**ASTRONOMY 271 WINTER 2008**

This course is an introduction to radiation in astrophysics. The material will be presented so as to develop the ability to read the astrophysical literature; there will not be enough time to go into great depth. The text is Rybicki and Lightman, *Radiative Processes in Astrophysics*.

Schedule:

1. Basic Definitions and Results
2. Low Energy Line Photons Emission/Absorption
3. High Energy Line Photons Emission/Absorption
4. Classical Theory of Radiation I: Retarded Potentials
5. Classical Theory of Radiation II: Larmor Formula
6. Thermal Sources of Radiation: free-free
7. Thermal Sources of Radiation: solid dust particles
8. Atomic Spectroscopy
9. Molecular Spectroscopy
10. Synchrotron Emission I
11. Synchrotron Emission II
12. Compton Effect
13. Scattering: Electrons + particles
14. Masers
15. Stars
16. Disks
17. AGN
18. ISM and IGM

Grading: There will be a written final exam.

Useful references will be supplied at various times.

**Astronomy 271 – Study Problems – Set No. 1**

1. What are the units of the Einstein $B's$?

2. What is the relationship between the Einstein $A$ and the oscillator strength, $f$?

3. Fabian et al. ( 2007, ApJ, 661, 102 ) report 21 cm mapping of nearby galaxies. How well can you reproduce their statement that a flux of 0.5 mJy in a beam of diameter 10" over a velocity width of 2.5 km $s^{-1}$ corresponds to an atomic hydrogen column density of 1.5 $\times$ $10^{19}$ $cm^{-2}$? Remember that a beam size is usually quoted as a Full Width Half Maximum.

4. Dent et al. (2005, MNRAS, 359, 663) report CO emission for nearby stars. If the gas has an excitation temperature of 100 K, what minimum mass of CO do you derive from their observations toward 49 Cet? Use their result that in the $J$ = 3-2 transition that the integrated intensity is 0.34 K km $s^{-1}$ for a star at a distance of 61 pc. You need to adopt a beam size; take 10".

5. Prochaska et al. (2001, ApJS, 137, 21) report data for quasar absorption line systems. Use the data in their Figure 2 to re-derive the Ni II column density given in their Table 3. The relevant atomic physics is given in their Table 2.

**Astronomy 271 – Study Problems – Set No. 2**

1. If the source function in a stellar atmosphere, $S$, varies as $a + b\tau$, then we showed in class that the emergent intensity is:

$$I = a + b\,\mu \tag{1}$$

where

$$\mu = \cos\theta \tag{2}$$

and $\theta$ is the angle of the line of sight relative to the normal of the atmosphere. Use these results to explain why the limb darkening in the Sun is much more pronounced in the ultraviolet compared to the infrared.

2. Assume a quasar has a power law emission such that its specific luminosity, $L_\nu$, varies as $\nu^{-1}$. Assume that the quasar is surrounded by atomic hydrogen and that all of the ionizing photons are absorbed by this gas and that ultimately every ionization leads to a recombination that produces a Lyman $\alpha$ photon. Compute the expected equivalent width of the Lyman $\alpha$ emission line.

3. Assume a quasar is powered by accretion onto a black hole. Compute the minimum mass black hole necessary to explain a source with a luminosity of $10^{13}$ $L_\odot$. If this source is powered by spherically symmetric accretion of fully ionized gas in free-fall, is it optically thin or thick to electron scattering? Assume that 10% of the gravitational energy released by the accreted mass is converted into luminous energy.

4. Assume Eddington-limited accretion onto a black hole. If a black hole has an initial mass of 1 $M_\odot$, how long does it take to grow to a black hole of mass $10^9$ $M_\odot$?

**Astronomy 271 – Study Problems – Set No. 3**

1. The radio flux from $\zeta$ Pup at 6 cm is 1.4 mJy. If the distance to the star is 500 pc and its outflow speed is 3000 km $s^{-1}$, estimate the mass loss rate in units of $M_\odot$ $yr^{-1}$.

2. Song et al. (2005, Nature, 436, 363) report a flux at 10 $\mu$m of 71.5 Jy is for the main-sequence star BD +20 307. Assume a distance of 90 pc, a grain temperature of 650 K and a grain opacity at this wavelength of 1000 $cm^2$ $g^{-1}$. What is the minimum mass of the dust? How does this compare to the mass of the Earth's Moon?

3. The solar corona may have a base electron density of $10^8$ $cm^{-3}$ at T = $2 \times 10^6$ K. Assume that the corona has an inner radius equal to that of the Sun, the corona is isothermal and that it obeys the equation of hydrostatic equilibrium. Compute the X-ray free-free emission from this model corona and compare with the total luminosity of the Sun.

**Astronomy 271 – Study Problems – Set No. 4**

1. Show that the total free-free energy radiated by an electron during an encounter with a proton is typically less than the electron's initial kinetic energy.

2. In deriving the relationship between the rate of radiative association and the cross section for photo-ionization, we made two approximations. We ignored stimulated emission that must occur during radiative recombination and we approximated the Planck function as a Wien function. Show that these two approximations cancel each other.

3. Consider an optically thin cloud of dust. Assume that the density of the dust varies as $R^{-q}$ while the emissivity of the dust varies as $\nu^{+p}$. Show that the specific luminosity of the dust cloud varies as:

$$L_\nu \propto \nu^{-3+2q+0.5pq-0.5p} \tag{1}$$

Note that if $q = 1$, then $L_\nu$ varies as $\nu^{-1}$, independent of $q$.

4. Assume an ionized nebular with a temperature of 10,000 K. If the O III 5007 line is equally strong with H$\beta$, derive the ratio for $n(O^{+2})/n(H^{+})$. You might use Tables 3.6 and 4.2 in *Astrophysics of Gaseous Nebulae and Active Galactic Nuclei* by Osterbrock and Ferland.

**Astronomy 271 – Study Problems – Set No. 5**

1. Assume an ionized gas of 10,000 K. What is the ratio of the free-free optical depth to the pulsar dispersion measure? If a source has an optical depth at frequency, $\nu$ that is less than 1, derive an expression for the maximum electron density from the value of the dispersion measure. Then derive a lower bound to the size of the intervening region between us and the pulsar.

2. Assume a radio pulsar could be discovered in a short period, eccentric orbit around the massive black hole at the center of the Milky Way. How could you use the pulsar timing to measure the predicted general relativistic advance of the periastron?

3. Consider the Crab Nebula and assume the magnetic field has a magnitude of 0.001 Gauss. What is the lifetime of an electron that produces a 10 keV X-ray? How does this compare with the estimated age of the Crab of $\sim$1000 years?

4. Assume a magnetic field of $5 \times 10^7$ Gauss. What wavelength of light corresponds to emission at the nonrelativistic cyclotron frequency? On the basis of the energy density and hydrostatic equilibrium, show that main-sequence stars are unlikely to possess magnetic fields of this magnitude but white dwarfs might.

**Astronomy 271 – Study Problems – Set No. 6**

1. Assume that the sky has an optical depth of 0.1 along the line of sight. What is the surface brightness of the sky as illuminated by the Sun?

2. If the Moon has an albedo of 0.05, compute its surface brightness and compare with your estimate for the sky as given above.

3. The Rayleigh scattering cross section varies as $\lambda{-4}$. Qualitatively explain why the clear sky appears white near the horizon but blue near the zenith.

4. Consider an optically thin cloud in the very center of a spherical galaxy. Show that this cloud cannot appear as a bright reflection nebulosity but only as dark patch.

## Extra Study Problem for the HST COS Proposal

The key document is the COS Handbook:

$$http://www.stsci.edu/hst/cos/documents/handbooks/current/cos\_cover.html \quad (1)$$

What is the correct way to insure COS safety when performing moderate-resolution observations near Lyman $\alpha$? How to include the sky background?

What is the sky background?

At line center, the flux from the Sun in Lyman $\alpha$ is approximately $3.4 \times 10^{11}$ photons cm$^2$ s$^{-1}$ Å$^{-1}$ (Figure 2a in Lemaire et al. 1978, ApJ, 223, L55). This result is achieved by correcting for absorption in the upper atmosphere of the Earth; this correction amounts to about a factor of 2. In energy units, this photon flux corresponds to a result, $F_\lambda$, of 5.6 erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$. The mean intensity, $J_\lambda$ is:

$$J_\lambda = \frac{F_\lambda}{4\pi} \quad (2)$$

Therefore, $J_\lambda = 0.45$ erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$ ster$^{-1}$. Equivalently, then $J_\lambda = 1.06 \times 10^{-11}$ erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$ arcsec$^{-2}$. The mean intensity provides an upper limit to the intensity of the scattered light. There may be times when there is sufficient atomic hydrogen in the upper atmosphere that the specific intensity may be as high as the mean intensity.

The Lyman $\alpha$ emission line from the Sun is about 1 Å broad. In the upper atmosphere, the hydrogen has a much narrower profile and the scattering only occurs over a relatively narrow spectral interval. We can write that

$$\phi(v) \propto \exp^{-\frac{mv^2}{2\,k\,T}} = \exp^{-\frac{v^2}{b^2}} \quad (3)$$

where:

$$b = \left(\frac{2kT}{m}\right)^{1/2} \quad (4)$$

When $v = 0.833b$, the line profile falls to half of its value at line center. Therefore, the FWHM of the line is $1.67b$. At $T = 2000$ K, this means that the FWHM of the line corresponds to 9.6 km s$^{-1}$.

**Problem No. 1**: The COS Handbook says to adopt a temperature of 2000 K for the atomic hydrogen in the upper atmosphere and the line then has a "width" of 3 km s$^{-1}$.

[Section §10.3.4, paragraph 2]. I cannot reproduce their result.

2. A FWHM of 9.6 km s$^{-1}$ in velocity corresponds to a FWHM of 0.039 Å. Therefore, at line center, the maximum intensity of scattered light, $I$, would be:

$$I = \int J_\lambda \, d\lambda \approx J_\lambda \, \Delta\lambda_{FWHM} \tag{5}$$

This yields $I = 4.1 \times 10^{-13}$ erg cm$^{-2}$ s$^{-1}$ arcsec$^{-2}$.

**Problem No. 2**: The COS handbook (same section and paragraph) quotes a maximum surface brightness of $6.3 \times 10^{-13}$ erg cm$^{-2}$ s$^{-1}$ arcsec$^{-2}$. This is 50% larger than computed above. However, it is possible that scattering over the line extends further than the FWHM. This discrepancy is interesting but probably not physically interesting.

COS has an entrance aperture of angular diameter of 2.5 arcsec which corresponds to a solid angle of 4.9 arcsec$^2$. Therefore, using the COS-Handbook value of the surface brightness, the total flux into the COS aperture, $F_{total}$, is about $3.1 \times 10^{-12}$ erg cm$^{-2}$ s$^{-1}$.

If all the energy from the sky acted like a point source then the apparent flux, $F_{apparent}$, would be:

$$F_{apparent} = \frac{F_{total}}{\Delta\lambda_{FWHM}} \tag{6}$$

This means that $F_{apparent} \approx 8.0 \times 10^{-11}$ erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$. COS uses the unit of FEFU =$10^{-15}$ erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$ so $F_{apparent}$ = 80,000 FEFU. This is much greater than the safety limit.

However, the light from an extended source does not follow the same optical path through the spectrograph as does the light from a point source. According to the COS handbook, the "aperture width" for the G130M mode is 1.12 Å. Therefore, the point source-equivalent flux, $F_{equivalent}$, from the sky appears as:

$$F_{equivalent} = \frac{F_{total}}{1.12} \tag{7}$$

or $F_{equivalent} = 2.8 \times 10^{-12}$ erg cm$^{-2}$ s$^{-1}$ Å$^{-1}$. In COS units, this flux is 2800 FEFU.

According to Figure 5.2, the sensitivity to a point source with G130M is 0.014 counts s$^{-1}$ resel$^{-1}$ s$^{-1}$ FEFU$^{-1}$. We therefore expect that at maximum, the count rate from the geocornal light could be 39 counts s$^{-1}$ resel$^{-1}$.

**Problem No. 3**: This expected count rate from the maximally-bright geocorona is a factor of 2 larger than given in the COS Handbook (last paragraph in §10.3.4) which states that it is 20 counts $s^{-1}$ $resel^{-1}$.

A particular complication is that according to Table 11.3, 40 counts per $resel^{-1}$ $s^{-1}$ is just at the limit when a source must be screened. Therefore, all observations with G130M are potentially hazardous to COS since the geocoronal Lyman $\alpha$ scattering could be very bright.

The true COS limits are given in Tables 11.1 and 11.2. The maximum allowed count rate is 100 $resel^{-1}$ $s^{-1}$.

## Lecture 1 – Basic Definitions and Results

We are interested in the flow of radiation; what is often called "radiative transfer". We follow the intensity, $I_\nu$ (units of joule m$^{-2}$ s$^{-1}$ Hz$^{-1}$ ster$^{-1}$) and describe the sources and sinks of photons. The first sink term is absorption. In this process, a photon is destroyed and an atom (or molecule) goes from a lower energy state to a higher energy state.

$$h\nu + X \rightarrow X^* \tag{1}$$

where $X$ denotes the atom in the lower energy level and $X^*$ denotes the atom in the upper energy level. If $ds$ denotes an increment in path length, then the absorption coefficient, $\kappa_\nu$, (units of m$^{-1}$) is defined such that:

$$dI_\nu \;=\; -\kappa_\nu\, I_\nu\, ds \tag{2}$$

Therefore:

$$\frac{dI_\nu}{ds} \;=\; -\kappa_\nu\, I_\nu \tag{3}$$

If the atom has a cross section at frequency $\nu$ of $\sigma_\nu$ (units of m$^2$) and the density of atoms is $n$ (m$^{-3}$), then we note that

$$\kappa_\nu \;=\; n\,\sigma_\nu \tag{4}$$

Often, we also use the opacity, $\chi_\nu$ defined so that:

$$\chi_\nu \;=\; \frac{\kappa_\nu}{\rho} \tag{5}$$

where $\rho$ is the mass density of the material. Since $\rho = \mu\, n$ where $\mu$ is the mean molecular weight, then

$$\chi_\nu \;=\; \frac{\sigma_\nu}{\mu} \tag{6}$$

If we define the mean free path for a photon of frequency $\nu$ as $l_\nu$, then

$$l_\nu \;=\; \kappa_\nu^{-1} \;=\; \frac{1}{n\sigma_\nu} \tag{7}$$

If light with intensity, $I_\nu^0$ is incident upon a medium of uniform $\kappa_\nu$ and if we define $s = 0$ to be the boundary of the medium, then the intensity as a function of $s$ is given as the solution to the differential equation (3):

$$I_\nu \;=\; I_\nu^0\, e^{-\kappa_\nu\, s} \tag{8}$$

Another important parameter to introduce is the dimensionless quantity, the optical depth, $\tau_\nu$. We define

$$d\tau_\nu \;=\; \kappa_\nu\, ds \tag{9}$$

For a homogeneous medium, then

$$\tau_\nu \;=\; \kappa_\nu\, s \tag{10}$$

The attenuation of the light is modest if $\tau_\nu < 1$; such a situation is described as being "optically thin". The attenuation of the light is large if $\tau_\nu > 1$; such a situation is described as being optically thick.

In addition to absorption, light can be produced by emission. Schematically, this occurs when an atom (or molecule) undergoes a transition from an an upper energy level ($X^*$) to a lower energy level ($X$). Thus:

$$X^* \rightarrow X \;+\; h\nu \tag{11}$$

We define the emissivity, $\epsilon_\nu$ as the rate at which energy is emitted per unit solid angle. Then, if we only include the source term:

$$dI_\nu \;=\; \epsilon_\nu\, ds \tag{12}$$

or

$$\frac{dI_\nu}{ds} \;=\; \epsilon_\nu \tag{13}$$

If we include both the source term and the sink term, then we may write that

$$\frac{dI_\nu}{ds} \;=\; -\kappa_\nu\, I_\nu \;+\; \epsilon_\nu \tag{14}$$

This expression is called the equation of transfer and is the fundamental equation used to describe the flow of radiation energy.

The equation of transfer is often re-written in the following fashion. Divide the equation by $\kappa_\nu$, and use the definition of the optical depth to get:

$$\frac{dI_\nu}{d\tau_\nu} \;=\; -I_\nu \;+\; \frac{\epsilon_\nu}{\kappa_\nu} \tag{15}$$

We introduce a new quantity called the source function, $S_\nu$ (units of joule m$^{-2}$ s$^{-1}$ Hz$^{-1}$ ster$^{-1}$) such that

$$S_\nu \;=\; \frac{\epsilon_\nu}{\kappa_\nu} \tag{16}$$

Thus the equation of transfer becomes:

$$\frac{dI_\nu}{d\tau_\nu} \;=\; -I_\nu \;+\; S_\nu \tag{17}$$

If we measure through a uniform medium ($S_\nu$ is constant) in terms of optical depth instead of physical distance, then the solution to the equation of transfer for the emergent intensity from a uniform medium of optical depth, $\tau$ is

$$I_\nu^0 \;=\; S_\nu \left(1 \,-\, e^{-\tau_\nu}\right) \tag{18}$$

Note that if the medium is optically thin ($\tau_\nu < 1$), then

$$I^0_\nu \approx S_\nu \, \tau_\nu \tag{19}$$

while if the medium is optically thick ($\tau_\nu > 1$), then

$$I^0_\nu \approx S_\nu \tag{20}$$

An opaque medium (that is, an optically thick medium) is a "black body". Therefore, for black body radiation, we may set, $S_\nu$ equal to the Planck function or

$$S_\nu \; = \; \frac{2h\nu^3}{c^2} \, \frac{1}{exp\left(\frac{h\nu}{kT}\right) \, - \, 1} \tag{21}$$

For an opaque object, the emergent intensity is independent of its composition and depends only upon its temperature.

Two related quantities to the intensity of radiation are the mean intensity and the flux or flow of energy. The mean intensity, $J_\nu$, is defined as the average over solid angle of the intensity. Therefore, at any particular location, we define

$$J_\nu \; = \; \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I_\nu(\theta, \phi) \, sin \, \theta \, d\theta \, d\phi \tag{22}$$

Note that $J_\nu$ and $I_\nu$ have the same units.

If we define $\theta$ relative to the $Z$-axis, then the flux of energy along the $Z$ axis, $F_\nu$, is given by the expression

$$F_\nu \; = \; \int_0^{2\pi} \int_0^{\pi} I_\nu(\theta, \phi) \, cos \, \theta \; sin \, \theta \, d\theta \, d\phi \tag{23}$$

The flux has units of joule m$^{-2}$ s$^{-1}$ Hz$^{-1}$ or watts m$^{-2}$ Hz$^{-1}$.

In an isotropic radiation field where the light moves equally in all directions, let $I_\nu(\theta, \phi)$ $= I^0_\nu$. Then:

$$J_\nu \; = \; I^0_\nu \tag{24}$$

and

$$F_\nu \; = \; 0 \tag{25}$$

In an isotropic radiation field, there is no flux because there is no *net* transport of energy.

Another important case is where a surface is radiating out into space. Assume a situation (such as the surface of a star or an aperture looking into a cavity.) In this case we assume that $I_\nu$ is only a function of $\theta$ and is independent of $\phi$. We write that

$$I_\nu(\theta) \; = \; I^0_\nu \tag{26}$$

for $0 \leq \theta \leq \pi/2$. and

$$I_\nu(\theta) \; = \; 0 \tag{27}$$

Therefore

$$F_\nu \; = \; 2\pi \int_0^{\frac{\pi}{2}} I_\nu^0 \, cos\,\theta \; sin\,\theta \, d\theta \tag{28}$$

By setting $x = \sin\,\theta$, then it is easy to evaluate the integral and we find that

$$F_\nu \; = \; \pi \, I_\nu^0 \tag{29}$$

From above, we may therefore write that for a black body,

$$F_\nu \; = \; 2\,\pi \frac{h\nu^3}{c^2} \, \frac{1}{exp\left(\frac{h\nu}{kT}\right) \; - \; 1} \tag{30}$$

Often we are interested in the flow of all the energy and not just the energy at a particular frequency. We can write for the total flux, $F$, that

$$F \; = \; \int_0^\infty F_\nu \, d\nu \tag{31}$$

The integrated flux has units of watts m$^{-2}$. For radiation from a plane surface, then

$$F \; = \; \int_0^\infty 2\,\pi \frac{h\nu^3}{c^2} \, \frac{1}{exp\left(\frac{h\nu}{kT}\right) \; - \; 1} \, d\nu \tag{32}$$

With the substitution

$$x \; = \; \frac{h\nu}{kT} \tag{33}$$

Then:

$$F \; = \; 2\pi \frac{k^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x \; - \; 1} \, dx \tag{34}$$

Consider an atom with an upper energy level $U$ and a lower energy level, $L$. We now want to describe the emission and absorption by this atom. In order to do this exactly, we need to understand how the transition occurs by using quantum mechanics. However, even without a detailed understanding of the system, we can determine some general properties of a spectral line.

Assume that the atom can undergo a "spontaneous" transition from the upper to the lower level with a rate (units of s$^{-1}$) of $A_{UL}$ where this quantity $A_{UL}$ is often called the "Einstein A". The mean lifetime of the atom is the upper level is the inverse of the spontaneous decay rate or $A_{UL}^{-1}$. The units of this mean lifetime are $s$. The emissivity of the atom is given by the expression:

$$\epsilon_\nu \; = \; \frac{1}{4\pi} \, A_{UL} \, h\nu \, n_U \;\, \phi(\Delta\nu) \tag{35}$$

where $\phi(\Delta\nu)$ (units of $\mathrm{Hz}^{-1}$) is called the line profile. The rate of production of line photons depends upon the number of atoms in the upper level, the energy per photon, the Einstein $A$. We also include the factor $\frac{1}{4\pi}$ because we are interested in the production of photons into each solid angle as well as the total rate of production of photons. Finally, we include $\phi(\Delta\nu)$, the line profile, because we want to know the spectral energy distribution of the emission at different frequencies. The line photons are not all emitted at exactly the same frequency. Instead there is a spread of frequencies, and the function $\phi(\Delta\nu)$ describes this spread.

Because the line photons are emitted near the frequency $\nu$, we can define the frequency offset from line center, $\Delta\nu$ as

$$\Delta\nu \; = \; \nu \, - \, \nu_0 \tag{36}$$

where $\nu_0$ is the frequency at line center. Then we expect that

$$\int_{-\infty}^{+\infty} \phi(\Delta\nu)\, d\Delta\nu \; = \; 1 \tag{37}$$

Alternatively, we may write this equation as:

$$\int_{0}^{+\infty} \phi(\Delta\nu)\, d\nu \; = \; 1 \tag{38}$$

since:

$$d\nu \; = \; d\Delta\nu \tag{39}$$

The difference between these two equations is the lower limit of the integral. Since the emission always occurs at frequencies relatively near $\nu_0$ so it does not make a real difference whether we integrate $\Delta\nu$ to $-\infty$ (physically slightly unrealistic) or to $-\nu_0$ (physically realistic but mathematically more complex).

