 7\times 10^{-8} \quad \alpha^{-7/10} \quad \dot{M}_{16}^{11/20} \quad r_{10}^{-15/8} \quad f^{11/5} \quad \text{g cm}^{-3}$$

   where $\dot{M}_{16}$ is the mass accretion rate in units of 10^{16} g sec^{-1}, r_{10} is radius in units of 10^{10} cm, and $f = (1 - \sqrt{r_0/r})^{1/4}$, where r_0 is the inner edge of the disk (which was denoted r_i in lecture).

   Next, find the ratio of v_r to v_K. Finally, show that for all reasonable choices of parameters the ratio is smaller than unity.

   Take the minimum plausible neutron star mass to be about 0.5 $M_\odot$, the maximum plausible accretion rate to be that which gives $\dot{M}c^2 \sim 10L_{\rm Edd}$, and the maximum size of the accretion disk to be $\sim$1 AU.

6. **Spectrum of a Shakura-Sunyaev accretion disk (20 pts)**

(a) Write an integral expression for the spectral luminosity L_ν (erg s^{-1} Hz^{-1}) of the accretion disk. Do this by treating each annulus in the disk as a blackbody radiator with temperature $T_{\rm eff}(r)$ as defined in Problem 3. Don't bother trying to evaluate the integral analytically.

For reference, the Planck function is:

$$P(\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

(b) Make the following approximations to calculate L_ν analytically: (i) Approximate the Planck function by $P(\nu) = 2\pi h\nu^3 c^{-2} e^{-h\nu/kT}$; (ii) In the expression for $T(r)$, take the factor $\left(1-\sqrt{r_0/r}\right)^{1/4}$ to be approximately unity; (iii) Carry out the integration from $r=0$ to $r=\infty$, even though a real disk obviously has narrower limits. Show that $L_\nu \propto \nu^{1/3}$.