14 Timescales in Stellar Interiors

Having dealt with the stellar photosphere and the radiation transport so relevant to our observations of this region, we're now ready to journey deeper into the inner layers of our stellar onion. Fundamentally, the aim we will develop in the coming chapters is to develop a connection between M, R, L, and T in stars (see Table 14 for some relevant scales).

More specifically, our goal will be to develop equilibrium models that describe stellar structure: $P(r)$, $\rho(r)$, and $T(r)$. We will have to model gravity, pressure balance, energy transport, and energy generation to get everything right. We will follow a fairly simple path, assuming spherical symmetric throughout and ignoring effects due to rotation, magnetic fields, etc.

Before laying out the equations, let's first think about some key timescales. By quantifying these timescales and assuming stars are in at least short-term equilibrium, we will be better-equipped to understand the relevant processes and to identify just what stellar equilibrium means.

14.1 *Photon collisions with matter*

This sets the timescale for radiation and matter to reach equilibrium. It depends on the **mean free path** of photons through the gas,

(227) $$\ell = \frac{1}{n\sigma}$$

So by dimensional analysis,

(228) $$\tau_\gamma \approx \frac{\ell}{c}$$

If we use numbers roughly appropriate for the average Sun (assuming full

Table 3: Relevant stellar quantities.

Quantity	Value in Sun	Range in other stars
M	2×10^{33} g	$0.08 \lesssim (M/M_\odot) \lesssim 100$
R	7×10^{10} cm	$0.08 \lesssim (R/R_\odot) \lesssim 1000$
L	4×10^{33} erg s^{-1}	$10^{-3} \lesssim (L/L_\odot) \lesssim 10^6$
T_{eff}	5777 K	3000 K $\lesssim (T_{\text{eff}}/mathrmK) \lesssim$ 50,000 K
ρ_c	150 g cm^{-3}	$10 \lesssim (\rho_c/\text{g cm}^{-3}) \lesssim 1000$
T_c	1.5×10^7 K	$10^6 \lesssim (T_c/\text{K}) \lesssim 10^8$

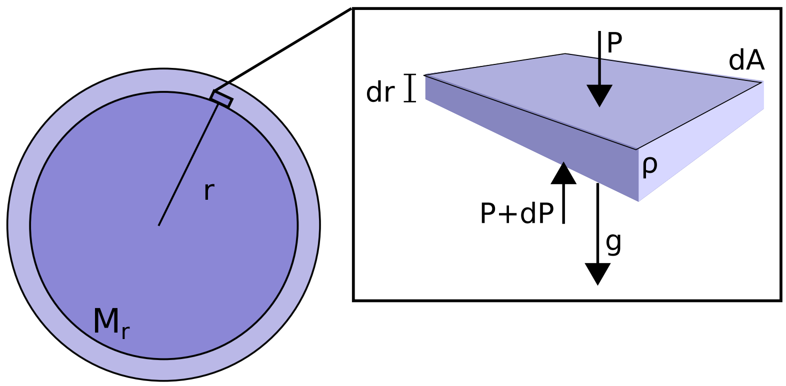

Figure 28: The state of hydrostatic equilibrium in an object like a star occurs when the inward force of gravity is balanced by an outward pressure gradient. This figure illustrates that balance for a packet of gas inside of a star

ionization, and thus Thomson scattering), we have

(229)
$$\ell = \frac{1}{n\sigma}$$
(230)
$$= \frac{m_p}{\rho \sigma_T}$$
(231)
$$= \frac{1.7 \times 10^{-24}\text{ g}}{(1.4\text{ g cm}^{-3})(2/3 \times 10^{-24}\text{ cm}^{-2})}$$
(232)
$$\approx 2\text{ cm}$$

So the matter-radiation equilibration timescale is roughly $\tau_\gamma \approx 10^{-10}$ s. Pretty fast!

14.2 *Gravity and the free-fall timescale*

For stars like the sun not to be either collapsing inward due to gravity or expanding outward due to their gas pressure, these two forces must be in balance. This condition is known as hydrostatic equilibrium. This balance is illustrated in Figure 28

As we will see, gravity sets the timescale for fluid to come into mechanical equilibrium. When we consider the balance between pressure and gravity on a small bit of the stellar atmosphere with volume $V = Adr$ (sketched in Fig. 28), we see that in equilibrium the vertical forces must cancel.