An important example of line broadening is that produced by Doppler motions of the atoms in the system. If we observe emission from a gas, then along the line of sight, some atoms will be approaching us and others receding. We expect that:

$$\frac{\Delta\nu}{\nu} \; = \; -\frac{v_r}{c} \tag{40}$$

where $v_r$ is the radial velocity of the gas atom and $c$ is the speed of light. In a 1-dimensional Maxwell-Boltzmann distribution,

$$f(v_r)\, dv_r \; \propto \; exp\left(-\frac{mv_r^2}{2kT}\right)\, dv_r \tag{41}$$

where $m$ denotes the mass of the atom. The velocity distribution is symmetric around its mean value which here we take to be 0 m $\mathrm{s}^{-1}$. We therefore expect that

$$f(\Delta\nu)\, d(\Delta\nu) \; \propto \; \exp\left(-\frac{mc^2\Delta\nu^2}{2kT\nu^2}\right)\, d(\Delta\nu) \tag{42}$$

We expect that

$$\phi(\Delta\nu) \propto f(\Delta\nu) \tag{43}$$

Therefore, with the normalization condition, we find for a line undergoing thermal broadening that:

$$\phi(\Delta\nu) \;=\; \sqrt{\frac{mc^2}{2\pi kT\nu^2}}\; exp\left(-\frac{mc^2\Delta\nu^2}{2kT\nu^2}\right) \tag{44}$$

In addition to emission, the atom can absorb light. We define the "Einstein B" such that $B_{LU}$

$$\kappa_\nu \;=\; n_L\, B_{LU}\, h\nu\, \frac{1}{4\pi}\, \phi(\Delta\nu) \tag{45}$$

With this definition, $B_{LU}$ has a similar appearance to $A_{UL}$. Warning: while everyone agrees about the definition of the Einstein $A$, different authors do or do not include the $\frac{1}{4\pi}$ term in their definition of the Einstein "B". The units of $B_{LU}$ (or $B_{UL}$) are different from those of $A_{UL}$. In particular, the units of $B_{LU}$ are equal to the units of $A_{UL}$ divided by an intensity. Thus $B_{LU}$ has units of $\mathrm{m}^2\ \mathrm{s}^{-1}\ \mathrm{joule}^{-1}$. You may also think of the cross section in the line and write,

$$\sigma_\nu \;=\; B_{LU}\, h\nu\, \frac{1}{4\pi}\, \phi(\Delta\nu) \tag{46}$$

As pointed out by no lesser an authority than Einstein, we must also allow for the possibility of stimulated emission. That is, as with all harmonic oscillators, there can be forced oscillations. We therefore, assume that there may be "stimulated emission" which is the reverse of absorption. In stimulated emission, we expect that:

$$h\nu \;+\; X^* \!\rightarrow h\nu \;+\; h\nu \;+\; X \tag{47}$$

In this scheme, there is conservation of energy, and a photon is produced from an atom which is already excited. We denote the coefficient for stimulated emission as $B_{UL}$, and it acts like a "negative absorption" Thus, it contributes to the opacity as:

$$\kappa_\nu \;=\; -n_U\, B_{UL}\, h\nu\, \frac{1}{4\pi}\, \phi(\Delta\nu) \tag{48}$$

We now estimate the relationship between the Einstein $A$ and $B's$. Consider the two level atom. In a steady state, the rate at which atoms leave level $U$ equals the rate at which they enter level $U$. The rate at which they leave $U$ depends both upon the rate of spontaneous emission and the rate of stimulated emission. The rate per unit volume of spontaneous emission is

$$\frac{1}{4\pi}\, A_{UL}\, n_U\;\, \phi(\Delta\nu) \tag{49}$$

The rate per unit volume of stimulated emission is the result of photons arriving from all directions and therefore depends upon the mean intensity, $J_\nu$. We can write that the total rate per unit volume of stimulated emission is

$$J_\nu \, n_U \, B_{UL} \, \frac{1}{4\pi} \, \phi(\Delta\nu) \tag{50}$$

The rate per unit volume at which atoms enter level $U$ is given by the rate per unit volume of absorptions from level $L$ and is given by the expression:

$$J_\nu n_L \, B_{LU} \, \frac{1}{4\pi} \, \phi(\Delta\nu) \tag{51}$$

Therefore, in a steady state,

$$\frac{1}{4\pi} \, A_{UL} \, n_U \; \phi(\Delta\nu) \; + \; J_\nu \, n_U \, B_{UL} \, \frac{1}{4\pi} \, \phi(\Delta\nu) \; = \; J_\nu n_L \, B_{LU} \, \frac{1}{4\pi} \, \phi(\Delta\nu) \tag{52}$$

This equation can be re-written as:

$$A_{UL} \, n_U \; + \; J_\nu \, n_U \, B_{UL} \; = \; J_\nu n_L \, B_{LU} \tag{53}$$

We can re-arrange the terms to find:

$$J_\nu \; = \; \frac{A_{UL} \, n_U}{n_L B_{LU} \; - \; n_U B_{UL}} \tag{54}$$

or

$$J_\nu \; = \; \frac{\frac{A_{UL}}{B_{UL}}}{\frac{n_L B_{LU}}{n_U B_{UL}} \; - \; 1} \tag{55}$$

This relationship is derived for a steady state at any temperature. We may therefore set $J_\nu$ equal to the Planck function at any temperature. This gives the following:

$$\frac{2h\nu^3}{c^2} \, \frac{1}{exp\left(\frac{h\nu}{kT}\right) \; - \; 1} \; = \; J_\nu \; = \; \frac{\frac{A_{UL}}{B_{UL}}}{\frac{n_L B_{LU}}{n_U B_{UL}} \; - \; 1} \tag{56}$$

The solution to this expression is that

$$\frac{2h\nu^3}{c^2} \; = \; \frac{A_{UL}}{B_{UL}} \tag{57}$$

and

$$exp\left(\frac{h\nu}{kT}\right) \; = \; \frac{n_L B_{LU}}{n_U B_{UL}} \tag{58}$$

which is the same as:

$$exp\left(-\frac{h\nu}{kT}\right) \; = \; \frac{n_U B_{UL}}{n_L B_{LU}} \tag{59}$$

In atomic spectroscopy, it is often found that a level is "degenerate". This means that there might be more than one "sublevel" at the same energy. If $g$ is used to denote the number of sublevels in a level, then we have both $g_L$ and $g_U$ for the lower and upper energy levels, respectively. We can generalize the usual Boltzmann relationship so that

$$\frac{n_U}{n_L} \; = \; \frac{g_U}{g_L} \, exp\left(-\frac{h\nu}{kT}\right) \tag{60}$$

Therefore:

$$g_L \, B_{LU} \; = \; g_U \, B_{UL} \tag{61}$$

At this point, we have now established that there is a simple, relationship between the Einstein $A$ and the Einstein $B's$. First, any atom which has a high value of $A$, that is any atom which can be a strong emitter, also must have a large value of $B$ which means that it also is a strong absorber. Second, of necessity, $B_{UL}$ is not zero and is positive. The process of stimulated emission must occur if thermodynamic equilibrium can be achieved. The Einstein $A$ and $B's$ are properties of the atom and *not* of the gas temperature or pressure.

The total opacity of the gas depends upon the difference between true absorptions and stimulated emissions. We may write that

$$\kappa_\nu \; = \; n_L \, B_{LU} \, h\nu \, \frac{1}{4\pi} \, \phi(\Delta\nu) \; - \; n_U \, B_{UL} \, h\nu \, \frac{1}{4\pi} \, \phi(\Delta\nu) \tag{62}$$

Collecting terms, this means that:

$$\kappa_\nu \; = \; n_L \, B_{LU} \, h\nu \, \frac{1}{4\pi} \, \phi(\Delta\nu) \left(1 \; - \; \frac{n_U \, B_{UL}}{n_L \, B_{LU}}\right) \tag{63}$$

In thermodynamic equilibrium, we therefore have that:

$$\kappa_\nu \; = \; n_L \, B_{LU} \, h\nu \, \frac{1}{4\pi} \, \phi(\Delta\nu) \left(1 \; - \; exp\left[-\frac{h\nu}{kT}\right]\right) \tag{64}$$

This is the "normal" condition of a gas.

A gas need not be in thermodynamic equilibrium. If, by some process the gas can be kept out of equilibrium and if, in fact, the population in the upper energy level can be kept high by some process, then, $\kappa_\nu$ can become negative. That is, there can be more stimulated emissions than absorptions. Therefore, when we write that

$$I_\nu \; = \; I_\nu^0 \, e^{-\kappa_\nu \, s} \tag{65}$$

we find a net effect in the emergent intensity compared to the incident intensity, $I_\nu^0$, *not* a decrease. A device which can do this is called a laser = light amplification [by] stimulated emission [of] radiation.

## Lecture 2 – Low Energy Line Photons: Emission/Absorption

To illustrate some of the formal results from radiative transfer, we consider low energy photons where $h\nu < kT$. We start by considering the 21 cm line of hydrogen (see Spitzer *Physical Processes in the Interstellar Medium*). This line arises from the hyperfine splitting of the hydrogen atom. The proton and electron spins can be parallel ($F = 1$; statistical weight = $2F + 1$ or 3) or anti-parallel ($F = 0$ and statistical weight = 0.) The frequency of the line is 1.420 Ghz (corresponding to a wavelength of 21.11 cm) and the Einstein $A$ is $2.869 \times 10^{-15}$ s$^{-1}$ which corresponds to a mean lifetime in the upper level of $\sim 1.1 \times 10^7$ years.

Consider the case where there a neutral cloud of gas in front of a continuum source (such as a quasar or galactic H II region). Let $I_{cont}$ denote the continuum source. The intensity that we detect, $I_\nu$, is a combination of attenuation of the background plus the emission from the cloud. Therefore:

$$I_\nu \; = \; S_\nu \left(1 \, - \, e^{-\tau_\nu}\right) \, + \, I_{cont} \, e^{-\tau_\nu} \tag{1}$$

where $S_\nu$ is the source function in the cloud. We can re-write this equation as:

$$I_\nu \; = \; (S_\nu \, - \, I_{cont}) \left(1 \, - \, e^{-\tau_\nu}\right) \, + \, I_{cont} \tag{2}$$

If $S_\nu > I_{cont}$, the line is seen in emission; if $S_\nu < I_{cont}$, the line is seen in absorption. If $S_\nu$ = $I_{cont}$, no line is detected. Thus whether the cloud adds or removes energy into the beam depends upon the relative size of the source function. We implicitly define the "brightness temperature" of a source, $T_b$, as:

$$I_{cont} \; = \; \frac{2\nu^2 \, k \, T_b}{c^2} \; = \; \frac{2 \, k \, T_b}{\lambda^2} \tag{3}$$

Furthermore, if the gas has temperature, $T$, then:

$$S_\nu \; = \; \frac{2 \, \nu^2 \, k \, T}{c^2} \tag{4}$$

Therefore, the criterion for producing an absorption line is that $T < T_b$ while an emission line results if $T > T_b$. This is consistent with our everyday notion that energy flows from the hot reservoir to the cold reservoir. There is no energy exchange, or no spectral line formed, in the special case that $T = T_b$.

Consider an observation where the background continuum is negligible. [There is always some continuum from the 2.7 K microwave background.] In the case where $\tau >> 1$, at least at line center, then the observed intensity measures the temperature of the gas. In this way, we know that in the Milky Way, the characteristic temperature of the neutral hydrogen is $\sim$100 K. In order to have confidence that we are measuring the intensity, we need to resolve the source. One way to tell if we resolve the source is to map it.

If the source is optically thin, we may write:

$$I_\nu \approx S_\nu \, \tau_\nu \tag{5}$$

We define the "column density", $N$, with units of cm$^{-2}$, as:

$$N = \int n \, ds \tag{6}$$

where $n$ is the density and $ds$ is an increment of distance measured along the line of sight. For the 21 cm line of hydrogen, since $h\nu << kT$, the correction for stimulated emission is very important, and:

$$\tau_\nu = N_L \frac{B_{LU} \, h\nu}{4\pi} \, \phi(\Delta\nu) \, \frac{h\nu}{kT} \tag{7}$$

Furthermore, since $h\nu << kT$, there are approximately three times as many atoms in the upper level as in the lower level. Therefore, we can relate the column density in the lower level to the total column density, $N(H)$, as:

$$N(H) = 4 \, N_L \tag{8}$$

Furthermore, we can write that:

$$B_{UL} = \frac{B_{LU}}{3} \tag{9}$$

Therefore, using the Einstein relation between the $A_{UL}$ and $B_{UL}$, we find that:

$$\tau_\nu = \frac{N(H)}{4} \left( \frac{3 \, A_{UL} \, c^2}{2 \, h \, \nu^3} \right) \frac{(h\nu)^2}{4\pi kT} \, \phi(\Delta\nu) \tag{10}$$

If we integrate over all values of $\Delta\nu$, then:

$$\int I_\nu \, d(\Delta\nu) = S_\nu \int \tau_\nu \, d(\Delta\nu) = \frac{3}{4} \, N(H) \, \frac{A_{UL}}{4\pi} \, h\nu \tag{11}$$

Recognizing that 3/4 of the atoms are in the upper level, this expression is equivalent to finding the intensity from the emissivity multiplied by the path length.

We now use:

$$I_\nu = \frac{2 \, k \, T_b}{\lambda^2} \tag{12}$$

Therefore, we can write:

$$N(H) = \frac{32\pi}{3 \, \lambda^2} \, \frac{k}{A_{UL} \, h\nu} \int T_b \, d(\Delta\nu) \tag{13}$$

Using the conversion between frequency and velocity:

$$\Delta\nu = \nu \, \frac{V}{c} \tag{14}$$

and assuming that we consider brightness temperature as a function of velocity, then this expression becomes:

$$N(H) \;=\; \frac{32\pi}{3}\,\frac{k}{h\,c\,\lambda^2\,A_{UL}}\int T_b\,dV \;=\; 1.823\times 10^{13}\int T_b\,dV \tag{15}$$

where $V$ is measured in cm s$^{-1}$.

We can use 21 cm data to measure the mass of atomic hydrogen, $M_{total}$, within a galaxy at distance $D$ from the Sun. If $\Omega$ is the telescope beam, then the projected area of the galaxy in the telescope beam is $\Omega\,D^2$. If $m_H$ denotes the mass of a hydrogen atom, then:

$$M_{total} \;=\; \Omega\,D^2\,N(H)\,m_H \tag{16}$$

This expression is independent of the gas temperature and density. We do not need to resolve the source to measure its mass. The brightness temperature can be a function of the telescope beam.

Absorption lines at 21 cm also have been observed. Typically, we measure the optical depth through the line. Note that because of the importance of stimulated emission, $\tau$ depends upon the gas temperature. Therefore, the interpretation of the 21 cm optical depth rests upon knowing the gas temperature.

The hyperfine level of atomic hydrogen is a particularly simple line to consider; there are more complex radio lines as well. An important example is the rotational lines of CO which are usually seen in emission (see, for example Goldsmith 1972, ApJ, 176, 597). The rotational quantum number $J$ ranges upwards from 0. The statistical weight of each level is $2J+1$ while the energy of each level is approximately given by

$$E \;=\; h\,c\,B\,J\,(J+1) \tag{17}$$

where $B = 1.9313$ cm$^{-1}$ for $^{12}$CO. [Note that the isotope shifts for the rotational transitions are quite significant since the moment of inertia of $^{13}$CO is quite different from that of $^{12}$CO.] Thus the transition between the $J=1$ and $J=0$ level occurs at a frequency of about 115 GHz or wavelength of about 2.7 mm. The Einstein $A$ for emission between $(J+1)$ and $J$ is:

$$A \;=\; \frac{64\,\pi^4\,\nu^3\,\mu^2}{3\,h\,c^3}\left(\frac{J+1}{2\,J+3}\right) \tag{18}$$

where $\mu$ denotes the dipole moment of the molecule. For $^{12}$CO, we can take $\mu = 0.1098$ Debye (Chackerian & Tipping 1983, J. Mol. Spectroscopy, 99, 431); a Debye equals $10^{-18}$ statcoulomb-cm. Thus the Einstein$A$ for the $J=$ 1-0 transition of this molecule is $7.1\times 10^{-8}$ s$^{-1}$ implying a mean lifetime in the upper level of about 0.45 year. The lifetimes in the higher rotational levels are notably shorter.

The CO population is distributed over a number of rotational levels. If we assume a single excitation temperature, $T$, then the partition function, $\zeta$ is:

$$\zeta \;=\; \sum_{J=0}^{J=\infty}(2J+1)\,\exp\left(-\frac{h\,c\,B\,J(J+1)}{kT}\right) \tag{19}$$

Approximating this sum as an integral, then

$$\zeta \;=\; \frac{kT}{h\,c\,B} \tag{20}$$

Remember that the column density in the $J'th$ level, $N(J)$, is related to the total column of CO, $N(CO)$, by:

$$N(J) \;=\; \frac{(2J\,+\,1)\,\exp - \left[\frac{h\,c\,B\,J(J+1)}{kT}\right]}{\zeta}\,N(CO) \tag{21}$$

If the line is optically thick, then the intensity is just determined by the source function and, in LTE, by the Planck function. If the line is optically thin, then:

$$I_\nu \;=\; \frac{2\nu^2}{c^2}\,k\,T_{mb} \;=\; S_\nu\,\tau_\nu \tag{22}$$

where $T_{mb}$ is the "main-beam" Rayleigh-Jeans brightness temperature. In the Rayleigh-Jeans approximation, then

$$T_{mb} \;=\; T\,\tau_\nu \tag{23}$$

If, again, we integrate over the entire spectral line, then for high temperatures where the correction for stimulated emission is simple, we expect for the $J$ = 2-1 transition that:

$$\int \tau_{2,1} dV \;=\; B_{12}\,N(1)\,\frac{h\nu}{4\pi}\,\frac{h\nu}{kT}\,\frac{c}{\nu} \tag{24}$$

We approximate:

$$N(1) \approx \frac{3}{\zeta}\,N(CO) \;=\; \frac{3\,h\,c\,B}{k\,T}\,N(CO) \;=\; \frac{3\,h\,\nu}{4\,k\,T}\,N(CO) \tag{25}$$

To derive this result, remember that the energy of the $J = 2$ level is $6h\,c\,B$ while the energy of the $J = 1$ level is $2h\,c\,B$ so that for the 2-1 transition, $h\nu = 4h, c, B.$ while

$$B_{12} \;=\; \frac{5}{3}\,B_{21} \;=\; \frac{5}{3}\,\frac{c^2}{2h\nu^3}\,A_{21} \tag{26}$$

Therefore:

$$B_{12} \;=\; \frac{64\,\pi^4\,\mu^2}{9\,h^2\,c} \tag{27}$$

Then:

$$\int \tau_{2,1} dV \;=\; \frac{4\pi^3}{3}\,\frac{\mu^2\,h\,\nu^2}{(kT)^2}\,N(CO) \tag{28}$$

Therefore for the 2-1 transition:

$$N(CO) \;=\; 3\,k^2\,T \int \frac{T_{mb}\,dV}{4\pi^3\,\mu^2\,h\nu^2} \tag{29}$$

Note that unlike atomic hydrogen, the total column of CO depends upon the temperature.

As a first approximation, a telescope beam might be described as a azimuthally-symmetric Gaussian such that the increment of sensitivity, $d\Omega$ as a function of offset angle, $\theta$, from the pointing is given by

$$d\Omega \;=\; 2\pi \,\exp\left(-\frac{\theta^2}{\theta_0^2}\right)\,\theta\,d\theta \tag{30}$$

With this defintion, then

$$\Omega \;=\; \pi\theta_0^2 \tag{31}$$

If we define $\theta_{FWHM}$ as the Full Width Half Maximum of the beam, then

$$\theta_{FWHM} \;=\; 2\sqrt{\ln 2}\,\theta_0 \tag{32}$$

Consequently, the effective area on the sky, $A$, of the telescope beam for a source at distance, $D$, is

$$A \;=\; \pi\,D^2\,\frac{\theta_{FWHM}^2}{4\ln 2} \tag{33}$$

If the mass of a CO molecule is $m_{CO}$, then the total mass of the molecular gas, $M(CO)$, is:

$$M(CO) \;=\; N(CO)\,A\,m_{CO} \tag{34}$$

## Lecture 3 – High Energy Line Photons: Emission/Absorption

We now consider high energy photons where $h\nu > kT$ so that stimulated emission is not too important. Again, a useful reference is Spitzer's *Physical Processes in the Interstellar Medium.* First consider a cold cloud in front of a continuum source with intensity $I_0$, such as occurs in the interstellar medium or in the absorption line spectra of quasars. The intensity at the Earth, $I_\nu$, is:

$$I_\nu \;=\; I_0\,\exp(-\tau_\nu) \tag{1}$$

We define the residual intensity of an absorption line, $r_\nu$, as:

$$r_\nu \;=\; \frac{I_\nu}{I_0} \tag{2}$$

Because stimulated emission is unimportant, then:

$$\tau_\nu \;=\; N_L\,\frac{B_{LU}\,h\nu}{4\,\pi}\,\phi(\Delta\nu) \tag{3}$$

where $N_L$ is the column density in the lower energy level. Instead of using the Einstein $B$, we often use, instead the dimensionless line oscillator strength, $f$, defined such that:

$$\frac{\pi\,e^2}{m_e\,c}\,f \;=\; \frac{B_{LU}\,h\nu}{4\,\pi} \tag{4}$$

where $m_e$ is the mass of an electron of charge $e$.

We often measure the equivalent width, $W_\nu$, measured in units of Hz, of a line defined as:

$$W_\nu \;=\; \int \frac{I_0\,-\,I_\nu}{I_0}\,d(\Delta\nu) \tag{5}$$

More frequently, astronomers use $W_\lambda$, measured in units such as Å, such that

$$\frac{W_\lambda}{\lambda} \;=\; \frac{W_\nu}{\nu} \tag{6}$$

where $\lambda\,\nu = c$. From above, we can write that:

$$W_\nu \;=\; \int (1\,-\,\exp(-\tau_\nu))\,d(\Delta\nu) \;=\; \int (1\,-\,r_\nu)\,d(\Delta\nu) \tag{7}$$

In the optically thin case where at all frequencies $\tau_\nu << 1$, the solution is simple:

$$W_\nu \;=\; N_L\,\frac{\pi\,e^2}{m_e\,c}\,f \tag{8}$$

or

$$W_\lambda \;=\; N_L\,\frac{\pi\,e^2\,\lambda^2}{m_e\,c^2}\,f \tag{9}$$

One test of whether a line is optically thin is to examine its residual intensity. Another test is to compare lines with different oscillator strengths from the same lower level.

The analysis can be more difficult if the line is optically thick at the center. In this case, the line is described as "saturated". While a lower bound to the column density can be determined from the equivalent width, its true value may be difficult to determine. One approach is to assume a Gaussian line profile so that:

$$\phi(\Delta\nu) \;=\; \frac{1}{\sqrt{\pi}\, b'}\, \exp\left(-\left[\frac{\Delta\nu}{b'}\right]^2\right) \tag{10}$$

Usually, instead of expressing the line-width in frequency units, we do so in velocity units. We define $b$, such that:

$$b \;=\; b'\left(\frac{c}{\nu}\right) \tag{11}$$

We can write that:

$$\tau(\Delta\nu) \;=\; \tau_0\, \exp\left(-\left[\frac{\Delta\nu}{b'}\right]^2\right) \tag{12}$$

A "curve of growth" is essentially a plot of $W_\lambda$ vs. $\tau_0$. The basic idea is that $W_\lambda$ increases linearly until $\tau_0 \sim 1$. Then, $W_\lambda$ flattens and large changes in $N_L$ or $\tau_0$ lead to only small changes in $W_\lambda$. In this regime, it is difficult to make a good measure of $N_L$. However, the line width serves as a good measure of $b$. That is:

$$W_\lambda \;=\; \frac{b\,\lambda}{c} \int_{-\infty}^{\infty} \left[1 \;-\; \exp(-\tau_0 \exp[-x^2])\right]\, dx \tag{13}$$

Asymptotically:

$$W_\lambda \approx \frac{2\,b\,\lambda}{c}\, (\ln \tau_0)^{1/2} \tag{14}$$

All spectral lines have very weak "damping wings". This can be caused by natural broadening or perturbations from neighbors. We write that in the far wings:

$$\phi(\Delta\nu) \;=\; \frac{\delta/\pi}{\delta^2 \;+\; (\Delta\nu)^2} \tag{15}$$

where

$$\delta \;=\; \frac{1}{4\pi} \sum_L A_{UL} \tag{16}$$

where the sum is performed into all lower levels. Therefore, in the far wings of the line:

$$\tau(\Delta\nu) \;=\; N_L\, \frac{\pi\, e^2}{m_e\, c}\, f\, \frac{\delta}{\pi(\Delta\nu)^2} \tag{17}$$

When the column density is large enough, the line is said to be on the damping portion of the curve of growth. One can show that $W_\lambda$ varies as $N_L^{1/2}$, but, typically, the line can be resolved and therefore the equivalent measure is not performed.

An interesting example of the curve-of-growth is the determination of $D/H$ in the interstellar medium. Typically, the $H$ lines are damped while the $D$ lines can lie on the optically thin portion of the curve of growth. Thus even though there is a factor of $10^5$ difference in the abundances of the two species, the relative column densities can be accurately determined.

Another important contemporary example of the curve-of-growth is the analysis of quasar absorption line systems. One class is the "Damped Lyman $\alpha$ where the hydrogen line is so broad because of the relatively high column density of atomic hydrogen.

The analysis of the curve of growth for stellar atmospheres (see, for example. Bohm-Vitense, *Introduction to Stellar Astrophysics, v. II: Stellar Atmospheres* is somewhat more complex than for the interstellar medium because the "cold" intervening material emits as well as absorbs. Consider an infinite plane parallel atmosphere. If we measure optical depth downwards through the atmosphere with $\tau = 0$ at the "top", then assume for simplicity that the source function, $S$, can be written as:

$$S = a + b\,\tau \tag{18}$$

Looking upwards out from the atmosphere, define $\mu$ as the cosine of the angle relative to the normal. Therefore, if $ds$ is the element of path length, we can write:

$$\kappa\, ds = -\frac{d\tau}{\mu} \tag{19}$$

Then for the different lines of sight through the atmosphere, we can write that the equation of transfer along each direction for $I(\tau, \mu)$, is:

$$\mu \frac{dI}{d\tau} = I - S \tag{20}$$

or:

$$\frac{dI}{d\tau} - \frac{I}{\mu} = -\frac{S}{\mu} \tag{21}$$

Note that $S$ does not vary as $\mu$ but only $\tau$ while $I$ depends upon both $\tau$ and $\mu$. This equation can be solved by multiplying by $e^{-\tau/\mu}$ to give:

$$\frac{d}{d\tau}\left(I\, e^{-\tau/\mu}\right) = -\frac{S}{\mu}\, e^{-\tau/\mu} \tag{22}$$

The solution is to integrate on both sides between $\tau_1$ and $\tau_2$ to give:

$$I(\tau_2, \mu)\, e^{-\tau_2/\mu} - I(\tau_1, \mu)\, e^{-\tau_1/\mu} = -\int_{\tau_1}^{\tau_2} \frac{S}{\mu}\, e^{-\tau/\mu}\, d\tau \tag{23}$$

To find the intensity at the surface, we take $\tau_1 = 0$. If the atmosphere is infinte, but $I$ does not exponentially increase with $\tau_2$, then the above equation becomes:

$$I(\mu) \;=\; \int_0^\infty \frac{S}{\mu}\, e^{-\tau/\mu}\, d\tau \tag{24}$$

With $S$ given above, then at $\tau = 0$, then:

$$I(\mu) \;=\; a \,+\, b\,\mu \tag{25}$$

The flux at the surface of the star is:

$$F \;=\; 2\,\pi \int_0^1 (a \,+\, b\mu)\; \mu\, d\mu \;=\; \pi\left(a \,+\, \frac{2}{3}\, b\right) \;=\; \pi\, S\left(\tau \;=\; \frac{2}{3}\right) \tag{26}$$

Thus the flux from the star is determined by the source function at optical depth 2/3.

We observe absorption lines in the spectrum of the star because optical depth 2/3 in the line occurs at a higher physical distance than optical depth 2/3 in the continuum. Consequently, in the line, the received flux is less than in the nearby continuum as long as $b > 0$ which is the same as the temperature increasing with depth in the atmosphere. [Note we are ignoring scattering which can complicate the analysis.] If the line is relatively "weak" then the line is formed only slightly higher than the continuum. As a result, in a Taylor series expansion, the strength of the line depends linearly upon the line opacity which typically depends linearly on the concentration of the atom in the lower level and thus linearly upon the abundance. The line is said to be on the linear portion of the curve of growth. However, if the line is very opaque, then optical depth 2/3 is reached at essentially the "top"of the atmosphere. In this case, the strength of the line depends only upon the source function at the top of the atmosphere compared to the source function where the continuum achieves optical depth 2/3. Thus the line strength is approximately independent of the abundance of the absorbing atoms and the line is described as saturated.

Emission lines with $h\nu > kT$ often are optically thin. In this case:

$$I_\nu \;=\; S_\nu\, \tau_\nu \tag{27}$$

Since $S_\nu = \epsilon_\nu/\kappa_{nu}$ and $\tau_\nu = \kappa_\nu\, L$ where $L$ denotes the path length through the medium, then:

$$I_\nu \;=\; \epsilon_\nu\, L \tag{28}$$

Often, emission lines are formed very far from thermodynamic equilibrium. An important example are the recombination lines of hydrogen. In a gaseous nebula (H II region, Planetary Nebula, quasar) the hydrogen is largely ionized (see, for example, Osterbrock and Ferland *Astrophysics of Gaseous Nebulae and Active Galactic Nuclei.* The gas recombines into excited levels and then cascades through the different levels until the atom reaches the ground state. The efficiency of the emission depends upon the line. For example, about

0.4 of all recombinations lead to $H\alpha$ corresponding to the $n = 3$ to $n = 2$ transition at 6563 Å while about 0.1 of all recombinations lead to $H\beta$ corresponding to the $n = 4$ to $n = 2$ transition at 4861 Å. Thus it is expected that the flux in the 6563 line is about 3 times greater than the flux in the 4861 line.

Ionized nebulae also display emission of "forbidden lines". These are lines with low values of the Einstein A so that in high density regions, the atom often is collisionally de-excited before it emits. In astrophysics, however, the densities are often low enough that collisional excitation is followed by radiative de-excitation. Thus:

$$e + X \rightarrow e + X^* \tag{29}$$

followed by:

$$X^* \rightarrow X + h\nu \tag{30}$$

An important example is excitation from the ground state of $O^{+2}$. Remember that the notation for the energy level is:

$$^{2S+1}L_J \tag{31}$$

where $S$ is the total spin, $L$ represents the total orbital angular momentum and $J$ the total vector angular momentum (orbital + spin). The $L$ angular momentum is given as letters such that $L = 0$ is denoted by $S$; $L = 1$ is denoted as $P$; $L = 2$ is denoted as $D$ and $L = 3$ is denoted as $F$. The six-electron $O^{+2}$ ion has an electronic ground state configuration of $1s^2\ 2s^2\ 2p^2$. The lowest energy level is $^3P$ with 3 fine structure levels. The lowest fine structure level is $^3P_0$ and then $^3P_1$ and $^3P_2$. The statistical weight of each level is given by $(2J + 1)$ or 1, 3 and 5 for the three different terms. The $^1D$ level has no fine structure. Note that since its total spin is 0, then $J = 2$ and its statistical weight is 5. The forbidden transitions are:

$$^1D \rightarrow {}^3P_2 \tag{32}$$

at 5007 Å [air] and $A_{UL} = 0.0210\ \mathrm{s}^{-1}$. Also

$$^1D \rightarrow {}^3P_1 \tag{33}$$

at 4959 Å [air] and $A_{UL} = 0.0071\ \mathrm{s}^{-1}$. The transition between $^1D$ and $^3P_0$ is very forbidden [very low value of $A$] and essentially never seen. Because 4959 Å and 5007 Å have the same upper level, we expect that the emission in an optically thin system yields:

$$F(5007)/F(4959) \approx 3.0 \tag{34}$$

The $^1S$ energy level is even higher and the transition:

$$^1S \rightarrow {}^1D \tag{35}$$

at 4363 Å [air] with $A_{UL} = 1.60\ \mathrm{s}^{-1}$ is observed as well. The flux ratio:

$$F(4363)/F(5007) \tag{36}$$

is a temperature diagnostic since it is sensitive to the gas temperature.

**Lecture 4 – Classical Theory of Radiation I: Retarded Potentials**

The classical theory of radiation is derived from Maxwell's equations (see, for example, Griffith *Introduction to Electrodynamics*). Instead of using the electric and magnetic fields, we use the scaler and vector potentials. The first step is to introduce the "retarded potentials". Information cannot travel faster than the speed of light, and therefore the potentials are determined by the charge and current densities at the "retarded time", $t_r$ defined as

$$t_r \;=\; t - \frac{d}{c} \tag{1}$$

where $t$ is the time at the location of the observer and $d$ is the distance between the observer and the distance of the charge contributing to the potential. If $\rho$ defines the charge density and $\vec{J}$ defines the current density, while the scaler potential is $\phi$ and the vector potential is $\vec{A}$, then:

$$\phi(\vec{r}, t) \;=\; \int \frac{\rho(\vec{r}', t_r)}{d}\, dV' \tag{2}$$

where $dV'$ is an element of volume and the charge density is measured at the retarded time. Similarly:

$$\vec{A}(\vec{r}, t) \;=\; \frac{1}{c}\int \frac{\vec{J}(\vec{r}', t_r)}{d}\, dV' \tag{3}$$

These results are stated without proof as being consistent with Maxwell's equations and the usual derivation of the fields from the potentials.

Now consider the application to the potentials from a moving charge, $q$. A key point is the effective volume of the moving charge density is not the same as the effective volume of the same charge density if the particle is stationary. Break down the motion of the volume element into a radial and transverse components. If $ds$ denotes the distance along the radial direction and $dA$ the projected area, then the element of volume is $dA\, ds$. To first order, there is no effect on $dA$ by the radial motion, but there is on $ds$. Imagine light leaving both from the "back" of the volume element and the "front" of the volume element. The light from the "front" part has less distance to travel compared to the "light" from the back. If the volume element is not moving, this does not matter. However, if the volume element is moving with a radial speed $v_r$, then the during the time it takes for the light to get from the back to the front of the volume element, $ds'/c$ equals the time the distance the volume element travels divided by its speed or

$$\frac{ds'}{c} \;=\; \frac{ds' - ds}{v_r} \tag{4}$$

Therefore, with a little algebra:

$$ds' \;=\; \frac{ds}{1 - v_r/c} \tag{5}$$

Note that this correction is independent of the size of $ds$. Therefore, even "shrinking" an extended charge to a tiny volume, we need to include this effect. If $\hat{d}$ denotes a unit vector in the $\vec{d}$ direction so that $\hat{d} = \vec{d}/d$, then:

$$v_r \;=\; \hat{d}\cdot\vec{v} \tag{6}$$

We may therefore write that

$$\phi(\vec{r}, t) \;=\; \left(\frac{q}{d \;-\; \vec{d}\cdot\vec{v}/c}\right)_{t\,-\,d/c} \tag{7}$$

Similarly:

$$\vec{A} \;=\; \left(\frac{q\vec{v}/c}{d \;-\; \vec{d}\cdot\vec{v}/c}\right)_{t\,-\,d/c} \tag{8}$$

These are the Lienart-Wiechart potentials.

Now consider a model of an oscillating electric dipole of an application of these potentials. We assume two tiny spheres separated by distance $h$ aligned along the $Z$-axis. The upper sphere located at $(0, 0, h/2)$ has charge $q$ and the lower sphere located at $(0, 0, -h/2)$, has charge $-q$ where

$$q \;=\; q_0 \cos(\omega\, t) \tag{9}$$

The dipole moment, $\vec{p}$, is:

$$\vec{p} \;=\; p_0 \cos(\omega\, t)\, \hat{z} \tag{10}$$

where $p_0 = q_0\, h$. Assume that the observer lies at distance $R$ from the origin at polar angle $\theta$. Since it is not moving, the contribution to the scaler potential from the upper sphere is:

$$\phi_U \;=\; \frac{q_0 \cos[\omega(t \;-\; d_U/c)]}{d_U} \tag{11}$$

By the law of cosines,

$$d_U \;=\; \sqrt{R^2 \;-\; R\,h\,\cos\theta \;+\; h^2/4} \tag{12}$$

Similarly, the contribution to the potential for the lower sphere is:

$$\phi_L \;=\; -\frac{q_0 \cos[\omega(t \;-\; d_L/c)]}{d_L} \tag{13}$$

where:

$$d_L \;=\; \sqrt{R^2 \;+\; R\,h\,\cos\theta \;+\; h^2/4} \tag{14}$$

We now make three approximations. First, we assume that we are far from the dipole in the sense that $R >> h$. As a result, we may write that:

$$d_U \approx R\left(1 \;-\; \frac{h}{2\,R}\cos\theta\right) \tag{15}$$

and

$$d_L \approx R\left(1 + \frac{h}{2\,R}\cos\theta\right) \tag{16}$$

We also assume that $h << \lambda$ where $\lambda = 2\,\pi\,c/\omega$. Therefore, we can approximate:

$$\cos[\omega(\,t\, -\, d_U/c)]) \approx \cos\left(\omega(t\, -\, R/c)\, +\, \frac{h\,\omega}{2\,c}\cos\theta\right) \tag{17}$$

Using the trigonometric identify for the cosine of the sum of two angles and then using that $h/\lambda << 1$, then:

$$\cos[\omega(t\, -\, d_U/c)] \approx cos[\omega(t\, -\, R/c)]\, -\, \frac{h\,\omega}{2\,c}\cos\theta\sin[\omega(t\, -\, R/c)] \tag{18}$$

Similarly, we can expand the cosine term for $\phi_L$ so that:

$$\cos[\omega(t\, -\, d_L/c)] \approx \cos[\omega(t\, -\, R/c)]\, +\, \frac{h\omega}{2\,c}\cos\theta\sin[\omega(t\, -\, R/c)] \tag{19}$$

We now write for the potential from the dipole that:

$$\phi\, =\, \phi_U\, +\, \phi_L \approx \frac{q_0\,h\,\cos\theta}{R}\left(-\frac{\omega}{c}\sin(\omega[t\, -\, R/c])\, +\, \frac{1}{R}\cos(\omega[t\, -\, R/c])\right) \tag{20}$$

In regions where $R >> \lambda$, this expression can be approximated as:

$$\phi \approx\, -\frac{p_0\omega}{c}\frac{\cos\theta}{R}\sin[\omega(t\, -\, R/c)] \tag{21}$$

This result for the potential depends only upon $p_0$ and therefore is independent of the details of the dipole moment.

We next evaluate the vector potential. Assuming a very thin wire, then the current, $\vec{I}$ can be derived from the current density and we write:

$$\vec{I}\, =\, \frac{dq}{dt'}\,\hat{z}\, =\, -q_0\,\omega\sin(\omega t')\,\hat{z} \tag{22}$$

Consequently, the vector potential is:

$$\vec{A}\, =\, \int_{-h/2}^{h/2} \frac{[-q_0\omega\sin[\omega(t\, -\, d/c)]\,\hat{z}}{c\,d}\,dz \tag{23}$$

To first order, we can write $d \approx R$ and therefore:

$$\vec{A}\, =\, -\frac{p_0\,\omega}{c\,R}\sin[\omega(t\, -\, R/c)]\,\hat{z} \tag{24}$$

To put $\vec{A}$ into spherical coordinates, we use:

$$\hat{z} \;=\; \cos\theta\,\hat{R} \;-\; \sin\theta\,\hat{\theta} \tag{25}$$

Or:

$$\vec{A} \;=\; -\frac{p_0\omega\cos\theta}{c\,R}\sin[\omega(t \;-\; R/c)]\hat{R} + \frac{p_0\omega\sin\theta}{c\,R}\sin[\omega(t \;-\; R/c)]\hat{\theta} \tag{26}$$

With the potentials, we can now determine the fields. In spherical coordinates without any azimuthal variation, we write that

$$\vec{\nabla}\phi \;=\; \frac{\partial\phi}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial\phi}{\partial\theta}\hat{\theta} \tag{27}$$

Far from the dipole, we take only the terms that vary as $R^{-1}$ and therefore:

$$\vec{\nabla}\phi \approx \frac{p_0\omega^2\cos\theta}{c^2R}cos[\omega(t \;-\; R/c)]\hat{R} \tag{28}$$

We also find that:

$$\frac{\partial\vec{A}}{\partial t} \;=\; -\frac{p_0\omega^2\cos\theta}{c\,R}\cos[\omega(t \;-\; R/c)]\hat{R} + \frac{p_0\omega^2\sin\theta}{c\,R}\cos[\omega(t \;-\; R/c)]\,\hat{\theta} \tag{29}$$

We therefore find for the electric field that:

$$\vec{E} \;=\; -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \;=\; -\frac{p_0\omega^2\sin\theta}{c^2\,R}\cos[\omega(t \;-\; R/c)]\,\hat{\theta} \tag{30}$$

Since $\vec{A}$ is independent of azimuth, then the magnetic field is:

$$\vec{B} \;=\; \vec{\nabla}\times\vec{A} \;=\; \frac{1}{R}\left(\frac{\partial(R\,A_\theta)}{\partial R} \;-\; \frac{\partial A_R}{\partial\theta}\right)\hat{\phi} \tag{31}$$

Keeping only the terms that vary as $R^{-1}$, then:

$$\vec{B} \;=\; -\frac{p_0\omega^2\sin\theta}{R\,c^2}\cos[\omega(t \;-\; R/c)]\hat{\phi} \tag{32}$$

We have found that $\vec{E}$ and $\vec{B}$ are mutually perpendicular, in phase, vary as $1/R$ and have the same amplitude as expected for spherical light waves. Thus we have found that the oscillating dipole emits light at frequency $\omega$. The Poynting vector, $\vec{S}$ is:

$$\vec{S} \;=\; \frac{c}{4\pi}\vec{E}\times\vec{B} \;=\; \frac{p_0^2\omega^4\sin^2\theta}{4\,\pi\,R^2\,c^3}\cos^2[\omega(t \;-\; R/c)]\,\hat{R} \tag{33}$$

Averaged over a cycle, then:

$$< \vec{S} > = \frac{p_0^2\omega^4\sin^2\theta}{8\pi R^2\,c^3}\hat{R} \tag{34}$$

To find the total power radiated, we integrate over a sphere surrounding the dipole. An element of area, $\vec{da}$, is:

$$\vec{da} \;=\; R^2 \sin\theta \, d\theta \, d\phi \hat{R} \tag{35}$$

Thus, the average power radiated by the dipole, $< P >$, is

$$< P > = \int_0^{2\pi} \int_0^{\pi} < \vec{S} > \cdot \vec{da} \tag{36}$$

Therefore:

$$< P > = \frac{p_0^2 \omega^4}{3\, c^3} \tag{37}$$

This result is a specific illustration of Larmor's formula which is that the instantaneous power, $P$, emitted by an accelerating charge, $q$, is

$$P \;=\; \frac{2\, q^2\, \dot{u}^2}{3\, c^3} \tag{38}$$

where $\vec{u}$ denotes the velocity vector of the charge.

**Lecture 5 – Classical Theory of Radiation II: Radiation Reaction**

We can use our formalism to describe the interaction of light with material. The general Larmor formula for the instantaneous power, $P$, emitted by an accelerating charge, $q$, is

$$P \;=\; \frac{2\,q^2\,\dot{u}^2}{3\,c^3} \tag{1}$$

where $\vec{u}$ denotes the velocity vector of the charge. First consider a free electron and an incident light wave with an electric field, $\vec{E}$, described as:

$$\vec{E} \;=\; E_0 \sin\omega_0\, t\, \hat{z} \tag{2}$$

Using Newton's second law for the response of the electron, we write:

$$m_e\ddot{\vec{x}} \;=\; q\,\vec{E} \tag{3}$$

Therefore:

$$\ddot{x}^2 \;=\; \frac{q^2\,E_0^2 \sin^2\omega_0 t}{m_e^2} \tag{4}$$

Averaging over a cycle, we find from Larmor's formula that the average power radiated is:

$$< P > = \frac{q^4 E_0^2}{3\,m_e^2} \tag{5}$$

We can also write that if $\sigma$ denotes the electron cross section that:

$$< P > = \;\sigma < |\vec{S}| > \tag{6}$$

where $\vec{S}$ denotes the Poynting vector or:

$$\vec{S} \;=\; \frac{c}{4\pi}\,\vec{E}\times\vec{B} \;=\; \frac{c}{4\pi}\,E_0^2\,\sin^2\omega_0 t\,\hat{n} \tag{7}$$

where $\hat{n}$ denotes a unit vector in the direction of the propagation of the light wave. Thus:

$$< |\vec{S}| > = \frac{c\,E_0^2}{8\,\pi} \tag{8}$$

Therefore:

$$\sigma \;=\; \frac{8\,\pi\,q^4}{3\,m_e^2\,c^4} \;=\; \frac{8\,\pi}{3}\,r_0^2 \tag{9}$$

where

$$r_0 \;=\; \frac{q^2}{m_e c^2} \tag{10}$$

and $r_0$ is the classical radius of the electron or $2.82 \times 10^{-13}$ cm. The electron scattering cross section is $6.65 \times 10^{-25}$ cm$^2$. In a gas of pure hydrogen, then the electron scattering opacity is $\sigma/m_H$ where $m_H$ denotes the mass of a hydrogen atom. Thus the electron scattering opacity for this gas is 0.40 cm$^2$ g$^{-1}$.