The small volume element has mass dm and so will feel a gravitational force equal to

(233) $$F_g = \frac{GM_r\,dm}{r^2}$$

where M_r is the mass of the star enclosed within a radius r,

(234) $$M(r) \equiv 4\pi \int_{r'=0}^{r'=r} \rho(r') r'^2 dr'$$

Assuming the volume element has a thickness dr and area dA, and the star has a uniform density ρ, then we can replace dm with $\rho dr dA$. This volume element will also feel a mean pressure which we can define as dP, where the pressure on the outward facing surface of this element is P and the pressure on the inward facing surface of this element is $P + dP$. The net pressure force is then $dPdA$, so

(235) $$F_P(r) = F_g(r)$$

(236) $$A\left(P(r) - P(r+dr)\right) = -\rho V g$$

(237) $$= \rho A dr g$$

(238)

which yields the classic expression for **hydrostatic equilibrium**,

(239) $$\frac{dP}{dr} = \rho(r) g(r)$$

where

(240) $$g \equiv -\frac{GM(r)}{r^2}$$

and $M(r)$ is defined as above.

When applying Eq. 239 to stellar interiors, it's common to recast it as

(241) $$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

In Eqs. 239 and 241 the left hand side is the pressure gradient across our volume element, and the right hand side is the gravitational force averaged over that same volume element. So it's not that pressure balances gravity in a star, but rather gravity is balanced by the gradient of increasing pressure from the center to the surface.

The gradient dP/dr describes the pressure profile of the stellar interior in equilibrium. What if the pressure changes suddenly – how long does it take

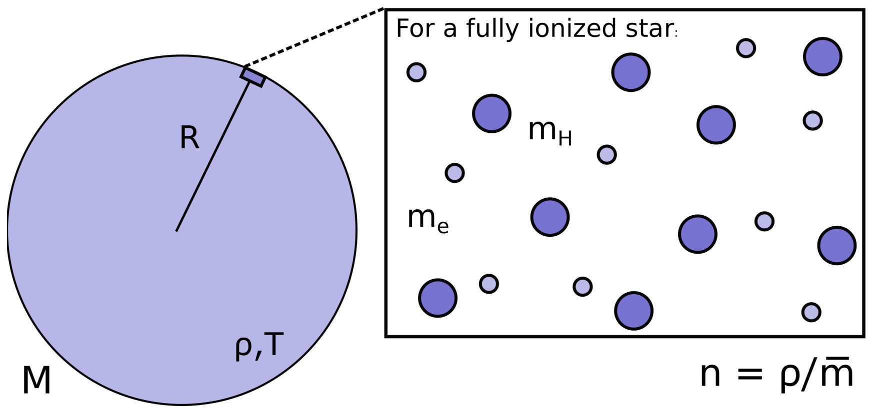

Figure 29: A simple model of a star having a radius R, mass M, constant density ρ, a constant temperature T, and a fully ionized interior. This simple model can be used to derive a typical free-fall time and a typical sound-crossing time for the sun.

us to re-establish equilibrium? Or equivalently: if nothing were holding up a star, how long would it take to collapse under its own gravity? Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M.

Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M. To order of magnitude, we can combine the following two equations

(242) $$a = -\frac{GM}{r^2}$$

and

(243) $$d = -\frac{1}{2}at^2.$$

Setting both r and d equal to the radius of our object R, and assuming a constant density $\rho = \frac{3M}{4\pi R^3}$, we find

(244) $$\tau_{ff} \sim \frac{1}{\sqrt{G\rho}}$$

which is within a factor of two of the exact solution,

(245) $$\tau_{ff} = \sqrt{\frac{3\pi}{32G\rho}}$$

Note that the free-fall timescale does not directly depend on the mass of an object or its radius (or in fact, the distance from the center of that object). It only depends on the density. Since $G \approx 2/3 \times 10^{-7}$ (cgs units), with $\langle \rho_\odot \rangle \approx 1$ g cm^{-3}, the average value is $\tau_{dyn} \sim 30$ min.

In real life, main-sequence stars like the sun are stable and long-lived structures that are not collapsing. Even if you have a cloud of gas that is collapsing under its own gravity to form a star, it does not collapse all the way to $R = 0$ thanks to its internal hydrostatic pressure gradient.

14.3 *The sound-crossing time*

We have an expression for the time scale upon which gravity will attempt to force changes on a system (such changes can either be collapse, if a system is far out of hydrostatic equilibrium and gravity is not significantly opposed by pressure, or contraction, if a system is more evenly balanced). What is the corresponding time scale upon which pressure will attempt to cause a system to expand?