An important application of electron scattering is the "Eddington limit". Consider a parcel of matter of density, $\rho$, cross sectional area $dA$ and height $dz$ at distance $D$ from a star of mass, $M_*$, and luminosity, $L_*$. The inward gravitational force on the star, $F_{grav}$ is:

$$F_{grav} = \frac{G\,M_*\,\rho\,dA\,dz}{D^2} \tag{11}$$

If we only consider the contribution by electron scattering, then the outward radiative force, $F_{rad}$, is determined by the rate at which photons of momentum $h\nu/c$ are scattered by free electrons:

$$F_{rad} = \left(\frac{L_*\,dA}{4\,\pi\,D^2\,c}\right)(\rho\chi\,dz) \tag{12}$$

If the gravitational force exceeds the radiative force, then the luminosity must be bounded such that:

$$L_* < \frac{4\,\pi\,G\,M_*\,c}{\chi} \tag{13}$$

This relationship is easily satisfied for the Sun. It can be used to place a lower bound to the masses of O-type stars whose luminosities can exceed $10^5$ $L_\odot$. This relationship is also important in sources powered by accretion such as black holes. Thus both X-ray binary stars and quasars are characterized by their luminosity relative to the Eddington limit.

We now consider the radiative reaction. When power is radiated away into space, the radiation is doing work on the accelerating charge. If $\vec{F}_{rad}$ denotes this radiative reaction, then we know that between two time intervals, $t_1$ and $t_2$ that:

$$-\int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{u}\,dt = \frac{2\,q^2}{3\,c^3}\int_{t_1}^{t_2} \dot{\vec{u}}\cdot\dot{\vec{u}}\,dt \tag{14}$$

We can integrate by parts:

$$\int_{t_1}^{t_2} \dot{\vec{u}}\cdot\dot{\vec{u}}\,dt = \left[\vec{u}\cdot\dot{\vec{u}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\vec{u}}\cdot\vec{u}\,dt \tag{15}$$

For an oscillator, we choose times so that term in the square brackets is zero. In this case:

$$\int_{t_1}^{t_2} \left(\vec{F}_{rad} - \frac{2\,q^2\,\ddot{\vec{u}}}{3\,c^3}\right)\cdot\vec{u}\,dt = 0 \tag{16}$$

Therefore:

$$\vec{F}_{rad} = \frac{2\,q^2\,\ddot{\vec{u}}}{3\,c^3} = m\tau\ddot{\vec{u}} \tag{17}$$

where:

$$\tau \; = \; \frac{2\,q^2}{3\,m\,c^3} \tag{18}$$

These arguments do not prove that this is the correct description of the radiative reaction, but they are self-consistent.

We can now apply these results to various circumstances. First, consider the emission from a simple harmonic oscillator where a classical particle moves like a spring. We can write for the restoring force, $\vec{F}$ that:

$$\vec{F} \; = \; -k\,\vec{x} \; = \; -m\,\omega_0^2\,\vec{x} \tag{19}$$

Therefore, the equation of motion of the particle is:

$$m\ddot{\vec{x}} \; = \; -m\omega_0^2\,\vec{x} \, + \, m\,\tau\ddot{\vec{u}} \tag{20}$$

We now make the approximation that the motion is only slightly damped in which case:

$$\ddot{\vec{u}} \; = \; -\omega_0^2\,\vec{u} \tag{21}$$

Since all the motion is along the $X$-axis, we can now write this equation as:

$$\ddot{x} \, + \, \omega_0^2\tau\,\dot{x} \, + \, \omega_0^2\,x \; = \; 0 \tag{22}$$

This is the usual equation for the damped harmonic oscillator. The amplitude is determined by the initial conditions, and we write that:

$$x \; = \; x_0\,e^{\alpha t} \tag{23}$$

where, in general, $\alpha$ is complex and we take the real portion of the solution. The differential equation becomes:

$$\alpha^2 \, + \, \left(\omega_0^2\tau\right)\alpha \, + \, \omega_0^2 \; = \; 0 \tag{24}$$

The solution is:

$$\alpha \approx \pm i\,\omega_0 \, - \, \frac{1}{2}\omega_0^2\tau \tag{25}$$

With the starting conditions that at $t = 0$, $x = x_0$ and $\dot{x} = 0$, then

$$x(t) \; = \; x_0 e^{-\Gamma t/2}\cos\omega_0 t \; = \; \frac{x_0}{2}\left(e^{-\Gamma t/2 \, + \, i\omega_0\,t} \, + \, e^{-\Gamma t/2 \, - \, i\omega_0 t}\right) \tag{26}$$

where

$$\Gamma \; = \; \omega_0^2\tau \; = \; \frac{2\,q^2\omega_0^2}{3\,m\,c^3} \tag{27}$$

We can Fourier transform this result with

$$x_F(\omega) \; = \; \frac{1}{2\pi}\int_0^\infty x(t)e^{i\omega t}\,dt \tag{28}$$

Therefore:

$$x_F \;=\; \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 \;-\; i(\omega \;+\; \omega_0)} \;+\; \frac{1}{\Gamma/2 \;-\; i(\omega \;-\; \omega_0)}\right] \tag{29}$$

We are only interested in positive values of $\omega$ where $x_F$ is appreciably different from zero. Therefore, we write:

$$x_F \approx \frac{x_0}{4\pi}\,\frac{1}{\Gamma/2 \,-\, i(\omega \;-\; \omega_0)} \tag{30}$$

Therefore:

$$x_F^2 \;=\; \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega \;-\; \omega_0)^2 \;+\; (\Gamma/2)^2} \tag{31}$$

We first use this expression to describe the shape of a spectral line. We can write that

$$\Delta\nu \;=\; \frac{\omega \;-\; \omega_0}{2\pi} \;=\; \frac{\Delta\omega}{2\pi} \tag{32}$$

We see that far from the line that the intensity of the emission varies as $(\Delta\nu)^{-2}$. The normalization of the line profile yields:

$$\phi(\Delta\nu) \;=\; \frac{\Gamma/(4\pi^2)}{(\Delta\nu)^2 \;+\; (\Gamma/4\pi)^2} \;=\; \frac{\delta/\pi}{(\Delta\nu)^2 \;+\; \delta^2} \tag{33}$$

The correspondence between the classical damping line damping and the quantum mechanical case is achieved by setting:

$$\delta \;=\; \Gamma/(4\pi) \;=\; \left(\sum_L A_{UL}\right)/(4\pi) \tag{34}$$

**Lecture 6 – Free free emission**

We now consider the power and spectrum radiated when an electron is accelerated by a nucleus of charge $Z$. We assume that the nucleus is stationary and that the electron radiates because of its acceleration caused by the electric charge of the nucleus. We assume that the electron has an impact parameter $b$ and an initial speed $v$ in the $X$ direction. We assume that the acceleration induced by the nucleus is a relative small perturbation on the electron's classical motion. The collision time, $t_{coll}$, is

$$t_{coll} \;=\; \frac{b}{v} \tag{1}$$

If $\Delta v$ denotes the change in speed, then we can approximate that the acceleration, $a$, is:

$$a \approx \frac{\Delta v}{t_{coll}} \tag{2}$$

We estimate $\Delta v$ with the simple approximation that the electron continues at constant speed along the $X$ axis and only acquires a $Y$ component of speed:

$$a_y \;=\; -\frac{Z\,q^2\,b}{m_e\,(b^2\,+\,x^2)^{3/2}} \tag{3}$$

With $dt = dx/v$, then:

$$\Delta v \;=\; \int_{-\infty}^{\infty} a_y\,dt \;=\; -\,\frac{Z\,q^2\,b}{v}\int_{-\infty}^{\infty}\frac{dx}{m_e\,(b^2\,+\,x^2)^{3/2}} \tag{4}$$

With the substition that $x = b\,y$, and $y = \tan\theta$ then:

$$\Delta v \;=\; -\frac{2\,Z\,q^2}{m_e\,b\,v}\int_{0}^{\infty}\frac{dy}{(1\,+\,y^2)^{3/2}} \;=\; -\frac{2\,Z\,q^2}{m_e\,b\,v} \tag{5}$$

Using the Larmor formula that:

$$P \;=\; \frac{2\,q^2\,a^2}{3\,c^3} \tag{6}$$

The time-averaged power radiated during a collision is:

$$P \;=\; \frac{2\,q^2(\Delta v)^2\,v^2}{3\,c^3\,b^2} \;=\; \frac{8\,Z^2\,q^6}{3\,c^3\,m_e^2\,b^4} \tag{7}$$

The total energy radiated during the collision, $E$ is:

$$E \;=P\,t_{coll} \;=\; \frac{8\,Z^2\,q^6}{3\,c^3\,m_e^2\,b^3\,v} \tag{8}$$

We are interested in the spectrum of the emission. We can write that

$$\ddot{x}_F \;=\; \int_{-\infty}^{\infty} \ddot{x}(t)\, e^{i\omega t}\, dt \tag{9}$$

We recognize that $x(t)$ is only appreciable different from 0 during the collision time. For large values of $\omega$, the exponential oscillates through many cycles during the collision and the integral is essentially 0. For small values of $\omega$ the integral is a constant. Therefore equal power is emitted in all frequency bins up to $\omega_{max}$ where:

$$\omega_{max} \approx \frac{\pi}{t_{coll}} \;=\; \frac{\pi v}{b} \tag{10}$$

We therefore write for the energy radiated per frequency, $dE/d\omega$, that:

$$\frac{dE}{d\omega} \;=\; \frac{E}{\omega_{max}} \tag{11}$$

so that:

$$\frac{dE}{d\omega} \;=\; \frac{8\, Z^2\, q^6}{3\,\pi\, c^3\, m_e^2\, v^2\, b^2} \tag{12}$$

Since $\omega = 2\pi\nu$, then:

$$\frac{dE}{d\nu} \;= 2\pi \frac{dE}{d\omega} \;=\; \frac{16\, Z^2\, q^6}{3\; m_e^2\, c^3\, v^2\, b^2} \tag{13}$$

To compute the spectrum from an ensemble of particles, we integrate over all possible collisions. We can write that the emissivity, $\epsilon_\nu$ is such that:

$$4\pi\epsilon_\nu \;=\; n_i\, n_e \;<\sigma v>\; \frac{dE}{d\nu} \tag{14}$$

We write that

$$d\sigma \;=\; 2\pi\, b\, db \tag{15}$$

We therefore get that:

$$4\pi\epsilon_\nu \;=\; n_i n_e \int_{b_{min}}^{b_{max}} 2\pi\, b\, db\, v\, \frac{dE}{d\nu} \tag{16}$$

Or:

$$4\pi\epsilon_\nu \;=\; n_i\, n_e \;< \frac{1}{v} >\; \frac{32\pi\, Z^2\, q^6}{3\, m_e^2\, c^3} \ln \frac{b_{max}}{b_{min}} \tag{17}$$

Since the limits on $b$ only enter logarithmically, then do not need to be evaluated too exactly. The usual approach is to take

$$b_{max} \approx \frac{v}{\omega} \tag{18}$$

In the nonrelativistic limit, the usual approach is to adopt a minimum value of $b$ determined by when $\Delta v$ becomes comparable to $v$ and this entire approach breaks down. In this case:

$$b_{min} \;=\; \frac{4\,Z\,q^2}{m\,v^2} \tag{19}$$

Alternatively, we assume that the minimum size is given by the de Broglie wavlength. In this case:

$$b_{min} \;=\; \frac{h}{m\,v} \tag{20}$$

The emissivity is often written as:

$$4\pi\epsilon_\nu \;=\; n_i n_e < \frac{1}{v} > \frac{32\pi^2\,Z^2\,q^6}{3\sqrt{3}m_e^2\,c^3}\,g_{ff} \tag{21}$$

where $g_{ff}$ is the Gaunt factor and:

$$g_{ff} \;=\; \frac{\sqrt{3}}{\pi}\ln\frac{b_{max}}{b_{min}} \tag{22}$$

The normalized Maxwell-Boltzmann distribution of speeds for a single particle is:

$$f(v)\,dv \;=\; 4\pi\,v^2\left(\frac{m_e}{2\pi\,k\,T}\right)^{3/2} e^{-m_e\,v^2/(2\,k\,T)}\,dv \tag{23}$$

In order to emit a photon of energy $h\nu$, the minimum speed, $v_{min}$, is given by:

$$v_{min} \;=\; \left(\frac{2\,h\nu}{m_e}\right)^{1/2} \tag{24}$$

Then

$$4\pi\epsilon_\nu \;=\; n_i\,n_e\frac{128\,\pi^3\,Z^2\,q^6}{3\sqrt{3}\,m_e^2\,c^3}\,g_{ff}\,\left(\frac{m_e}{2\pi k\,T}\right)^{3/2}\int_{v_{min}}^{\infty} v\,e^{-m_e v^2/(2\,k\,T)}\,dv \tag{25}$$

The integral is easy to evaluate with the substitution that $u = (m_e\,v^2)/(2\,k\,T)$, and we get:

$$4\pi\epsilon_\nu \;=\; n_i\,n_e\,\frac{32\sqrt{2}\pi^{3/2}\,Z^2\,q^6}{3\sqrt{3}\,m_e^2\,c^3}\left(\frac{m_e}{k\,T}\right)^{1/2} e^{-h\nu/(kT)}\,g_{ff} \tag{26}$$

Numerically, this expression is

$$\epsilon_\nu \;=\; 5.44\times 10^{-39}\,Z^2\,n_e\,n_i\,T^{-1/2}\,e^{-(h\nu)/(kT)}\,g_{ff}\;\;\mathrm{erg\,cm^{-3}\,s^{-1}\,ster^{-1}\,Hz^{-1}} \tag{27}$$

An important application is the total energy emitted, $\Lambda$, (erg cm$^{-3}$ s$^{-1}$) by the gas:

$$\Lambda \;=\; 4\pi\int_0^\infty \epsilon_\nu\,d\nu \;=\; \left(\frac{2\pi k}{3\,m_e}\right)^{1/2}\frac{32\pi q^6}{3\,h\,m_e\,c^3}\,Z^2\,T^{1/2}\,n_e\,n_i\,\overline{g}_{ff} \tag{28}$$

Also, from Kirchoff's law, we can derive the free-free absorption coefficient:

$$B_\nu(T) \;=\; \frac{\epsilon_\nu}{\kappa_\nu} \tag{29}$$

In the radio regime, we can use the Rayleih-Jeans approximation so that

$$\kappa_\nu \;=\; \frac{\epsilon_\nu\, c^2}{2\,\nu^2\, k\, T} \tag{30}$$

Therefore:

$$\kappa_\nu \;=\; \frac{4\, q^6}{3\, m_e\, k\, c} \left(\frac{2\pi}{3\, k\, m_e}\right)^{1/2} \nu^{-2}\, T^{-3/2}\, Z^2\, n_e\, n_i\, g_{ff} \tag{31}$$

There are a very large number of astrophysical applications of these results. For H II regions, we see that the opacity increases at lower frequencies. Therefore, there is some frequency where we expect the ionized gas to be opaque. The flux, $F_\nu$, from such a region then is determined by the temperature and subtended solid angle, $\Omega$:

$$F_\nu \;=\; \frac{2\nu^2 kT}{c^2}\,\Omega \tag{32}$$

Thus, in this model, we expect that $F_\nu$ varies as $\nu^2$. At sufficiently high frequencies, the H II region is optically thin. Therefore if $L$ is the physical path length through the cloud, we expect that:

$$F_\nu \;=\; \epsilon_\nu\, L\, \Omega \tag{33}$$

In this case, except for the slow frequency variation of $g_{ff}$, $F_\nu$ is independent of frequency.

An important example of considering the entire free-free emission from a gas is when the matter is very hot; typically more than $10^7$ K. The total free-free emission increases with temperature and the line emission can become negligible when the matter becomes nearly completely ionized. The X-ray emission from clusters of galaxies is often largely dominated by free-free emission.

A third example is the radio emission from the winds around hot stars. The mass loss from hot stars is important in their evolution and the return of matter into the interstellar medium. The outflow speed, $V$, can be measured from the shape of their P Cygni lines. Also, the line profiles can be modelled to estimate the mass loss rate. Assume a spherically symmetric mass loss where the density is $n$ and $\dot{N}$ is the mass loss rate in an ion. Then if there is no ionization or recombination, we can write that

$$n \;=\; \frac{\dot{N}}{4\pi r^2 V_*} \tag{34}$$

If $N$ is the column density of the matter in the line of sight derived from the line profile, then

$$N \;=\; \int_{R_*}^{\infty} n\, dr \tag{35}$$

Consequently, we can write that:

$$\dot{N} \;=\; 4\pi\, R_*\, V\, N \tag{36}$$

Another approach to estimating the mass loss rate is less sensitive to the details of the physical conditions in the outflow: this is to observe the free-free emission from the circumstellar envelope. This approach is described in Lamers & Cassinelli. Assume that the circumstellar gas is photoinized and therefore has a characteristic tempearture of $10^4$ K. The opacity varies as $n_e^2 L \nu^{-2}$ and therefore at low enough frequencies, the inner portion of the circumstellar envelope is optically thick while the outer portion is optically thin. Let $b$ denote the impact parameter (as a function of mass loss parameters and frequency) where the gas becomes optically thin. If the source lies at a distance $D$ from us, then from the inner portion of the object, we can write that the observed flux, $F_\nu$, is:

$$F_\nu \;=\; \frac{2\nu^2 k_B T}{c^2}\frac{\pi b^2}{D^2} \tag{37}$$

From the outer portion of the object, we can write that if $R$ denotes the impact parameter of each line of sight that since $n$ varies as $R^{-2}$ and the free-free opacity varies as $n^2$ that the integral along the line of sight which is proportional to $\tau$ gives:

$$\tau \;=\; \left(\frac{b}{R}\right)^3 \tag{38}$$

With the approximation that

$$1 \,-\, e^{-\tau} \approx \tau \tag{39}$$

Then the flux from the outer portion of the envelope is given by the expression:

$$F_\nu \;=\; \frac{2\nu^2 k_B T}{c^2}\int_b^\infty \frac{2\pi R}{D}\,\frac{dR}{D}\left(\frac{b}{R}\right)^3 \tag{40}$$

Therefore

$$F_\nu \;=\; \frac{4\nu^2 k_B T}{c^2}\frac{\pi b^2}{D^2} \tag{41}$$

Thus the outer envelope contributes twice the flux as the inner envelope. If we can compute the flux from the inner envelope, then we can estimate the total mass loss rate from the star. Thus we need to estimate $b$.

We can write that if the line of sight defines the $Z$ axis that

$$\tau \;=\; \int_{-\infty}^{+\infty} \frac{K_{FF}\, n^2}{\nu^2\, T^{3/2}}\, dz \tag{42}$$

where $K_{FF}$ is a "constant" that includes the free-free Gaunt factor as a function of frequency. In *cgs* unis, we can write (see Spitzer 1978, *Physical Processes in the Interstellar Medium*)

$$K_{FF} \;=\; 0.1731 \left(1 \,+\, 0.130 \log \frac{T^{3/2}}{\nu}\right) \tag{43}$$

With

$$n \;=\; \frac{\dot{M}}{4\,\pi\,\mu\,V_*\,R^2} \;=\; \frac{\dot{M}}{4\,\pi\,\mu\,V_*\,(b^2 \,+\, z^2)} \tag{44}$$

Therefore:

$$\tau \;=\; \frac{K_{FF}\dot{M}^2}{16\,\pi^2\,\nu^2\,T^{3/2}\,\mu^2\,V_*^2} \int_{-\infty}^{+\infty} \frac{dz}{(b^2 \,+\, z^2)^2} \;=\; \frac{K_{FF}\dot{M}^2}{16\,\pi^2\,\nu^2\,T^{3/2}\,\mu^2\,V_*^2}\,\frac{2}{b^3} \int_0^{\infty} \frac{dy}{(1 \,+\, y^2)^2} \tag{45}$$

With the substitution that $y = \tan\, u$, then the integral becomes

$$\int_0^{\infty} \frac{dy}{(1 \,+\, y^2)^2} \;=\; \int_0^{\frac{\pi}{2}} \cos^2\, du \;=\; \frac{\pi}{4} \tag{46}$$

If we define $b$ by the requirement that $\tau \,=\, 1$, then:

$$b \;=\; \left(\frac{K_{FF}\dot{M}^2}{32\,\pi\,\nu^2\,T^{3/2}\,\mu^2\,V_*^2}\right)^{1/3} \tag{47}$$

The total radio flux from the circumstellar envelope is predicted to be

$$F_\nu \;=\; \frac{6\,\nu^2\,k_B\,T}{c^2}\,\frac{\pi}{D^2}\left(\frac{K_{FF}\dot{M}^2}{32\,\pi\,\nu^2\,T^{3/2}\,\mu^2\,V_*^2}\right)^{2/3} \tag{48}$$

Thus, the spectrum. $F_\nu$, is predicted to vary as $\nu^{2/3}$. Furthermore, we find that:

$$\dot{M} \;=\; 4\,\left(\frac{1}{3}\right)^{3/4} \frac{F_\nu^{3/4}\,c^{3/2}\,D^{3/2}\,\mu\,V_*}{\pi^{1/2}\,\nu^{1/2}\,k_B^{3/4}\,K_{FF}^{1/2}} \tag{49}$$

Data for some mass losing hot stars are given in Abbott, Bieging & Churchwell (1981, ApJ, 250, 645)

## Lecture 7 – Solid Grains

Small solid particles are pervasive in cool astrophysical environments and even in some hot ones. We therefore want to understand the emission of this material. Typically, the emission occurs in the infrared while absorption is observed from the infrared through X-rays. The optical properties depend upon the grain size, composition and shape, and there are a huge range of possibilities.

First consider spherical grains. The full Mie theory is fairly complicated mathematically, but the idea is straightforward enough. One imagines a plane parallel electromagnetic wave incident upon a sphere. One considers the propagation of the waves produced by matching the boundary conditions of the fields at the surface of the sphere. For grains of radius $a$, we define a dimensionless parameter, $x$:

$$x \;=\; \frac{2\pi a}{\lambda} \tag{1}$$

We write that $Q$ is the ratio of the cross section to $\pi a^2$, the results for $x << 1$ that:

$$Q_{scat} \;=\; \frac{8\,x^4}{3}\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right)^2 \tag{2}$$

and

$$Q_{abs} \;=\; -4\,x\,Im\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right) \tag{3}$$

where $m$ denotes the complex index of refraction of the material in the sphere so that

$$m \;=\; n\,-\,i\,k \tag{4}$$

where $n$ and $k$ and the real and imaginary parts of this index. The meaning of this index of refraction is that if $\vec{k}_w$ denotes the wave vector of a wave propagating in the direction $\vec{r}$ so that:

$$\vec{k}_w \;=\; (n\,-\,i\,k)\,\vec{k}_0 \tag{5}$$

where $|\vec{k}_0| = \omega/c$ and $\vec{k}_0$ points in the direction of the propagation of the wave. The electric field for this wave can be written as:

$$\vec{E} \;=\; \vec{E}_0\,e^{-k\,(\vec{k}_0\cdot\vec{r})}\;e^{i(n\vec{k}_0\cdot\vec{r}\,-\,\omega\,t)} \tag{6}$$

Thus $k$ measures the attenuation of the wave through the medium or true absorption. For $x$ >>1, we expect that both $Q_{abs}$ and $Q_{scat}$ approach 1. Thus, for large grains, it is the surface area that controls the scattering and absorption while for small grains, it is the volume. For small particles the total cross section is:

$$\sigma_{abs} \;=\; Q_{abs}\,\pi\,a^2 \;=\; -\frac{8\,\pi^2\,a^3}{\lambda}\,Im\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right) \tag{7}$$

Since the mass of a grain, $M_{gr}$, is:

$$M_{gr} \;=\; \frac{4\pi}{3}\, a^3\, \rho_s \tag{8}$$

Then:

$$\sigma_{abs} \;=\; -\frac{6\, M_{gr}\, \pi}{\lambda \rho_s}\, Im\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right) \tag{9}$$

After some algebra, one can show that:

$$Im\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right) \;=\; -\frac{6\, n\, k}{(n^2\,-\,k^2\,+\,2)^2\,+\,4\, n^2\, k^2} \tag{10}$$

In a classical theory of conductors, we expect that at long wavelengths that

$$n \approx k \approx \left(\frac{\lambda\, \sigma_0}{c}\right)^{1/2} \tag{11}$$

where $\sigma_0$ denotes the DC conductivity of the material. Therefore, at long wavelengths, we expect that:

$$\sigma_{abs} \propto \lambda^{-2} \tag{12}$$

To compute the emission from grains, we often use the opacity per gram or $\chi_\nu$ which is $\sigma_{abs}/M_{gr}$. Consider emission from an optically thin region, we can write for the observed flux, $F_\nu$ that:

$$F_\nu \;=\; \epsilon_\nu\, L\, \Omega \tag{13}$$

where $L$ denotes the path length through the medium and $\Omega$ denotes the solid angle subtended by the source. If we the object lies at distance from us $D$, then if $A$ is the projected area of the source on the sky:

$$A \;=\; \Omega\, D^2 \tag{14}$$

Using:

$$\epsilon_\nu \;=\; \kappa_\nu\, B_\nu(T) \;=\; \chi_\nu\, \rho\, B_\nu(T) \tag{15}$$

where $\rho$ denotes the space density of the dust. Then we find that:

$$F_\nu \;=\; \chi_\nu\, B_\nu(T)\, \frac{A\, L\, \rho}{D^2} \tag{16}$$

We can write for the mass of the dust, $M_{dust}$:

$$M_{dust} \;=\; \rho\, A\, L \tag{17}$$

Therefore for an optically thin region:

$$M_{dust} \;=\; \frac{F_\nu\, D^2}{\chi_\nu\, B_\nu(t)} \tag{18}$$

At low frequencies, this expression can be approximated as:

$$M_{dust} \approx \frac{F_\nu\, D^2\, \lambda^2}{2\, \chi_\nu\, k\, T} \tag{19}$$

Since we can often infer a temperature of the material from its spectrum, we can determine the mass of dust if we know the opacity.

It is important to calculate the grain temperature. In a steady state, we assume that the particle temperature is achieved by balancing the rate of absorption of energy with the rate of emission. Therefore, if $J_\nu$ is the mean intensity of the radiation, then:

$$\int_0^\infty 4\pi\, J_\nu\, Q_\nu(abs)\, \pi\, a^2\, d\nu \;=\; \int_0^\infty 4\pi\, B_\nu(T) Q_\nu(abs)\, \pi a^2\, d\nu \tag{20}$$

Consider a few simple cases. First, assume a large grain with $Q_\nu(abs) = 1$ for all frequencies of interest in orbit at distance $D$ around a star of radius, $R_*$ and temperature, $T_*$. This equation can be re-written as:

$$\int_0^\infty J_\nu\, d\nu \;=\; \int_0^\infty B_\nu(T)\, d\nu \tag{21}$$

Remembering the definition of the mean intensity, then far from the star:

$$\int_0^\infty J_\nu \;=\; \int_0^\infty \frac{F_\nu}{4\pi} \;=\; \frac{4\pi R_*^2\, \sigma_{SB} T_*^4}{4\pi\, (4\pi\, D^2)} \;=\; \frac{\sigma_{SB} T^4}{\pi} \tag{22}$$

or

$$T \;=\; \frac{T_*}{\sqrt{2}} \left(\frac{R_*}{D}\right)^{1/2} \tag{23}$$

This is the same expression as the mean temperature of a planet of zero albedo.

Often, the grains are relatively small compared to the wavelength at which they emit, and therefore we should not take $Q_\nu$ as constant. For simplicity, one approach is to assume that $Q_\nu$ varies as a power law so that:

$$Q_\nu \;=\; Q_0 \left(\frac{\nu}{\nu_0}\right)^p \tag{24}$$

If the star radiates like a blackbody, then the equation for thermal equilibrium becomes:

$$\int_0^\infty \frac{\nu^p\, 4\pi\, R_*^2\, \pi\, B_\nu(T_*)}{4\pi\, (4\pi D^2)}\, d\nu \;=\; \int_0^\infty \nu^p\, B_\nu(T)\, d\nu \tag{25}$$

We need to evaluate $I$ where :

$$I \;=\; \int_0^\infty \frac{2h\nu^{(3+p)}}{e^{h\nu/kT} \,-\, 1}\, d\nu \tag{26}$$

Using a substitution of variables, we see that:

$$I = const\, T^{(4+p)} \tag{27}$$

Therefore:

$$T = T_* \left( \frac{R_*^2}{4\, D^2} \right)^{1/(4+p)} \tag{28}$$

Therefore, the temperature profile is flatter than for a pure blackbody. The small grains emit inefficiently compared to blackbodies and therefore they are warmer in order to return the absorbed energy into space.

Mass-losing red giants display infrared excesses, and here we describe how one can interpret the available data. Let $\dot{M}_{gr}$ denote the mass loss rate of grains which we assume to be in a steady state with a constant outward radial velocity, $V$. At distance $r$ from the star, the mass density of grains, $\rho_{gr}$ is:

$$\rho_{gr} = \frac{\dot{M}_{gr}}{4\pi r^2 V} \tag{29}$$

Knowing the temperature of the grains, we can compute the luminosity of the source. Assume that the grains have opacity $\chi_\nu$. We may then write that

$$\chi_\nu = \frac{Q_\nu\, \pi\, a^2}{4\pi/3\, \rho_s\, a^3} = \frac{3\, Q_\nu}{4\, \rho_s\, a} \tag{30}$$

Also, we have that:

$$L_\nu = 4\,\pi \int_0^\infty B_\nu(T_{gr}(r))\, (\rho\chi_\nu)\, 4\pi\, r^2\, dr \tag{31}$$

We can then write that:

$$L_\nu = \frac{3\,\pi\, Q_\nu\, \dot{M}_{gr}}{V\, \rho_s\, a} \int_0^\infty B_\nu(T_{gr}(r))\, dr \tag{32}$$

This becomes:

$$L_\nu = \frac{6\,\pi\, Q_\nu\, h\,\nu^3\, \dot{M}_{gr}}{V\, c^2\, \rho_s\, a} \int_0^\infty \frac{dr}{e^{\frac{h\,\nu}{k\,T}} - 1} \tag{33}$$

We define the variable, $x$, such that

$$x = \frac{h\,\nu}{k\, T_*} \left( \frac{4\, r^2}{R_*^2} \right)^{\frac{1}{4+p}} \tag{34}$$

or

$$r = \frac{1}{2} \left( \frac{k\, T_*}{h\,\nu} \right)^{\frac{4+p}{2}} R_*\, x^{\frac{4+p}{2}} \tag{35}$$

Therefore:

$$L_\nu \;=\; \left(\frac{6\,\pi\,Q_\nu\,h\,\nu^3\,\dot{M}_{gr}}{V\,c^2\,\rho_s\,a}\right)\left(\frac{R_*(4\,+\,p)}{4}\right)\left(\frac{k\,T_*}{h\,\nu}\right)^{\frac{4+p}{2}}\left(\int_0^\infty \frac{x^{1+p/2}}{e^x\,-\,1}\,dx\right) \tag{36}$$

Since $Q_\nu$ varies as $\nu^p$, then we see that $L_\nu$ varies as $\nu^{1+p/2}$.

Note that in the small grain limit, we can write (Spitzer 1978, Physical Processes in the Interstellar Medium) that:

$$Q_\nu \;=\; -4\,\frac{2\pi\,a\,\nu}{c}\,Im\left(\frac{m^2\,-\,1}{m^2\,+\,2}\right) \tag{37}$$

where $m$ denotes the index of refraction of the grain material at frequency, $\nu$. If $m$ is independent of $\nu$, then we can re-write the equation for the infrared emission as:

$$L_\nu \;=\; \left(\frac{60\pi^2\,h\nu^4\dot{M}_{gr}\,R_*}{V\,c^3}\right)\left(\frac{k\,T_*}{h\,\nu}\right)^{\frac{5}{2}}\left(\int_0^\infty \frac{x^{\,1.5}}{e^x\,-\,1}\,dx\right)\left(-Im\frac{m^2\,-\,1}{m^2\,+\,2}\right) \tag{38}$$

In this case, $L_\nu$ is independent of the grain size and depends upon the grain composition.

## Lecture 8 – Atomic Spectroscopy

This discussion is mainly focused on emission lines. *Astrophysics of Gaseous Nebulae and Active Galactic Nuclei* by Osterbrock and Ferland is a very useful reference.

To understand the emission line spectrum of a system of atoms, we need to know their ionization state, energy levels, transition probabilities and populations among the different states. In regions of "high" density, we often assume that the time scales are short enough that the system achieves thermal equilibrium and therefore the temperature can be used. In this case, the ionization balance is described by the Saha equation. Remember, however, that even using the Saha equation can be nontrivial because the ionization depends upon the partition functions. We expect that for number densities denoted by $n$, total number $N$ in volume $V$ so that $n = N/V$ and that:

$$n_T(X) = \sum n(X^{+j}) \tag{1}$$

$$\frac{N(X^{+j+1})\,N_e}{N(X^{+j})} = \frac{\zeta_{j+1}\zeta_e}{\zeta_j}\,e^{-I_0/kT} \tag{2}$$

where $I_0$ denotes the ionization potential and $\zeta$ denotes the partition function for each particle. In this formulation, for the electron, when we include its spin:

$$\zeta_e = \frac{2}{h^3}\,(2\pi m_e kT)^{3/2}\;V \tag{3}$$

while $\zeta$ for the ions is more complicated. For hydrogen, for example, when the nuclear spin is included then:

$$\zeta_+ = \frac{2}{h^3}\,(2\pi m_H kT)^{3/2}\;V \tag{4}$$

and for the neutral atom:

$$\zeta_0 = \frac{4}{h^3}\,(2\pi m_H kT)^{3/2}\left(\sum_1^{j_{max}} j^2 e^{-E_j/kt}\right)V \tag{5}$$

where $E_j$ is the $j'th$ energy level with statistical weight $4j^2$. Note that for the hydrogen atom, the sum in the right hand side of this equation diverges unless there is a finite value of $j_{max}$. Typically, this is given by the density of the medium. The third equation we need is that for hydrogen:

$$n_e = n(H^+) \tag{6}$$

Even for hydrogen, the full Saha calculation can become fairly complex when $H_2$ and $H^{-1}$ are included and be particularly important in lower temperature stars. At low temperatures, we can write for a pure hydrogen gas that:

$$\frac{[n(H^+)]^2}{n(H)} = \frac{(2\pi m_e kT)^{3/2}}{h^3}\,e^{-I_0/kT} \tag{7}$$

In lower density regions such as the interstellar medium, the state of the gas is usually very far from thermal equilibrium. We often still approximate the kinetic energy of the atoms and electrons by a Maxwellian distribution, but the states of ionization and the populations of the energy levels can be very far from their equilibrium values. One common approximation is to consider the steady state balance of ionization and equilibrium. In this case, we write the photo-ionization rate, $\Gamma$ ($\mathrm{s}^{-1}$) as:

$$\Gamma \;=\; \int_{\nu_0}^{\infty} 4\pi \frac{J_\nu}{h\nu} \sigma_\nu \, d\nu \tag{8}$$

where $J_\nu$ denotes the mean intensity of the ionization radiation, $\sigma_\nu$ denotes the cross section for photoionization, and $h\nu_0 = I_0$. The steady state condition is that:

$$\Gamma \, n(H) \;=\; n_e \, n(H^+) \, \alpha(T) \tag{9}$$

where $\alpha(T)$ ($\mathrm{cm}^3$ $\mathrm{s}^{-1}$) denotes the rate of radiative recombination and is effectively $< \sigma epv >$, the collision rate coefficient between ions and electrons where the effective cross section is $\sigma_{ep}$.

We can compute the rate of recombination from the rate of photo-ionization. Ignoring stimulated recombination, we can write that:

$$\alpha(T) \;=\; \Gamma \frac{n(H)}{n_e n(H^+)} \tag{10}$$

We evaluate $\Gamma$ for the case where the mean intensity is given by the Planck function and we use the Saha equation to derive the relative fraction of ionized and neutral hydrogen. We therefore find that in the situation where $I_) >> kT$ that

$$\alpha(T) \;=\; (2\pi m_e \, k\, T)^{-3/2} \, e^{I_0/kT} \, (8\pi h^3) \int_{\nu_0}^{\infty} \sigma \left( \frac{\nu^2}{c^2} \right) \, e^{-h\nu/kT} \, d\nu \tag{11}$$

In the simple, but incorrect, approximation that:

$$\sigma_\nu \;=\; \sigma_0 \, \left( \frac{\nu_0}{\nu} \right)^2 \tag{12}$$

then the integral can be evaluated exactly and we find that:

$$\alpha(T) \approx \frac{8\pi \, \sigma_0 \, h^2 \, \nu_0^2}{(2\pi m_e)^{3/2} \, c^2} \left( \frac{1}{kT} \right)^{1/2} \tag{13}$$

With $\sigma_0 = 6 \times 10^{-18}$ $\mathrm{cm}^2$, then $\alpha = 1.6 \times 10^{-13}$ $\mathrm{cm}^3$ $\mathrm{s}^{-1}$ at $T$ = 10,000 K. A more accurate number is about $3 \times 10^{-13}$ $\mathrm{cm}^3$ $\mathrm{s}^{-1}$, but this answer is not too bad. At lower temperatures, the recombination rate is larger.

In the interstellar medium, the mean free path for an ionizing photon can be very short – much less than a parsec. Therefore, it is often a good approximation to assume that around a star or other source of ultraviolet radiation of a "Stromgren Sphere". In this case, the total rate of ionizations from the star, $I_*$ is balanced by the total rate of recombinations. The radius of the sphere, $r_S$, is given implicitly by the equation:

$$\frac{4\pi}{3}\, r_S^3\, n_e^2\, \alpha(T) \;=\; I_* \tag{14}$$

Inside the H II region, the gas temperature is typically near 10,000 K, but the ionization fraction is not determined by the Saha equation.

The emission lines from ionized nebulae include both the recombination lines of hydrogen and helium and the excitation of forbidden lines. The states of ionization of the minor elements such as oxygen are controlled by the balance between photo-ionization and radiative recombination. Of course each element has its own cross sections. Ions such as $O^{+2}$ have low lying energy levels which can be collisionally excited. The levels may be radiatively de-excited and therefore not populated according to thermodynamic equilibrium. Some of the most famous optical lines are O III (from $O^{+2}$) at 5007 Å and 4959 Å.

$O^{+2}$ has 6 electrons whose hydrogenic configuration is $1\, s^2\, 2s^2\, 2p^2$. The open shell $2p$ electrons can be configured in different ways. The lowest level is $^3P$ with fine structure levels of $^3P_2$, $^3P_1$ and $^3P_0$. In the usual notation that the superscript refers to $2S + 1$ where $S$ is the total spin, the letter refers to the total orbital angular momentum and the subscript refers to the total angular momentum, $J$. The upper electronic level is $^1D$ with no fine structure. The transition 5007 Å corresponds to $^1D$ to $^3P_2$ and 4959 Å corresponds to $^1D$ to $^3P_1$. The transition $^1D$ to $^3P_0$ occurs at 4932 Å but is so highly forbidden that it is rarely seen. Since 5007 and 4959 result from the same upper level, the intensity ratio is controlled only by the relative values of the Einstein $A's$ and is therefore expected to be 3.

The $O^{+2}$ ion also has a $^1S$ level which lies above the $D^1$ level. The wavelength of the transition between $^1S$ and $^1D$ is 4363 Å. This line is often used to infer the temperature of the gas since the relative rate of excitation into the $^1S$ level relative to the rate of excitation into the $^1D$ level is sensitive to the gas temperature.

There are also density diagnostics in the gas. A particular set is O II ($O^+$) at 3726 Å and 3729 Å. The transitions are $^2D_{5/2}$ to the ground state $^4S_{3/2}$ at 3729 Å and$^2D_{3/2}$ to the ground state at $^4S_{3/2}$ at 3726 A. At low densities, the emission is just given by the ratio of collisional excitations into the upper levels which just depends upon the ratio of the statistical weights. Thus the 3729 AA line is expected to be 1.5 times stronger than the 3726 Å line in the low density limit. In the high density limit, the ratio of the line strengths depends upon the number in the level multiplied by the Einstein A. The Einstein A from the 3729 Å line is about 0.23 times as strong as that for the 3726 Å line and therefore the expected line ratio in the high density limit is = 0.34.

Emission line strengths have been used to infer elemental abundances within ionized nebulae. The optical line intensities are sensitive to the gas temperature which is characteristically near 10,000 K. While the infrared lines are quite insensitive to gas temperature, they can be sensitive to the gas density. For example, the O III transitions at 88 $\mu$m from the lowest ground state ($^3$P$_1$ to $^3$P$_0$ and 52 $\mu$m from the two excited fine structure levels ($^3$P$_2$ to $^3$P$_1$) are good measures of the oxygen abundance if the density is low enough to ignore collisional de-excitation of the upper level.

The prime source of heating within a nebula is from photo-ionization. We can write that the rate of heating, $\Lambda$ (erg s$^{-1}$) is:

$$\Lambda \;=\; \int_{\nu_0}^{\infty} 4\pi \frac{J_\nu}{h\nu}\, \sigma_\nu \,(h\nu \;-\; h\nu_0)\; d\nu \tag{15}$$

Thus the thermal balance is determined by writing:

$$\Lambda\, n(H) \;=\; \sum n_e\, n(X^{+i})\,(\Delta E)\;<\sigma_{eX} v> \tag{16}$$

where we assume that ion $X^{+i}$ is collisionally excited into an energy level that is $\Delta E$ above the ground state which is then radiated as photons. The coefficient for the collision rate is given as $< \sigma_{eX} v >$. Note that the heating rate is suite similar to the ionization rate which is proportional to the recombination rate. Therefore, the heating varies approximately as $n^2$ as does the cooling. Consequently, the temperature is not sensitive to density. However, the value of $< \sigma v >$ typically is temperature sensitive. The main coolants from an ionized nebula lie at energy levels tyically much higher than $kT$. Consequently, relatively few electrons have enough energy to excite an ion into an excited state and the fraction of such electrons is temperature-sensitive. A result is that a very wide variety of ionized nebulae have “characteristic” temperatures near 10,000 K.

## Lecture 10 – Pulsars

Pulsars are fascinating objects and they probe physics at the extreme. There are compelling arguments that radio pulsars are magnetized, rotating neutron stars. Assume that the magnetic field of the neutron star can be described as that from a magnetic dipole, $\vec{m}$, that is oriented at angle $\alpha$ relative to the spin axis. Outside of the star there is a time-varying magnetic field, and we can calculate the radiation from this field as a time-varying magnetic dipole. Previously, we considered electric dipole radiation. Note that if we imagine a current, $I$, in a loop of cross sectional area, $a$, then

$$\vec{m} \;=\; \frac{I\,a\,\hat{n}}{c} \tag{1}$$

where $\hat{n}$ is a unit vector normal to the surface area of the loop and $c$ is the speed of light.

In cgs units, we can write for a unit vector, $\hat{r}$ in the direction $\vec{r}$ that:

$$\vec{B} \;=\; \frac{3\hat{r}(\hat{r}\cdot\vec{m}) \;-\; \vec{m}}{|\vec{r}|^3} \tag{2}$$

Alternatively, if the dipole is oriented along the $Z$ axis, so that

$$\vec{m} \;=\; m_0\,\hat{z} \tag{3}$$

with $m_0 = (I\,a)/c$, and using:

$$\hat{z} \;=\; \hat{r}\cos\theta \;-\; \hat{\theta}\sin\theta \tag{4}$$

then in spherical coordinates, we can write that

$$\vec{B} \;=\; \frac{m_0}{|\vec{r}|^3}\left(2\cos\theta\,\hat{r} \;+ \sin\theta\,\hat{\theta}\right) \tag{5}$$

Thus for a star of radius, $R_*$, the maximum magnetic field, $B_0$ is given by:

$$B_0 \;=\; \frac{2\,m_0}{R_*^3} \tag{6}$$

We can calculate the radiation from a magnetic dipole in a manner analogous to that of the electric dipole. However, we need only compute the retarded vector potential, $\vec{A}$ since the scaler potential is zero for a magnetic dipole. Skipping the details, for now, the instantaneous radiated power, $P$, is:

$$P \;=\; \frac{2|\ddot{\vec{m}}|^2}{3\,c^3} \tag{7}$$

The projection of the magnetic dipole, $m_0 \cos\alpha$ along the rotation axis is constant while the magnitude of the time-varying portion of the magnetic dipole is $m_0 \sin\alpha$. If the star rotates with angular velocity $\omega$, then:

$$|\ddot{\vec{m}}| \; = \; \omega^2 \, m_0 \, \sin\alpha \tag{8}$$

Therefore, we can write for the total power radiated that:

$$P \; = \; \frac{\omega^4 \, B_0^2 \, R_*^6 \sin^2\alpha}{6 \, c^3} \tag{9}$$

If $I$ denotes the moment of inertia of the star, then if homogeneous, we can write that:

$$I \; = \; \frac{2}{5} \, M_* \, R_*^2 \tag{10}$$

Consequently, the rotational energy, $E_{rot}$, is

$$E_{rot} \; = \; \frac{1}{2} \, I \, \omega^2 \; = \; \frac{1}{5} \, M_* \, R_*^2 \, \omega^2 \tag{11}$$

By considering the period of the pulsar, $T$, so that $\omega = (2\pi)/T$, then if the spin down of the pulsar radiates into free space, we expect that:

$$P \; = \; \frac{8 \, \pi^4 \, B_0^2 \, R_*^6 \, \sin^2\alpha}{3 \, c^3 \, T^4} \; = \; -\frac{dE_{rot}}{dt} \tag{12}$$

Also:

$$-\frac{dE_{rot}}{dt} \; = \; -\frac{8\pi^2 \, M_* \, R_*^2}{5 \, T^3} \frac{dT}{dt} \tag{13}$$

Thus

$$\frac{dT}{dt} \; = \; \frac{1}{T} \, \frac{5 \, \pi^2 \, B_0^2 \, R_*^4 \sin^2\alpha}{3 \, c^3 \, M_*} \tag{14}$$

It is possible to make very exact measurements of a pulsar's value of $T$ and $dT/dt$ and therefore, assuming values for $M_*$, $R_*$ and $\alpha$, it is possible to estimate $B_0$. Typical values are $B_0 \approx 10^{12}$ Gauss.