The pressure time scale in a system can be characterized using the sound speed (as sound is equivalent to pressure waves in a medium). This isothermal sound speed is given by the relation

$$(246) \quad c_s = \sqrt{\frac{P}{\rho}}$$

Although gas clouds in the interstellar medium may be reasonably approximated as isothermal, the same is not true for stars. We will ignore that fact for now, but will return to this point later.

Referring back to Figure 29, we can define the sound-crossing time for an object as the time it takes for a sound wave to cross the object. Using a simple equation of motion $d = vt$ and approximating $2R$ just as R we can then define a sound-crossing time as

$$(247) \quad \tau_s \sim R\sqrt{\frac{\rho}{P}}$$

Using the ideal gas equation, we can substitute $\frac{\rho}{\bar{m}}kT$ for P and get an expression for the sound crossing time in terms of more fundamental parameters for an object:

$$(248) \quad \tau_s \sim R\sqrt{\frac{\bar{m}}{kT}}$$

Unlike the free-fall time we derived earlier, the sound-crossing time depends directly upon the size of the object, and its temperature. At the center of the Sun, $T_c \approx 1.5 \times 10^7$ K and $\bar{m} \sim m_p$ and so the sound-crossing timescale is roughly 30 min.

Note that by Eq. 245 we see that τ_s is also approximately equal to the free-fall timescale τ_{ff}. For an object not just to be in hydrostatic equilibrium but to remain this way, the pressure must be able to respond to changes in gravity, and vice versa. This response requires that a change in one force is met with a change in another force on a timescale that is sufficiently fast to restore the force balance. In practice, this means that for objects in hydrostatic equilibrium, the free-fall time is more or less equivalent to the sound-crossing time. In that way, a perturbation in pressure or density can be met with a corresponding response before the object moves significantly out of equilibrium.

14.4 *Radiation transport*

If photons streamed freely through a star, they'd zip without interruption from the core to the stellar surface in $R_\odot / c \approx 2$ s. But as we saw above in Eq. 232, the photons actually scatter every ∼1 cm. With each collision they "forget" their history, so the motion is a random walk with N steps. So for a single photon[7] to reach the surface from the core requires

$$\ell \sqrt{N} \sim R_\odot \tag{249}$$

which implies that the **photon diffusion timescale** is

$$\tau_{\gamma,\mathrm{diff}} \sim \frac{N\ell}{c} \sim \frac{R_\odot^2}{\ell}\frac{1}{c} \tag{250}$$

or roughly 10^4 yr.

14.5 *Thermal (Kelvin-Helmholtz) timescale*

The thermal timescale answers the question, How long will it take to radiate away an object's gravitational binding energy? This timescale also governs the contraction of stars and brown dwarfs (and gas giant planets) by specifying the time it takes for the object to radiate away a significant amount of its gravitational potential energy. This is determined by the **Kelvin-Helmholtz timescale**. This thermal time scale can generally be given as:

$$\tau_{KH} = \frac{E}{L}, \tag{251}$$

where E is the gravitational potential energy released in the contraction to its final radius and L is the luminosity of the source. Approximating the Sun as a uniform sphere, we have

$$\tau_{KH} \sim \frac{G M_\odot^2}{R_\odot}\frac{1}{L_\odot} \tag{252}$$

[7] This is rather poetic – of course a given photon doesn't survive to reach the surface, but is absorbed and re-radiated as a new photon $\sim (R_\odot/\ell)^2$ times. Because of this, it may be better to think of the timescale of Eq. 250 as the **radiative energy transport timescale**.

which is roughly 3×10^7 yr.

Before nuclear processes were known, the Kelvin-Helmholtz timescale was invoked to argue that the Sun could be only a few 10^7 yr old – and therefore much of geology and evolutionary biology (read: Darwin) must be wrong. There turned out to be missing physics, but τ_{KH} turns out to still be important when describing the contraction of large gas clouds as they form new, young stars.

The time that a protostar spends contracting depends upon its mass, as its radius slowly contracts. A 0.1 $M_\odot$ star can take 100 million years on the Hayashi track to finish contracting and reach the main sequence. On the other hand, a 1 $M_\odot$ star can take only a few million years contracting on the Hayashi track before it develops a radiative core, and then spends up to a few tens of millions of years on the Henyey track before reaching the main sequence and nuclear burning equilibrium. The most massive stars, 10 $M_\odot$ and above, take less than 100,000 years to evolve to the main sequence.