This theory of pulsar spin-down makes an exact prediction. We can write that

$$\dot{\omega} = \; K \, \omega^3 \tag{15}$$

Thus:

$$\ddot{\omega} \; = \; 3 \, K \, \omega^2 \, \dot{\omega} \tag{16}$$

Therefore:

$$\ddot{\omega} \; = \; \frac{n \, \dot{\omega}^2}{\omega} \tag{17}$$

where $n$, the "braking index", is predicted to equal 3. Observationally, about 5 pulsars have had accurate measurements made of the braking index, and it is typically less than 3. The idealized model that we have discussed does not work perfectly.

Radio pulsars appear to be powered by their spin-down. However, there exist the "anomalous X-ray pulsars" which emit about 100 times as much power as allowed from the spin down of a neutron star. These objects are now thought to be "magnetars"; pulsars with extremely large magnetic fields, and the energy of the systems is probably derived from dissipation of this magnetic energy.

Observations of pulsars show that the pulses at low frequencies arrive later than the pulses at high frequencies. This can be explained as a plasma dispersion effect. Assume a plane parallel wave propagating through an ionized medium. We assume that the electric field in the wave is of the form:

$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{18}$$

where:

$$\vec{E}_0 = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} \tag{19}$$

with $E_x$, $E_y$ and $E_z$ being constants. We describe the wave vector as $\vec{k}$ so that

$$\vec{k}\cdot\vec{r} = k_x \, x + k_y \, y + k_z \, z \tag{20}$$

where, again, $k_x$, $k_y$ and $k_z$ are constants. With this approach, then we can write that:

$$\vec{\nabla}\cdot\vec{E} = (i \, k_x \, E_x + i \, k_y \, E_y + i \, k_z \, E_z) \; e^{i(\vec{k}\cdot\vec{r} - \omega t)} = i \, \vec{k}\cdot\vec{E} \tag{21}$$

Similarly, with a little algebra, we find that:

$$\vec{\nabla}\times\vec{E} = i\vec{k}\times\vec{E} \tag{22}$$

and

$$\frac{\partial \vec{E}}{\partial t} = -i\omega\vec{E} \tag{23}$$

With this approach, Maxwell's differential equations can be reconfigured as algebraic equations. For the electric field, we can write that the equation relating the field to the local density:

$$\vec{\nabla}\cdot\vec{E} = 4\pi \, \rho \tag{24}$$

becomes:

$$i\vec{k}\cdot\vec{E} = 4\pi\rho \tag{25}$$

where $\rho$ is the charge density. Similarly,

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{26}$$

becomes:

$$\vec{k}\times\vec{E} \;=\; \frac{\omega}{c}\vec{B} \tag{27}$$

while for the magnetic field, we write that:

$$i\vec{k}\cdot\vec{B} \;=\; 0 \tag{28}$$

and

$$i\vec{k}\times\vec{B} \;=\; \frac{4\pi}{c}\vec{j} - i\frac{\omega}{c}\vec{E} \tag{29}$$

where $\vec{j}$ is the current density.

We now make consider a very simple case where instead of a vacuum, there are charges. We assume that only the electrons move and that their motion is dominated by the electric field since they are assumed to be at rest except for the field. In this case:

$$m_e\dot{\vec{v}} \;=\; -q\vec{E} \tag{30}$$

Assuming that the electron just oscillates as the electric field, then

$$\dot{\vec{v}} \;=\; -i\,\omega\,\vec{v} \tag{31}$$

so that

$$\vec{v} \;=\; \frac{q\vec{E}}{i\omega m_e} \tag{32}$$

We can write for the current density that:

$$\vec{j} \;=\; -q\,n_e\,\vec{v} \tag{33}$$

Therefore:

$$\vec{j} \;=\; \sigma\vec{E} \tag{34}$$

where the conductivity, $\sigma$, is given by:

$$\sigma \;=\; \frac{i\,n_e\,q^2}{\omega\,m_e} \tag{35}$$

The conservation of charge for a flow is:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} \;=\; 0 \tag{36}$$

In our expressions, this becomes:

$$-i\omega\rho + i\vec{k}\cdot\vec{j} \;=\; 0 \tag{37}$$

Therefore:

$$\rho \;=\; \frac{1}{\omega}\vec{k}\cdot\vec{j} \;=\; \frac{\sigma}{\omega}\vec{k}\cdot\vec{E} \tag{38}$$

We can now write Maxwell's equations with the charge density and current density terms as:

$$i\,\vec{k}\cdot\vec{E}\left(1 \;-\; \frac{4\pi\sigma}{i\omega}\right) \;=\; 0 \tag{39}$$

and:

$$i\vec{k}\times\vec{B} \;=\; -i\frac{\omega}{c}\vec{E}\left(1 \;-\; \frac{4\pi\sigma}{i\omega}\right) \tag{40}$$

and

$$i\,\vec{k}\cdot\vec{B} \;=\; 0 \tag{41}$$

and

$$i\vec{k}\times\vec{E} \;=\; i\frac{\omega}{c}\vec{B} \tag{42}$$

We have therefore re-arranged Maxwell's equations for a plasma to the equivalent of the equations in a vacuum if we define the complex dielectric constant, $\epsilon$, as

$$\epsilon \;=\; \left(1 \;-\; \frac{4\pi n_e q^2}{m_e\omega^2}\right) \;=\; \left(1 \;-\; \frac{\omega_p^2}{\omega^2}\right) \tag{43}$$

which implicitly defines the plasma frequency, $\omega_p$ such that:

$$\omega_p^2 \;=\; \frac{4\pi\, n_e\, q^2}{m_e} \tag{44}$$

If we assume that $\vec{k}$ propagates in the $Z$ direction while the electric field is in the $X$ direction and the magnetic field in the $Y$ direction, then we find for the solution to Maxwell's equations for the wave that:

$$c^2\,k^2 \;=\; \epsilon\,\omega^2 \tag{45}$$

Using the definition of $\epsilon$, this equation becomes:

$$k \;=\; \frac{\sqrt{\omega^2 \;-\; \omega_p^2}}{c} \tag{46}$$

The value of $k$ is imaginary and the wave is attenuated if $\omega < \omega_p$. Alternatively, we can write that:

$$\omega^2 \;=\; \omega_p^2 \;+\; c^2\,k^2 \tag{47}$$

For any wave, the group velocity, $v_g$ is given by:

$$v_g \; = \; \frac{\partial \omega}{\partial k} \approx c\sqrt{1 \, - \, \frac{\omega_p^2}{\omega^2}} \tag{48}$$

As a result, the waves propagate more slowly at the lower frequencies. If the pulsar is at distance $s$ from us, then the time the pulse, $t_p$, takes to reach us is:

$$t_p \; = \; \int \frac{ds}{v_g} \approx \frac{s}{c}\left(1 \, + \, \int \frac{\omega_p^2}{2\,\omega^2} \, ds\right) \tag{49}$$

Using the definition of the plasma frequency, we see that the relative delays in the pulsar pulse arrival as a function of time directly measure the dispersion measure, $D$, defined as:

$$D \; = \; \int n_e \, ds \tag{50}$$

## Lecture 11 – Synchrotron Radiation

There are many astrophysical environments with relativistic electrons in a magnetic field. The synchrotron emission from these systems can be very powerful and is the most common form of "non-thermal" emission that astronomers consider.

Assume a relativistic electron with charge, $q$, mass, $m_e$, and relativistic factor, $\gamma$, such that

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1}$$

If there is magnetic field, $\vec{B}$ and no electric field, then the energy of the particle is constant and:

$$\frac{d}{dt}(\gamma\, m_e\, \vec{v}) = \frac{q}{c}\,\vec{v}\times\vec{B} \tag{2}$$

Since the energy is constant, then $\gamma$ is constant and the equation becomes similar to the non-relativistic case:

$$\gamma\, m_e \frac{d\vec{v}}{dt} = \frac{q}{c}\,\vec{v}\times\vec{B} \tag{3}$$

We consider the components of $\vec{v}$ that are perpendicular, $\vec{v}_\perp$ and parallel, $\vec{v}_{||}$ to the magnetic field so that:

$$\vec{v} = \vec{v}_\perp + \vec{v}_{||} \tag{4}$$

The solution to the equation of motion is that $\vec{v}_{||}$ is constant while $\vec{v}_\perp$ undergoes circular motion. If we assume that $\vec{B}$ defines the $Z$ axis so that $\vec{B}$ is $B_0\hat{z}$ and write

$$\vec{v}_\perp = v_x\,\hat{x} + v_y\,\hat{y} \tag{5}$$

Then the equation of motion for the $X$ and $Y$ components of the velocity can be written as:

$$\gamma\, m_e \frac{dv_x}{dt} = \frac{v_y}{c}\, q\, B_0 \tag{6}$$

and

$$\gamma\, m_e \frac{dv_y}{dt} = -\frac{v_x}{c}\, q\, B_0 \tag{7}$$

Combining these equations, we get:

$$\frac{d^2 v_x}{dt^2} = \frac{q\, B_0}{\gamma\, m_e\, c}\,\frac{dv_y}{dt} = -\frac{q^2\, B_0^2}{\gamma^2\, m_e^2\, c^2}\, v_x \tag{8}$$

The solution to this equation is that

$$v_x = v_0 \cos(\omega_B\, t) \tag{9}$$

where:

$$\omega_B = \frac{q\, B_0}{\gamma\, m_e\, c} \tag{10}$$

and $v_0$ is a constant that is given by the initial conditions. This result is the same as for the nonrelativistic case if $\gamma = 1$. The motion of the electron is a helix. The "pitch angle" is the angle between the magnetic field and the velocity vector of the electron. For constant magnetic field, the pitch angle also is constant.

Since the electron is accelerated, it radiates. The relativistic generalization of Larmor's formula is that:

$$P \;=\; \frac{2\,q^2}{3\,c^3}\,\gamma^4\left(a_\perp^2\,+\,\gamma^2 a_\parallel^2\right) \tag{11}$$

where the $a's$ denote the components of the electron's acceleration that are parallel and perpendicular to its velocity. We take $a_\parallel = 0$ and:

$$a_\perp \;=\; \omega_B\, v_\perp \tag{12}$$

Therefore:

$$P \;=\; \frac{2\,q^4 B_0^2}{3\,m_e^2\,c^5}\,\gamma^2\,v_\perp^2 \tag{13}$$

We can consider some approximations. For example, in an ensemble of relativistic electrons, we might imagine that the distribution of pitch angles is uniform, Therefore, if $\theta$ denotes the angle between $v_\perp$ and $v_0$, then

$$< v_\perp^2 > = \; v_0^2\,\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\sin^2\theta)\,\sin\theta\,d\theta\,d\phi \;=\; \frac{2}{3}\,v_0^2 \tag{14}$$

For highly relativistic particles where $v_0 \approx c$, then:

$$P \approx \frac{4\,q^4\,B_0^2}{9\,m_e^2\,c^3}\,\gamma^2 \tag{15}$$

Since the energy of an electron is $\gamma m_e c^2$, then the characteristic time, $t_{syn}$, for an electron to lose its energy by synchrotron radiation is:

$$t_{syn} \;=\; \frac{\gamma\,m_e\,c^2}{P} \;\approx\; \frac{9\,m_e^3\,c^5}{4\,q^4\,B_0^2\,\gamma} \tag{16}$$

In addition to the total power emitted by a synchrotron electron, we want to describe its spectrum. Following the arguments in Rybicki and Lightman, we note the following. The emission from a relativistic electron is beamed into a cone of opening angle $\gamma^{-1}$. We assume that the pulse of light is only strong for a fraction of the orbital motion given by the time interval in the frame of the electron:

$$\Delta t \approx \frac{2}{\gamma\omega_B\sin\alpha} \tag{17}$$

where $\alpha$ is the angle between the magnetic field and the line of sight. Because of this, the pulse of light from each motion of the electron in the magnetic field is much shorter than the gyration time. It is useful to note that if we assume $v$ is nearly $c$, then we can write that the arrival time of the pulse, $\Delta t^A$ is "shrunk" by the interval

$$\Delta t^A \;=\; \Delta t \left(1 \,-\, \frac{v}{c}\right) \approx \frac{\Delta t}{2\gamma^2} \tag{18}$$

Therefore, the pulse arrives over a time interval:

$$\Delta t^A \approx \frac{1}{\gamma^3 \omega_B \sin\alpha} \tag{19}$$

From Fourier analysis, the upper cutoff frequency of the radiation, $\omega_c$ is:

$$\omega_c \approx (\Delta t^A)^{-1} \;=\; \gamma^3\, \omega_B \sin\alpha \tag{20}$$

We expect that $\nu_c = \omega_c/(2\pi)$. Because $\omega_B$ varies as $\gamma^{-1}$, we expect that $\nu_c$ varies as $\gamma^2$. Therefore, for a relativistic synchrotron electron, both the total power and the characteristic maximum frequency varies as $\gamma^2$ so that the power emitted per unit frequency, $P_\nu$, scales as a function $G(\nu/\nu_c)$.

Consider an observational case where the energy distribution of electrons, $N(E)\, dE$, is a power law such that:

$$N(E)\, dE \;=\; N_0\, E^{-p}\, dE \tag{21}$$

We can write for the total power emitted by this ensemble, $P_\nu(tot)$, that:

$$P_\nu(tot) \;=\; \int_{E_{min}}^{E_{max}} N_0\, E^{-p}\, G\left(\frac{\nu}{\nu_c}\right) dE \tag{22}$$

Make the substitution that $x = \nu/\nu_c$ and recognize that $\nu_c = K\, E^2$. Therefore, we can re-write this equation so that:

$$x \;=\; \frac{\nu}{K\, E^2} \tag{23}$$

so:

$$E \;=\; \left(\frac{\nu}{K\, x}\right)^{1/2} \tag{24}$$

and

$$dE \;=\; -\frac{1}{2}\left(\frac{\nu}{K}\right)^{1/2} x^{-3/2}\, dx \tag{25}$$

and

$$P_\nu(tot) \;=\; \nu^{(-p/2\,+\,1/2)} \int_{x_{min}}^{x_{max}} \frac{N_0}{2}\, x^{(p/2\,-\,3/2)}\, K^{(p/2\,-\,1/2)}\, G(x)\, dx \tag{26}$$

Therefore, if

$$P_\nu \propto \nu^{-s} \tag{27}$$

then:

$$s = \frac{p - 1}{2} \tag{28}$$

For many nonthermal radio sources, we do not independently know the magnetic field and the energy distribution of the electrons. One approach is to assume equipartition so that the energy density in the magnetic field, $B^2/(8\pi)$, equals the energy density in relativistic particles. However, this is highly uncertain. Nonthermal radio sources can be distinguished from thermal sources by (1) their spectral energy distribution, (2) their surface brightness, (3) their variability and (4) their polarization.

**Lecture 12 – Scattering**

So far, we have only considered direct emission as a photon source function. In fact, however scattering also contributes to the source function. The simplest case is isotropic scattering. For a flow of photons along the $X$-axis, through a medium with particle density, $n$, we can write that the optical depth, $d\tau$ is:

$$d\tau \;=\; n \,(\sigma_{scat} \,+\, \sigma_{abs})\; dx \tag{1}$$

$$\frac{dI}{dx} \;=\; -n\,(\sigma_{abs} \,+\, \sigma_{scat})\; I \,+\, \epsilon \,+\, n\,\sigma_{scat}\, J \tag{2}$$

where $J$ is the mean intensity of the radiation field so that

$$J \;=\; \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} I(\theta,\phi)\,\sin\theta\, d\theta\, d\phi \tag{3}$$

We can therefore re-write the equation of transfer as:

$$\frac{dI}{d\tau} \;=\; -I \,+\, (1\,-\,a)\, S_{em} \,+\, a\, J \tag{4}$$

where $a$ is the albedo of the particles defined so that:

$$a \;=\; \frac{\sigma_{scat}}{\sigma_{scat} \,+\, \sigma_{abs}} \tag{5}$$

and $S_{em}$ is the usual source function for emission so that

$$S_{em} \;=\; \frac{\epsilon}{n\,\sigma_{abs}} \tag{6}$$

In the common case that $S_{em} = B$, the Planck function, then

$$\frac{dI}{d\tau} \;=\; -I \,+\, (1\,-\,a)\, B \,+\, a\, J \;=\; -I \,+\, S \tag{7}$$

where $S$ is the generalized source function that includes scattering.

Consider a few simple examples. If $a = 1$ and the system is optically thin and $S$ is constant, then the intensity that we observe, $I_{obs}$ is:

$$I_{obs} \;=\; S\left(1 \,-\, e^{-\tau}\right) \,\approx S\,\tau \tag{8}$$

On a cloudless day, we can approximate the mean intensity of the light from the Sun that has luminosity, $L$, and distance from the Earth, $D$, as:

$$J \approx \frac{1}{(4\pi)^2}\,\frac{L}{D^2} \tag{9}$$

Thus:

$$I_{obs} \approx J\,\tau \approx \frac{\tau\,L}{16\pi^2\,D^2} \tag{10}$$

Note that we expect that $I_{obs} < J$.

As an example, model the sky as a plane parallel atmosphere with vertical scattering optical depth, $\tau$. If $\theta$ denotes the angle with respect to the zenith and if $\mu = \cos\,\theta$, then:

$$I(\theta) \;=\; J\left(1 \,-\, e^{-\tau/\mu}\right) \tag{11}$$

The flux received by a detector aimed at the zenith, $F$, is

$$F \;=\; \int_0^{2\pi}\int_0^{\pi/2} J\left(1 \,-\, e^{-\tau/\mu}\right)\cos\theta\sin\theta\,d\theta\,d\phi \tag{12}$$

Or:

$$F \;=\; \pi\,J \,-\, 2\,\pi\,J\int_0^{\pi/2} e^{-\tau/\mu}\cos\theta\sin\theta\,d\theta \;=\; \pi\,J \,-\, 2\,\pi\,J\int_1^{\infty} e^{-\tau x}\frac{dx}{x^3} \tag{13}$$

where $x = 1/\cos\,\theta$. Therefore:

$$F \;=\; \pi\,J \,-\, 2\,\pi\,J\,E_3(\tau) \tag{14}$$

This compares with the direct flux from the Sun, $F_*$, which is assumed to lie at zenith angle $\theta_*$ so that

$$F_* \;=\; 4\pi\,J\,e^{-\tau/\mu_*} \tag{15}$$

If, for example, $\tau = 0.1$ and $\theta_* = 45°$, then $F/F_* = 0.048$. Most of the flux we could detect would come directly from the Sun.

Consider a plane parallel atmosphere where $z$ and $\tau$ are measured downward from the top. In such an atmosphere, we can write that

$$dx \;=\; -\frac{dz}{\cos\theta} \;=\; -\frac{dz}{\mu} \tag{16}$$

Consequently, along each direction defined by $\theta$, the equation of transfer becomes:

$$\mu\frac{dI}{d\tau} \;=\; I \,-\,(1 \,-\, a)\,B \,-\, a\,J \tag{17}$$

In the two-stream approximation, we only consider light that moves in the directions $\mu = \pm 1$. We define $I_+$ as the upward intensity and $I_-$ as the downward intensity. We can write that

$$J \;=\; \frac{1}{2}\,(I_+ \,+\, I_-) \tag{18}$$

Each stream of radiation must satisfy the transfer equation so that

$$\frac{dI_+}{d\tau} \;=\; I_+ \,-\,(1 \,-\, a)\,B \,-\, a\,J \tag{19}$$

and

$$\frac{dI_-}{d\tau} = -I_- + (1 - a)\,B + a\,J \tag{20}$$

It is useful to define the flux, $H$, such that

$$H = \frac{(I_+ - I_-)}{2} \tag{21}$$

By summing the equations, we find that:

$$\frac{dI_+}{d\tau} + \frac{dI_-}{d\tau} = 2\,\frac{dJ}{d\tau} = 2\,H \tag{22}$$

Thus:

$$\frac{dJ}{d\tau} = H \tag{23}$$

Subtracting the two equations, we find that:

$$\frac{dI_+}{d\tau} - \frac{dI_-}{d\tau} = 2\,\frac{dH}{d\tau} = 2\,J - 2(1 - a)\,B - 2\,a\,J \tag{24}$$

Thus:

$$\frac{dH}{d\tau} = J(1 - a) - (1 - a)\,B \tag{25}$$

We can eliminate $H$ by writing:

$$\frac{d^2J}{d\tau^2} = \frac{dH}{d\tau} = J(1 - a) - (1 - a)\,B \tag{26}$$

If, for simplicity, $B$ is constant, then the solution to the differential equation is:

$$J = C_1\,e^{\sqrt{1-a}\tau} + C_2\,e^{-\sqrt{1-a}\tau} + B \tag{27}$$

where $C_1$ and $C_2$ are constants to be determined by the boundary conditions. In order to keep the mean intensity well bounded within the atmosphere, we take $C_1 = 0$. We find from our solution that:

$$\frac{dJ}{d\tau} = H = -\sqrt{1-a}\,C_2\,e^{-\sqrt{1-a}\tau} \tag{28}$$

At the surface of the atmosphere where $\tau = 0$, $I_- = 0$ because there is no incident radiation. Therefore

$$I_-(0) = 0 = J(0) - H(0) \tag{29}$$

Therefore:

$$C_2 + B = -\sqrt{1-a}\,C_2 \tag{30}$$

Therefore:

$$C_2 = -\frac{B}{1 + \sqrt{1-a}} \tag{31}$$

Then:

$$J = B - \frac{B}{1 + \sqrt{1-a}} e^{-\sqrt{1-a}\tau} \tag{32}$$

and

$$H = \frac{\sqrt{1-a}}{1 + \sqrt{1-a}} e^{-\sqrt{1-a}\tau} B \tag{33}$$

The intensity in the upwards direction, $I_+$ is:

$$I_+ = J + H = B - B\, e^{-\sqrt{1-a}\tau} \left( \frac{1 - \sqrt{1-a}}{1 + \sqrt{1-a}} \right) \tag{34}$$

There are a number of interesting results to be derived from this approximation. For example, deep in the atmosphere, we see that $J$ approaches $B$ which means that the mean intensity approaches the thermal source function. Also, deep in the atmosphere, we see that $H = 0$. This is a consequence of assuming that $B$ is constant. If there is a net flux, then $B$ cannot be constant. Consider $I_+$ at $\tau = 0$. If $a = 0$, then $I_+(0) = B$; the emergent intensity just equals the source function for an isothermal atmosphere. Another important result is that when $a \approx 1$, then $I_+(0) \approx 0$. This is important for understanding very strong "scattering" lines in stellar atmospheres. An example are the calcium H and K lines in the spectrum of the Sun where the residual intensity in the line is quite small compared to the continuum.

## Lecture 13 – Non-LTE Line Formation

The appearance of a spectral line formation depends upon the source function. In Local Thermodynamic Equilibrium, LTE, we take $S_\nu = B_\nu(T)$. In many environments this approximation works very well. However, there are often circumstances where the system is not well-described by LTE. In these cases, we need to consider the rates which control the level populations in order to determine the source function which then enters into the equation of transfer.

In a two level atom with lower and upper energy levels separated by energy $\Delta E$ with statistical weights $g_L$ and $g_U$, Einstein coefficient, $A_{UL}$, define collisional rate coefficient $C_{LU}$ for $L$ to $U$ and coefficient $C_{UL}$ for collisions from $U$ to $L$. We know that in a steady that that if $n$ is the density of colliders that:

$$n\,n_L\,C_{LU} \;=\; n\,n_U\,C_{UL} \tag{1}$$

Furthermore, we know that in LTE that:

$$\frac{n_U}{n_L} \;=\; \frac{g_U}{g_L}\,e^{-\Delta E/kT} \tag{2}$$

Therefore:

$$C_{LU} \;=\; C_{UL}\,\frac{g_U}{g_L}\,e^{-\Delta E/kT} \tag{3}$$

We expect that:

$$C_{UL} \;= <\sigma v> \tag{4}$$

where $\sigma$ is the cross section for de-excitation and $v$ is the speed of a collider. In principle, we should take $v$ in the center of mass of the atom and the collider. Often, however, the colliders are electrons which are much lighter than the atoms that are being excited and this correction is not necessary.

For an atom within a radiation field, we can write in a steady state that:

$$n_L\left(n\,C_{LU} \,+\, B_{LU}\,\overline{J}\right) \;=\; n_U\left(n\,C_{UL} \,+\, B_{UL}\,\overline{J} \,+\, A_{UL}\right) \tag{5}$$

where

$$\overline{J} \;=\; 4\pi\int_{-\infty}^{+\infty} J_\nu(\Delta\nu)\,\phi(\Delta\nu)\,d(\Delta\nu) \tag{6}$$

When the value of $n$ is high, then the system approaches LTE. However, when $n$ is low, the other terms in this expression become important and the level populations can deviate quite significantly from their LTE values.

Now consider situations where we investigate "high" and "low". In a star, we typically observe to continuum optical depth, $\tau_c$, of 2/3. We also have from the equation of hydrostatic equilibrium that if $z$ is measured outwards:

$$\frac{dp}{dz} \;=\; -\rho\,g \tag{7}$$

Remembering that if $\tau$ is measured inwards that:

$$d\tau \;=\; -\chi\,\rho\,dz \tag{8}$$

Then:

$$\frac{dp}{d\tau} \;=\; \frac{g}{\chi} \tag{9}$$

Although $\chi$ is not constant, in the region where the continuum is formed, we expect that:

$$p \approx \frac{2}{3}\,\frac{g}{\chi} \tag{10}$$

Using the ideal gas law so that $p = n\,k\,T$, we find that very approximately, the density where the continuum is formed, $n_{cr}$, is therefore:

$$n_{cr} \approx \frac{g}{k\,T\;\chi} \tag{11}$$

In this formulation, typically, the largest uncertainty is the opacity which is a strong function of temperature, density and composition. If electron scattering dominates and hydrogen is the main constituent, then $\chi = 0.40$ cm$^2$ g$^{-1}$. However, in the Sun, most of the atoms are neutral, and the main source of opacity is $H^-$ by the process:

$$h\nu \,+\, H^- \rightarrow H \,+\, e^- \tag{12}$$

The energy threshold for this process is only 0.75 eV and the cross section can be as high as 4 $\times$ 10$^{-17}$ cm$^2$ at 8500 Å. However, only a tiny fraction of the hydrogen is in the form of H$^-$ and the opacity is typically 0.01 to 1 cm$^2$ g$^{-1}$. Taking the acceleration by gravity at the sun of $g$ = 2.7 $\times$ 10$^4$ cm s$^{-2}$, $T$ = 7000 K and $\chi$ = 1 cm$^2$ g$^{-1}$, then $n_{cr} \sim 2 \times 10^{16}$ cm$^{-3}$. For comparison, the number density of molecules in the Earth's atmosphere at its surface typically is 2.7 $\times$ 10$^{19}$ cm$^{-3}$. For collisional de-excitation of an atom, the characteristic cross section might be 10$^{-16}$ cm$^2$. The average speed, $\overline{v}$, of a hydrogen atom is

$$\overline{v} \;=\; \left(\frac{8\,k\,T}{\pi m_H}\right)^{1/2} \tag{13}$$

This yields $\overline{v} = 1.2 \times 10^6$ cm s$^{-1}$. Thus a typical value of $C_{UL}$ is 10$^{-10}$ cm$^3$ s$^{-1}$ and $n\,C_{UL}$ in the solar photosphere is typically 10$^6$ s$^{-1}$. The sodium $D$ lines (at 5890 Å and 5896 Å) have $A_{UL}$ = 6.3 $\times$ 10$^7$ s$^{-1}$. Therefore, this strong line can be out of LTE even in a main sequence star like the Sun.

We can use this analysis to estimate the calcium abundance in the Sun from the H (3968.5 Å) and K (3933.6 Å) lines. The K line has a residual intensity near 0 at line center and about 0.5 at 5 Å from line center. Therefore, at 5 Å from line center, we assume that $a = 0.5$. Therefore, if $\rho_H$ and $\rho_{Ca}$ denote the mass density of hydrogen and calcium, respectively, with opacities of $\chi_H$ and $\chi_{Ca}$, then with $a \approx 0.5$, we expect that:

$$\rho_H \, \chi_H \approx \rho_{Ca} \, \chi_{Ca} \tag{14}$$

or

$$n(H) \, m_H \, \chi_H \ = \ n(Ca) \, m_{Ca} \, \chi_{Ca} \tag{15}$$

Let us further assume that essentially all the calcium is singly ionized and that essentially all of it is in the ground state. Therefore, from Lecture 3, we find that:

$$\chi(Ca^+)[\Delta\nu] \ = \ \frac{\pi e^2}{m_e c} \, f \, \frac{\delta}{\pi(\Delta\nu)^2} \frac{1}{m_{Ca}} \tag{16}$$

where:

$$\delta \ = \ \frac{A_{UL}}{4\pi} \tag{17}$$

With $\Delta\nu = 9.69 \times 10^{11}$ Hz, $f = 0.69$ and $A_{UL} = 1.5 \times 10^8$ s$^{-1}$, we have that: $\chi(Ca^+) =$ 1100 cm$^2$ g$^{-1}$. At the "top" of the photosphere, we take $\chi_H = 0.1$ cm$^2$ g$^{-1}$. Therefore:

$$\frac{n(Ca)}{n(H)} \ = \ \frac{\chi_H}{\chi_{Ca}} \frac{m_H}{m_{Ca}} \tag{18}$$

This expression computes to $n(Ca)/n(H) = 2 \times 10^{-6}$ which is the accepted number and considerably better than we should expect given the level of approximation that we used.

An important astrophysical situation where lines are very far from LTE are masers. These are environments where there is a population inversion so that there is a negative optical depth where

$$d\tau \ = \ n_L \, B_{LU} \, \phi(\Delta\nu) \ - \ n_U \, B_{UL} \, \phi(\Delta\nu) \tag{19}$$

which occurs when

$$n_U \, B_{UL} \ > \ n_L \, B_{LU} \tag{20}$$

or when:

$$\frac{n_U}{n_L} \ > \ \frac{g_U}{g_L} \tag{21}$$

Note from above, in a 2-level atom in a steady state, we have that:

$$\frac{n_U}{n_L} \ = \ \frac{n \, C_{LU} \ + \ B_{LU} \, \overline{J}}{n \, C_{UL} \ + \ B_{UL} \, \overline{J} \ + \ A_{UL}} \tag{22}$$

Remember that:

$$B_{LU} \ = \ \frac{g_U}{g_L} \, B_{UL} \tag{23}$$

and

$$C_{LU} \leq \frac{g_U}{g_L} \, C_{UL} \tag{24}$$

Consequently, in a two level system:

$$\frac{n_U}{n_L} \leq \frac{g_U}{g_L} \tag{25}$$

In order to invert the level population, we need to consider at least a 3 level system.

Some of the important features of astrophysical masers are the following. The surface brightness of a radial line can be very high. If $I_0$ is the background continuum, then:

$$I \;=\; I_0\, e^{-\tau} \;+\; S\left(1 \;-\; e^{-\tau}\right) \tag{26}$$

The general non-LTE source function is:

$$S_\nu \;=\; \frac{\epsilon_\nu}{\kappa_\nu} \tag{27}$$

Note that when $\tau$ is negative then so is $\kappa$, so that $I$ is always positive. Compelling evidence that an emission line is produced by a maser is provided by observations of very high surface brightnessess. If line flux is $F$ in solid angle $\Omega$, then:

$$I \;=\; \frac{F}{\Omega} \;=\; \frac{2\,k\,T_b}{\lambda^2} \tag{28}$$

Thus lines with high fluxes in small solid angles are masers. Brightness temperatures in excess of $10^{12}$ K have been observed in molecular lines such as OH.

Other important characteristic of masers is that they exhibit narrow lines, high polarization, and measurable time variability.

**Lecture 14 – Stellar Atmospheres**

One of the most important applications of radiative transfer is the description of stellar atmospheres. To begin, we assume that the atmosphere is plane parallel and grey in the sense that the opacity is independent of frequency. Furthermore, we assume that all the energy is carried by radiation with an integrated flux, $F$. If $T_e$ is the effective temperature of the star and $\sigma_{SB}$ is the Stephan-Boltzmann constant, then

$$F = \sigma_{SB} T_e^4 \tag{1}$$

In the atmosphere, we write that

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu \tag{2}$$

where $\mu = \cos\theta$ is defined relative to the vertical measured outwards. Integrating over all frequencies, this equation becomes:

$$\mu \frac{dI}{d\tau} = I - S \tag{3}$$

Note that locally, we can write that in local thermodynamic equilibrium that:

$$S = \int_0^\infty S_\nu \, d\nu = \int_0^\infty B_\nu \, d\nu = \frac{\sigma_{SB} T^4}{\pi} \tag{4}$$

Previously, we considered the two stream approximation. Here, we solve the equation with the Eddington approximation. We integrate over $4\pi$ steradians to find that

$$\frac{d}{d\tau} \int_0^{2\pi} \int_{-1}^1 I \, \mu \, d\mu d\phi = \int_0^{2\pi} \int_{-1}^1 I \, d\mu \, d\phi - \int_0^{2\pi} \int_{-1}^1 S \, d\mu \, d\phi \tag{5}$$

or:

$$\frac{dF}{d\tau} = 4\pi J - 4\pi S \tag{6}$$

where $J$ is the mean intensity. Assuming that the flux is constant through the atmosphere, then we find that:

$$J = S \tag{7}$$

We now multiply the equation of transfer by $\mu$ and integrate over $4\pi$ steradians to find that:

$$\frac{d}{d\tau} \int_0^{2\pi} \int_{-1}^1 I \, \mu^2 \, d\mu \, d\phi = \int_0^{2\pi} \int_{-1}^1 I \, \mu \, d\mu \, d\phi - \int_0^{2\pi} \int_{-1}^1 S \, \mu \, d\mu \, d\phi \tag{8}$$

We evaluate each of these terms. For the first quantity, we make the diffusion approximation that $I$ is nearly isotropic. In this case, we find that

$$\frac{d}{d\tau} \int_0^{2\pi} \int_{-1}^1 I \, \mu^2 \, d\mu \, d\phi \approx \frac{dJ}{d\tau} \int_0^{2\pi} \int_{-1}^1 \mu^2 \, d\mu \, d\phi = \frac{4\pi}{3} \frac{dJ}{d\tau} \tag{9}$$

By definition:

$$F = \int_0^{2\pi} \int_{-1}^{1} I \, \mu \, d\mu \, d\phi \tag{10}$$

Since $S$ is independent of angle, we also have that:

$$\int_0^{2\pi} \int_{-1}^{1} S \, \mu \, d\mu \, d\phi = 0 \tag{11}$$

Using $J = S$, the equation of transfer then reduces to:

$$\frac{4\pi}{3} \frac{dS}{d\tau} = F \tag{12}$$

or:

$$S = \frac{3}{4\pi} F \, \tau + C \tag{13}$$

where $C$ is a constant. Using our results for $S$ and $F$ given above, we find that

At the outer boundary of the stellar atmosphere where $\tau = 0$, we assume that there is no incoming radiation and therefore:

$$S = \frac{I}{2} = \frac{F}{2\,\pi} \tag{14}$$

Therefore using the boundary condition at $\tau = 0$, then

$$C = \frac{F}{2\pi} \tag{15}$$

or:

$$S = \frac{F}{\pi} \left( \frac{3}{4}\tau + \frac{1}{2} \right) \tag{16}$$

Using the results from above, we therefore find for the temperature that:

$$T^4 = T_e^4 \left( \frac{3}{4}\tau + \frac{1}{2} \right) \tag{17}$$

In an atmosphere with a net flux, we expect a temperature gradient.

The Eddington approximation is very useful as a place to begin to understand a star's atmosphere, but it is obviously incomplete. For example, flux is not exactly conserved through the atmosphere. Another uncertainty is that typically, in a real atmosphere, the opacity varies as a function of frequency. Consider now "real" space with $Z$ pointing outwards along the normal of the atmosphere so that:

$$d\tau_\nu = -\chi_\nu \, \rho \, dz \tag{18}$$

where $\rho$ is the mass density and $\chi_\nu$ is the opacity. In "real" space, the equation of transfer at each frequency is:

$$\mu \frac{dI_\nu}{dz} = -\chi_\nu \, \rho \, I_\nu + \epsilon_\nu \tag{19}$$

We divide by $\chi_\nu \rho$ and use that in LTE the source function is the Planck function to find that:

$$\frac{\mu}{\rho \chi_\nu} \frac{dI_\nu}{dz} = -I_\nu + B_\nu \tag{20}$$

Multiplying by $\mu$ and integrating over $4\pi$ steradians and over all freqeuncies, we get that:

$$\frac{1}{\rho} \int_0^\infty \int_0^\pi \int_0^{2\pi} \mu^2 \, \frac{1}{\chi_\nu} \frac{dI_\nu}{dz} \, d\nu \, \sin\theta \, d\theta \, d\phi = -F \tag{21}$$

Using the Eddington approximation and that $J = B$, this equation can be re-written as:

$$\frac{1}{3\rho} \int_0^\infty \frac{1}{\chi_\nu} \frac{dB_\nu}{dz} \approx -F \tag{22}$$

Therefore, flux will be preserved through the atmosphere is we use the Rosseland mean opacity, $\overline{\chi}$ defined such that:

$$\frac{1}{\overline{\chi}} = \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{dB_\nu}{dz} \, d\nu}{\int_0^\infty \frac{dB_\nu}{dz} \, d\nu} \tag{23}$$

Using:

$$\frac{dB_\nu}{dz} = \frac{\partial B_\nu}{\partial T} \frac{dT}{dz} \tag{24}$$

Then using the cancellation of $dT/dz$, we find that:

$$\frac{1}{\overline{\chi}} = \left( \int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} \, d\nu \right) / \left( \frac{\partial B}{\partial T} \right) \tag{25}$$

Since:

$$B = \frac{\sigma_{SB} T^4}{\pi} \tag{26}$$

Then:

$$\frac{\partial B}{\partial T} = \frac{4 \, \sigma_{SB} T^3}{\pi} \tag{27}$$

Therefore, the mean opacity to be used in the atmosphere is usually taken as:

$$\frac{1}{\overline{\chi}} = \frac{\pi}{4 \, \sigma_{SB} \, T^3} \int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu}{\partial T} \, d\nu \tag{28}$$

With a model atmosphere, it is possible to predict such observable quantities as the absorption line strengths, the shape of the continuum and limb darkening. There are, of course, a huge number of applications of these results. Consider the formation of a weak line.

Line opacities are necessary for abundance determinations. As a first approximation, we might write the source function as

$$S_\nu \; = \; a_\nu \; + \; \tau_\nu b_\nu \tag{29}$$

with

$$a_\nu \; = \; S_\nu(\tau_\nu = 0) \tag{30}$$

where

$$b_\nu \; = \; \frac{\partial S_\nu}{\partial \tau_\nu}(\tau_\nu = 0) \tag{31}$$

Or, for the moment, measuring $z$ downwards, then:

$$b_\nu \; = \; \frac{\partial S_\nu}{\partial z}\frac{\partial z}{\partial \tau_\nu} \; = \; \kappa_\nu^{-1}\frac{\partial S_\nu}{\partial z} \tag{32}$$

In the absence of scattering, the emergent flux is

$$F_\nu \; = \; \pi\left(a_\nu \; + \; \frac{2}{3}b_\nu\right) \; = \; \pi\left(a_\nu \; + \; \frac{2}{3}\,\kappa_\nu^{-1}\frac{\partial S_\nu}{\partial z}\right) \tag{33}$$

There are two "extremes". (i) The line can be very strong in which case $\kappa_\nu$ is large and $b_\nu$ is relatively small. Therefore, the flux in the line is controlled by $a_\nu$, the source function at the top of the atmosphere. (ii) If the line is weak, then the line opacity is only a small addition to the continuum. We write that the total opacity, $\kappa$, is given by the sum of the line opacity, $\kappa_L$ and the continuum opacity, $\kappa_L$ so that:

$$\kappa \; = \; \kappa_L \; + \; \kappa_C \tag{34}$$

Using a Taylor series expansion of the opacity, then the flux in the line, $F_L$ is

$$F_L \; = \; \pi\left(a \; + \; \frac{2}{3}\kappa_C^{-1}\left(1 \; - \; \frac{\kappa_L}{\kappa_C}\right)\frac{\partial S_\nu}{\partial z}\right) \tag{35}$$

While the flux in the continuum is:

$$F_C \; = \; \pi\left(a \; + \; \frac{2}{3}\kappa_C^{-1}\frac{\partial S_\nu}{\partial z}\right) \tag{36}$$

Therefore, the residual intensity, $r$, is given by:

$$r \; = \; \frac{F_C \; - \; F_L}{F_C} \; = \; \frac{\frac{2\pi}{3}\frac{\kappa_L}{\kappa_C}\frac{\partial S_\nu}{\partial z}}{\kappa_C F_C} \tag{37}$$

We can write that if the number density of absorbers in the lower level of the line is $n_L$, then

$$\kappa_L \;=\; n_L \frac{\pi e^2}{mc}\, f\, \phi(\Delta\nu) \left(1 \;-\; e^{-\frac{h\nu}{kT}}\right) \tag{38}$$

Similarly if the continuum depends upon a density of continuum absorbers, $n_C$, then

$$\kappa_C \;=\; n_C\, \sigma_C \left(1 \;-\; e^{-\frac{h\nu}{kT}}\right) \tag{39}$$

Finally, if we write the equivalent width as

$$W_\nu \;=\; \int_{-\infty}^{\infty} r_\nu d(\Delta\nu) \tag{40}$$

and using the normalization of the line broadening such that

$$\int_{-\infty}^{\infty} \phi(\Delta\nu) d(\Delta\nu) \;=\; 1 \tag{41}$$

Then

$$W_\nu \;=\; \frac{\frac{\pi e^2}{mc}\, f\, \frac{n_L}{n_C \sigma_C}\, \frac{2\pi}{3}\, \frac{\partial S_\nu}{\partial z}}{\kappa_C F_C} \tag{42}$$

Furthermore, since the continuum opacity dominates, we can write that:

$$\frac{1}{\kappa_C} \frac{\partial S_\nu}{\partial z} \approx \frac{\partial S_\nu}{\partial \tau_\nu} \tag{43}$$

An even further approximation is to take:

$$F_C \approx \pi\, S_\nu \tag{44}$$

With these approximations, we finally get:

$$W_\nu \approx \frac{2}{3}\, \frac{\pi e^2}{m\, c}\, f \frac{n_L}{n_C \sigma_C}\, \frac{\partial \log\, S_\nu}{\partial \tau_\nu} \tag{45}$$

This expression is a key result in determining abundances in a stellar atmosphere. When we observe weak lines, $W_\nu$ depends linearly upon the oscillator strength, $f$, and also directly upon $n_L/n_C$, the relative number of absorbers in the line compared to the number of absorbers in the continuum. By measuring the equivalent width, we then measure this ratio which, for the known temperature and density, allows us to estimate the abundance of the line forming element. Note, by the way, that this procedure requires knowing the temperature gradient in the atmosphere. If there is no gradient, there are no absorption lines.

## Lecture 15 – Passive Disks

For pre-main sequence stars or white dwarfs, we often need to consider a passive disk. These are systems where a dusty disk is illuminated by the central star, and then re-radiates in the infrared. In an active disk, dissipation of accretion energy is important.

Consider a flat disk whose thickness is less than the radius of the star. We can compute the temperature of the disk by assuming energy balance so that the rate of absorbing energy equals the rate of emitting energy. Assume the that emits isotropically. Each portion of the disk is illuminated by a hemisphere, which we will assume to appear as a semi-circle as seen from the disk. The height measured from the plane is $z$ and assume that the image of the star is subdivided into rectangles of height $dz$. We then consider the illumination of the disk by these rectangles of width, $2R$. We can write that:

$$R^2 + z^2 = R_*^2 \tag{1}$$

where $R_*$ denotes the radius of the star. If $D$ measures the distance from the star to the disk, if the star has effective temperature $T_e$ and therefore the surface intensity is $\sigma_{SB}T_e^4/\pi$, then on each side of the disk, the incident flux, $F_{in}$, is given by the intensity multiplied by the subtended solid angle multiplied by the cosine of the angle between the incident ray and the plane of the disk or $z/D$. Therefore:

$$F_{in} = \frac{\sigma_{SB}T_e^4}{\pi}\int_0^{R_*}\frac{2R}{D}\frac{dz}{D}\left(\frac{z}{D}\right) = \frac{2\,\sigma_{SB}\,T_e^4}{\pi\,D^3}\int_0^{R_*}\sqrt{R_*^2 - z^2}\,z\,dz \tag{2}$$

If there is a steady state so that the incident flux equals the outward flux which is siven by $F_{out} = \sigma_{SB}T^4$, then

$$\sigma_{SB}T^4 = \frac{2}{3\pi}\sigma_{SB}T_e^4\left(\frac{R_*}{D}\right)^3 \tag{3}$$

or:

$$T = T_*\left(\frac{2}{3\pi}\right)^{1/4}\left(\frac{R_*}{D}\right)^{3/4} \tag{4}$$

Thus, we expect that $T$ varies as $D^{-3/4}$.

Given the temperature as a function of radius from the star, we can compute the expected flux from the disk at Earth. Assume that we observe the disk at inclination angle, $i$, such that if it is face-on then $i = 0°$. The flux, $F_\nu$, from the disk at distance from Earth, $D_*$, and with inner radius, $D_{inner}$, and outer radius, $D_{outer}$, is:

$$F_\nu = \cos\, i\int_{D_{inner}}^{D_{outer}} B_\nu(T)\,\frac{2\pi\,D\,dD}{D_*^2} \tag{5}$$

With the usual Planck function, and the substitution:

$$x = \frac{h\,\nu}{k\,T} = \frac{h\nu}{k\,T_*}\left(\frac{3\pi}{2}\right)^{1/4}\left(\frac{D}{R_*}\right)^{3/4} \tag{6}$$

Therefore:

$$D = x^{4/3}\, R_*\left(\frac{2}{3\pi}\right)^{1/3}\left(\frac{k\,T_*}{h\nu}\right)^{4/3} \tag{7}$$

and

$$dD = \frac{4}{3}\,x^{1/3}\,dx\,R_*\left(\frac{2}{3\pi}\right)^{1/3}\left(\frac{k\,T_*}{h\nu}\right)^{4/3} \tag{8}$$

then:

$$F_\nu = \frac{4\,\pi\,h\,\nu^3\,\cos\,i}{c^2\,D_*^2}\int_{D_{inner}}^{D_{outer}}\frac{D\,dD}{e^x\,-\,1} \tag{9}$$

Or:

$$F_\nu = 12\,\pi^{1/3}\cos\,i\left(\frac{R_*}{D_*}\right)^2\left(\frac{2\,k\,T_*}{3\,h\,\nu}\right)^{8/3}\frac{h\nu^3}{c^2}\int_{x_{inner}}^{x_{outer}}\frac{x^{5/3}}{e^x\,-\,1}\,dx \tag{10}$$

This expression shows that over a range of frequency, we might expect that $F_\nu$ varies as $\nu^{1/3}$. The integral has a maximum value of 1.9, and therefore, there is a maximum allowed value of $F_\nu$. The disk can be detected as an infrared excess. For the star, we expect that on the Rayleihg-Jeans portion of Planck curve that:

$$F_\nu(*) = 2\,\pi\left(\frac{R_*}{D_*}\right)^2\frac{\nu^2\,k\,T_*}{c^2} \tag{11}$$

Thus $F_\nu(*)$ varies as $\nu^2$ and at low frequencies, the disk dominates the total emission from the system.

For white dwarfs, it seems that the disks are largely composed of dust grains and very flat. However, for pre-main-sequence stars, there is likely to be a large amount of gas in the disk. The usual assumption for a disk is that it is in vertical hydrostatic equilibrium. If $z$ denotes the distance from the disk, then we suppose that:

$$\frac{dp}{dz} = -\rho\,g \tag{12}$$

Using the ideal gas law and assuming that the disk is vertically, isothermal, this expression can be re-written as

$$\frac{d\rho}{dz} = -\rho\,\frac{\mu\,g}{k\,T} \tag{13}$$

where $\mu$ denotes the mean molecular weight of the gas. In the $Z$ direction, if we assume that the star dominates the gravitational acceleration, then:

$$g = \frac{G\,M_*}{D^2}\,\frac{z}{D} \tag{14}$$

Therefore, the hydrostatic equlibrium equation becomes:

$$\frac{d\log\,\rho}{dz} = -z\,\frac{G\,\mu\,M_*}{D^3\,k\,T} \tag{15}$$

The solution is:

$$\rho \;=\; \rho_0 \,\exp\left(-\frac{z^2}{H^2}\right) \tag{16}$$

where

$$H^2 \;=\; \left(\frac{2\,k_B\,T\,D^3}{G\,M_*\,\mu}\right) \tag{17}$$

The value of the density in the midplane, $\rho_0$ is determined by the surface density of the disk, $\Sigma$. We can write that:

$$\Sigma \;=\; \int_{-\infty}^{\infty} \rho\,dz \;=\; \sqrt{\pi}\,\rho_0\,H \tag{18}$$

Given the temperature variation of the flat disk given above, we expect that:

$$H \;=\; \left(\frac{2k_B}{G\,M_*\,\mu}\right)^{1/2} D^{3/2}\,T_*^{1/2}\,\left(\frac{2}{3\pi}\right)^{1/8}\,\left(\frac{R_*}{D}\right)^{3/8} \tag{19}$$

Thus $H$ varies as $D^{9/8}$ and thus the relative thickness of the disk, $H/D$, increases outwards. If the disk is flat, we require that $H < R_*$. The critical distance where this occurs, $D_{crit}$, is:

$$D_{crit} \;=\; \left(\frac{3\pi}{32}\right)^{1/9}\,\left(\frac{GM_*\mu}{k_BT_*R_*}\right)^{4/9}\;R_* \tag{20}$$

In their models for flared disks, Chiang & Goldreich (1997) define the "grazing angle", $\alpha$, as the angle between the surface of the disk and the line of sight to the star. Far from the star, in a Taylor series expansion, the local value of $\alpha$ is given by:

$$\alpha \approx D\,\frac{d}{dD}\left(\frac{H}{D}\right) \tag{21}$$

At location $D$, the line-of-sight to the star makes an angle $H/D$ with respect to the midplane of the disk. The deviation from the line-of-sight to the star is the angle $\alpha$.

The inward flux on any element of the disk is balanced by the outward flux. We assume that the light from the star all arrives at the same angle (unlike the flat disk where different portions of the star made different incident angles to the surface of the disk). In this case, the intensity from each portion of the disk is $\sigma_{SB}T_*^4/\pi$. The solid angle subtend by the illuminating hemisphere of the star is $\pi R_*^2/(2\,D^2)$. We then multiply by $\alpha$ to find the incident flux. The temperature is then found from:

$$T \;=\; \left(\frac{\alpha}{2}\right)^{1/4}\,\left(\frac{R_*}{D}\right)^{1/2}\;T_* \tag{22}$$

From above, we write that:

$$H^2 \;=\; \left(\frac{2\,k_B\,T\,D^3}{G\,M_*\,\mu}\right) \tag{23}$$

Set:

$$T \;=\; C_1\, D^{C_2} \tag{24}$$

Therefore:

$$H \;=\; \left(\frac{2\,k_B\,C_1}{G\,M_*\,\mu}\right)^{1/2}\; D^{\frac{3+C_2}{2}} \tag{25}$$

Then from equation (21), we have that:

$$\alpha \;=\; \left(\frac{1\,+\,C_2}{2}\right)\,\left(\frac{2\,k_B\,C_1}{G\,M_*\,\mu}\right)^{1/2}\; D^{\frac{1+C_2}{2}} \tag{26}$$

Then from equation (22), we write that:

$$C_1\,D^{C_2} \;=\; \left(\frac{1\,+\,C_2}{4}\right)^{1/4}\,\left(\frac{2k_B\,C_1}{GM_*\mu}\right)^{1/8}\; D^{\frac{1+C_2}{8}}\; R_*^{1/2}\,D^{-1/2}\,T_* \tag{27}$$

The terms in $D$ give $C_2$ = -3/7. Then, we can solve for $C_1$ to find that the disk temperature is given by the expression:

$$T \;=\; \left(\frac{1}{7}\right)^{2/7}\left(\frac{R_*}{D}\right)^{3/7}\,\left(\frac{2k_BT_*R_*}{GM_*\mu}\right)^{1/7}\; T_* \tag{28}$$

Since the disk is opaque, we can determine the flux from the source, $F_\nu$, by:

$$F_\nu \;=\; \frac{2\,\pi\,cos\,i}{D_*^2}\int_0^{R_{out}} B_\nu(T_{disk})\,D\,dD \tag{29}$$

where $R_{out}$ denotes the outer boundary of the disk where the temperature is $T_{out}$. With the dimensionless parameter, $x$:

$$x \;=\; \frac{h\nu}{k_BT} \tag{30}$$

From above:

$$F_\nu \;=\; \frac{28\pi}{3}\,\frac{cos\,i\,R_*^2}{D_*^2}\,\left(\frac{k_BT_*}{h\nu}\right)^{5/3}\,\frac{(k_BT_*)^3}{(hc)^2}\,\left(\frac{2\,k_BT_*R_*}{49\,G\,M_*\,\mu}\right)^{2/3}\,\int_0^{x_{out}}\frac{x^{11/3}}{e^x\,-\,1}\,dx \tag{31}$$

At high frequencies with $x_{out} > 10$, the observed flux is insensitive to the outer boundary condition since the integral in equation (31) is approximately 15. The spectrum is predicted to vary as $\nu^{-5/3}$. At low frequencies where $x_{out}$ <2, we may re-write the expression for the flux to find that:

$$F_\nu \;=\; \frac{28\pi}{11}\,\frac{cos\,i\,R_*^2}{\lambda^2\,D_*^2}\,\left(\frac{T_*}{T_{out}}\right)^{11/3}\,\left(\frac{2\,k_BT_*R_*}{49\,G\,M_*\,\mu}\right)^{2/3}\; k_B\,T_* \tag{32}$$

If the disk is opaque, then at low frequencies, $F_\nu$ varies as $\nu^2$.

**Lecture 17 – Accretion Power and Active Disks**

There are many astrophysical objects which are powered by accretion. If the accretor has mass $M_*$ and radius $R_*$ and if $\dot{M}$ is the accretion rate, then the total luminosity, $L_{total}$, can be

$$L_{total} \;=\; \frac{G\,M_*\,\dot{M}}{R_*} \tag{1}$$

For a black hole, the appropriate radius may be that of the "last stable orbit" at approximately $3R_S$ where

$$R_S \;=\; \frac{2\,G\,M_*}{c^2} \tag{2}$$

Furthermore, if the matter is approaching the black hole in a disk, then (see below), as it moves inwards, half the released gravitational energy leads to the material moving faster. Finally, some of the radiated energy shines into the black hole never to be seen again. As a result, we might expect that for black holes:

$$L_{total} \approx 0.1\dot{M}\,c^2 \tag{3}$$

Although accretion may proceed in a complex fashion, for example, it may be strongly modulated by the star's magnetic field, two simple models are accretion through a disk and spherical accretion. Consider accretion through a disk which is active is one where the energy is largely derived from the local dissipation of infall. Consider a flat disk where the distance from the star is denoted as $D$. The gas in the disk mainly moves in circular orbits, but there is an inward radial drift of material. This inward drift is caused by viscous torques on the gas in the disk. Assume that matter moves inwards from $D = dD$ to $D$. Because the gas is mainly moving in circular orbits, it moves faster as it drifts inwards. Let $E_{orb}$ denote the orbital energy. Then for mass element, $m$,

$$E_{orb} \;=\; \left(\frac{v^2}{2} \;-\; \frac{G\,M_*}{D}\right)\,m \tag{4}$$

In the approximately circular orbits, we have:

$$v^2 \;=\; \frac{G\,M_*}{D} \tag{5}$$

Thus:

$$E_{orb} \;=\; -\frac{1}{2}\,\frac{G\,M_*\,m}{D} \tag{6}$$

The rate that orbital energy that is released during this infall, $d\Delta E_{orb}/dt$ is:

$$\frac{d\Delta E_{orb}}{dt} \;=\; \frac{1}{2}\,\frac{G\,M_*\,\dot{M}}{D^2}\,\Delta D \tag{7}$$

For ordinary work, the power, $P$, varies as $F\,v$ where $F$ is the exerted force and $v$ is the speed. For rotational motion, the power, $P$, varies as $N\,\Omega$ where $N$ is the torque and $\Omega$ is the angular speed. We can write that:

$$\frac{dE_{torque}}{dt} \;=\; \frac{d(\Omega N)}{dD}\,\Delta D \tag{8}$$

With

$$\Omega \;=\; \left(\frac{G\,M_*}{D^3}\right)^{1/2} \tag{9}$$

In the disks, the viscous torque leads to:

$$N \;=\; -\,D^2\,\Omega\,\dot{M} \tag{10}$$

Therefore:

$$\frac{dE_{torque}}{dt} \;=\; \frac{G\,M_*\,\dot{M}}{D^2}\,\Delta D \tag{11}$$

Therefore, we can write that:

$$\frac{dE_{total}}{dt} \;=\; \frac{dE_{torque}}{dt} + \frac{dE_{grav}}{dt} \;=\; \frac{3}{2}\frac{G\,M_*\dot{M}}{D^2}\,\Delta D \tag{12}$$

If we make the assumption that the power is dissipated as light, and since there are two sides to the disk, if $T$ is the local effective temperature, the radiated power from a ring, $p_{rad}$, is

$$p_{rad} \;=\; 2\,(2\pi\,D\,\Delta D)\;\sigma_{SB}\,T^4 \tag{13}$$

We therefore find for an active disk that:

$$\sigma_{SB}\,T^4 \;=\; \frac{3}{8\,\pi}\left(\frac{G\,M_*\,\dot{M}}{D^3}\right) \tag{14}$$

This expression is only valid relatively far from the surface of the star since the viscous flow near the star is controlled by its rotational speed which must be less than the orbital speed. Since $T$ varies as $D^{-3}$, the SED (spectral energy distribution) predicted by an active disk is similar to that of a passive disk.

In addition to the emission from the disk, we typically need to consider the emission from the "boundary layer", the region where the gas decelerates to the rotational speed of the star or accreting object. There may be additional complications as well. For example, accretion often leads to jet form and highly relativistic particles may be accelerated and nonthermal emission produced. Continuum emission from the disk may lead to photo-ionization in the surrounding gas.

**Supplementary Material – Compton Effect**

The Compton effect is important in a number of astrophysical situations. The easiest place to start is the scattering of a photon off an electron at rest.

$$h\nu \; + \; e \rightarrow h\nu' \; + \; e \tag{1}$$

After the collision, the electron moves with speed $v$. We use the conservation of energy and momentum, the principles of relativity and the result that light comes in photons. The frequency of the photon before the collision is $\nu$ while after the collision, it is $\nu'$. The photon is scattered through angle $\alpha$ while the electron recoils at angle $\beta$ relative to the line of the photon's initial trajectory, denoted as the $X$-axis. We define the plane of the collision as the $X - Y$ plane.

Conservation of energy gives:

$$h\nu \; + \; m\,c^2 \; = \; h\nu' \; + \; \gamma\, m\, c^2 \tag{2}$$

The conservation of momentum along the $X$-axis gives:

$$\frac{h\nu}{c} \; = \; \frac{h\nu' \cos\alpha}{c} \; + \; m\gamma\, v\, \cos\beta \tag{3}$$

while the conservation of momentum along the $Y$-axis gives:

$$0 \; = \; \frac{h\nu' \sin\alpha}{c} \; - \; m\gamma\, v\, \sin\beta \tag{4}$$

Remember that in relativistic dynamics:

$$\gamma \; = \; \frac{1}{\sqrt{1 \; - \; \frac{v^2}{c^2}}} \tag{5}$$

We have written down 3 equations, with 3 unknowns. We can eliminate $\alpha$ and $\beta$ to find $\nu'$ as a function of $\nu$.

We find from equations (3) and (4) that:

$$\cos^2\beta \; = \; \left(\frac{1}{m\gamma v}\right)^2 \left( \left[\frac{h\nu}{c}\right]^2 \; - \; \frac{2h^2\nu\nu'\cos\alpha}{c^2} \; + \; \left[\frac{h\nu'\cos\alpha}{c}\right]^2 \right) \tag{6}$$

and

$$\sin^2\beta \; = \; \left(\frac{1}{m\gamma v}\right)^2 \left(\frac{h\nu'\sin\alpha}{c}\right)^2 \tag{7}$$

Therefore, combining equations (6) and (7) and re-arranging the terms:

$$(\gamma\, m\; v\, c)^2 \; = \; \left([h\nu]^2 \; - \; 2h^2\nu\nu'\cos\alpha \; + \; [h\nu']^2\right) \tag{8}$$

We can re-write the conservation of energy [equation (2)] to give:

$$\gamma^2 m^2 c^4 \;=\; \left([h\nu \,-\, h\nu'] \,+\, m\,c^2\right)^2 \tag{9}$$

or

$$\gamma^2 m^2 c^4 \;=\; h^2\left(\nu^2 \,+\nu'^2 \,-2\nu\nu'\right) \,+\, 2mc^2 h(\nu \,-\, \nu') \,+\, m^2 c^4 \tag{10}$$

Subtract equation (8) from equation (10), and we get:

$$\gamma^2 m^2 c^2\left(c^2 \,-\, v^2\right) \;=\; -2h^2\nu\nu'\,(1 \,-\, \cos\alpha) \,+\, 2mc^2 h(\nu \,-\, \nu') \,+\, m^2 c^4 \tag{11}$$

Using the definition of $\gamma$ given in equation (5), we find that the left hand side of equation (11) is:

$$\gamma^2 m^2 c^2\left(c^2 \,-\, v^2\right) \;=\; m^2 c^4 \tag{12}$$

Consequently, equation (11) simplifies to:

$$(1 \,-\, \cos\alpha) \;=\; \left(\frac{m\,c^2}{h}\right)\left(\frac{\nu \,-\, \nu'}{\nu\nu'}\right) \tag{13}$$

We can re-write this as

$$(1 \,-\, \cos\alpha) \;=\; \left(\frac{m\,c^2}{h}\right)\left(\frac{1}{\nu'} \,-\, \frac{1}{\nu}\right) \;=\; \left(\frac{m\,c^2}{h}\right)\left(\frac{\lambda'}{c} \,-\, \frac{\lambda}{c}\right) \tag{14}$$

Or if we define

$$\Delta\lambda \;=\; \lambda' \,-\, \lambda \tag{15}$$

and if we define the Compton wavelength,

$$\lambda_0 \;=\; \frac{h}{m\,c} \tag{16}$$

The numerical value of this Compton wavelength is 0.00243 nm. Then:

$$\Delta\lambda \;=\; \lambda_0\,(1 \,-\, \cos\alpha) \;=\; 2\,\lambda_0\,\sin^2\frac{\alpha}{2} \tag{17}$$

The cross section for the scattering event is energy dependent. If we define

$$x \;=\; \frac{h\nu}{m_e c^2} \tag{18}$$

and the Thompson cross section as $\sigma_T$, then

$$\sigma \;=\; \frac{3\,\sigma_T}{4}\left(\frac{1 \,+\, x}{x^3}\left[\frac{2\,x\,(1 \,+\, x)}{1 \,+\, 2\,x} \,-\, \ln(1 \,+\, 2\,x)\right] \,+\, \frac{\ln(1 \,+\, 2\,x)}{2\,x} \,-\, \frac{1 \,+\, 3\,x}{(1 \,+\, 2\,x)^2}\right) \tag{19}$$

It is a useful exercise to show that for $x << 1$ that:

$$\sigma \approx \sigma_T \left( 1 \,-\, 2\,x \,+\, \frac{26\,x^2}{5} \,+\, ... \right) \tag{20}$$

It is considerably easier to show that for $x >> 1$ that

$$\sigma \approx \frac{3\,\sigma_T}{8\,x}\; \ln(2\,x) \tag{21}$$

The calculation above was performed for the situation where the electron is a rest. Thus, by transforming into the electron's frame of reference, we can compute how photons scatter off electrons of arbitrary motion. It is therefore possible that there is a net transfer of energy from the electrons into the radiation field, if the electrons are "hot" and the photons are "cold". Consider first the simplest case of a 1-dimensional collision where the photon bounces backwards off the electron. In the laboratory frame, the photon energy before the collision, $E_B$ is:

$$E_B \;=\; h\nu \tag{22}$$

In the rest frame of the electron, with velocity $v$, and with $\beta = v/c$ and the usual definition of $\gamma$, this energy, $E_B'$ is:

$$E_B' \;=\; \gamma h\nu\,(1 \,+\, \beta) \;=\; h\nu' \tag{23}$$

The energy after the collision, $E_A'$ is:

$$E_A' \approx h\nu' \left( 1 \,-\, \frac{2h\nu'}{m_e c^2} \right) \;=\; h\nu'' \tag{24}$$

Then finally, the energy of the photon in the laboratory frame after the collision, $E_A$, has reversed direction so that

$$E_A \;=\; \gamma h\nu''\;(1 \,+\, \beta) \tag{25}$$

is:

$$E_A \approx h\nu\;\gamma^2 (1 \,+\, \beta)^2 \left( 1 \,-\, \frac{2\,\gamma(1 \,+\, \beta)\,h\nu}{m_e c^2} \right) \tag{26}$$

Consider now a more general case of an electron at the origin moving with velocitty $\vec{v}$ such that

$$\vec{v} \;=\; v_0\,(\cos\phi\,\sin\theta\,\hat{x} \,+\, \sin\phi\,\sin\theta\,\hat{y} \,+\, \cos\theta\,\hat{z}) \tag{27}$$

Light is incident upon the scattering electron in direction $-\hat{n}$ where

$$\hat{n} \;=\; \cos\phi'\,\sin\theta'\,\hat{x} \,+\, \sin\phi'\,\sin\theta'\,\hat{y} + \,\cos\theta'\,\hat{z} \tag{28}$$

In the electron's frame, the Dopper shift of the incident photon is:

$$\frac{\Delta\nu'}{\nu'} \;=\; \gamma \left( 1 \,+\, \frac{\vec{v}\cdot\hat{n}}{c} \right) \tag{29}$$

We observe a shift of the frequency of the scattered photon by an amount

$$\frac{\Delta\nu''}{\nu''} = \gamma \left(1 + \frac{\vec{v}\cdot\hat{z}}{c}\right) \tag{30}$$

Therefore, the total Doppler shift that we observe from the initial photon is:

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\nu'}{\nu'}\frac{\Delta\nu''}{\nu''} = \gamma^2 \left(1 + \frac{v_0}{c}\cos\theta\right)\left(1 + \frac{\vec{v}\cdot\hat{n}}{c}\right) \tag{31}$$

where:

$$\left(1 + \frac{\vec{v}\cdot\hat{n}}{c}\right) = \left(1 - \frac{v_0}{c}\left[\cos\phi\cos\phi\sin\theta\sin\theta' + \sin\phi\sin\phi'\sin\theta\sin\theta' + \cos\theta\cos\theta'\right]\right) \tag{32}$$

Now assume that the radiation field is isotropic in the laboratory frame. Ignoring the aberration of starlight so that radiation field is not isotropic in the electron's frame, and to find the average Doppler shift we must compute:

$$\overline{\frac{\Delta\nu}{\nu}} = \frac{\gamma^2}{(4\pi)^2}\int_0^\pi\int_0^\pi\int_0^{2\pi}\int_0^{2\pi}\frac{\Delta\nu}{\nu}\left(\theta,\theta',\phi,\phi'\right)\sin\theta\sin\theta'\,d\theta d\theta' d\phi d\phi' \tag{33}$$

The azimuthal integrals all come to 0 so

$$\overline{\frac{\Delta\nu}{\nu}} = \frac{\gamma^2}{4}\int_0^\pi\int_0^\pi\left(1 + \frac{v_0}{c}\cos\theta(1 - \cos\theta') - \left[\frac{v_0}{c}\right]^2\cos^2\theta\cos\theta'\right)\sin\theta\sin\theta' d\theta d\theta' \tag{34}$$

Therefore, in the nonrelativistic limit:

$$\overline{\frac{\Delta\nu}{\nu}} = \gamma^2 \approx \left(1 + \frac{v_0^2}{c^2}\right) \tag{35}$$

By including the aberration of starlight, the light seen by the electron is not isotropic but instead has a pattern:

$$d\Omega = \left(1 + \frac{v_0}{c}\cos\theta\right) \tag{36}$$

When including this term, the integral is increased by a factor of 4/3 since we now must also average over $\cos^2\theta$.

An interesting application of this result is the Zeldovich-Sunyaev effect. The hot gas within a cluster of galaxies scatters the isotropic microwave background radiation. The number of photons is conserved, but the spectrum is shifted to higher energies. The amount of the shift depends upon $v_0^2$ or the temperature of the gas and the optical depth through the hot gas. The Compton $y$-parameter is:

$$y = \frac{4\,k\,T}{m_e c^2}\,\tau \tag{37}$$

Another application is that the electrons in a radiation field lose energy to Compton scattering. If $U_{ph}$ denotes the energy density in the photon field and $U_B$ denotes the energy density in the magnetic field, then the ratio of synchrotron energy loss to Compton loss is:

$$\frac{P_{synch}}{P_{compt}} = \frac{U_B}{U_{ph}} \tag{38}$$