14.6 *Nuclear timescale*

The time that a star spends on the main sequence – essentially the duration of the star's nuclear fuel under a constant burn rate – is termed the **the nuclear timescale**. It is a function of stellar mass and luminosity, essentially analogous to the thermal time scale of Equation 251. Here, the mass available (technically, the mass difference between the reactants and product of the nuclear reaction) serves as the energy available, according to $E = mc^2$.

If we fuse 4 protons to form one He^4 nucleus (an **alpha particle**), then the fractional energy change is

$$\frac{\Delta E}{E} = \frac{4m_p c^2 - m_{He} c^2}{4 m_p c^2} \approx 0.007 \tag{253}$$

This is a handy rule of thumb: fusing H to He liberates roughly 0.7% of the available mass energy. As we will see, in more massive stars heavier elements can also fuse; further rules of thumb are that fusing He to C and then C to Fe (through multiple intermediate steps) each liberates another 0.1% of mass energy. But for a solar-mass star, the main-sequence nuclear timescale is

$$\tau_{nuc} = \frac{\Delta E}{E_{\rm tot}} \approx \frac{0.007 M_\odot c^2}{L_\odot} \approx 10^{11} \text{ yr} \tag{254}$$

which implies a main-sequence lifetime of roughly 100 billion years. The actual main-sequence lifetime for a $1M_\odot$ star is closer to 10 billion years; it turns out that significant stellar evolution typically occurs by the time ~10% of a star's mass has been processed by fusion.

14.7 *A Hierarchy of Timescales*

So if we arrange our timescales, we find a strong separation of scales:

$$\begin{array}{ccccccccc} \tau_{nuc} & \gg & \tau_{KH} & \gg & \tau_{\gamma,diff} & \gg & \tau_{dyn} & \gg & \tau_\gamma \\ 10^{11}\text{ yr} & \gg & 3\times 10^7\text{ yr} & \gg & 10^4\text{ yr} & \gg & 30\text{ min} & \gg & 10^{-10}\text{ s} \end{array}$$

This separation is pleasant because it means whenever we consider one timescale, we can assume that the faster processes are in equilibrium while the slower processes are static.

Much excitement ensues when this hierarchy breaks down. For example, we see convection occur on τ_{dyn} which then fundamentally changes the thermal transport. Or in the cores of stars near the end of their life, τ_{nuc} becomes much shorter. If it gets shorter than τ_{dyn}, then the star has no time to settle into equilibrium – it may collapse.

14.8 *The Virial Theorem*

In considering complex systems as a whole, it becomes easier to describe important properties of a system in equilibrium in terms of its energy balance rather than its force balance. For systems in equilibrium– not just a star now, or even particles in a gas, but systems as complicated as planets in orbit, or clusters of stars and galaxies– there is a fundamental relationship between the internal, kinetic energy of the system and its gravitational binding energy.

This relationship can be derived in a fairly complicated way by taking several time derivatives of the moment of inertia of a system, and applying the equations of motion and Newton's laws. We will skip this derivation, the result of which can be expressed as:

$$\frac{d^2 I}{dt^2} = 2\langle K \rangle + \langle U \rangle, \tag{255}$$

where $\langle K \rangle$ is the time-averaged kinetic energy, and $\langle U \rangle$ is the time-averaged gravitational potential energy. For a system in equilibrium, $\frac{d^2 I}{dt^2}$ is zero, yielding the form more traditionally used in astronomy:

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle \tag{256}$$

The relationship Eq. 256 is known as the Virial Theorem. It is a consequence of the more general fact that whenever $U \propto r^n$, we will have

$$\langle K \rangle = \frac{1}{n}\langle U \rangle \tag{257}$$

And so for gravity with $U \propto r^{-1}$, we have the Virial Theorem, Eq. 256.

When can the Virial Theorem be applied to a system? In general, the system must be in equilibrium (as stated before, this is satisfied by the second time derivative of the moment of inertia being equal to zero). Note that this is not necessarily equivalent to the system being stationary, as we are considering the time-averaged quantities $\langle K \rangle$ and $\langle U \rangle$. This allows us to apply the Virial Theorem to a broad diversity of systems in motion, from atoms swirling within a star to stars orbiting in a globular cluster, for example. The system also generally must be isolated. In the simplified form we are using, we don't consider so-called 'surface terms' due to an additional external pressure from a medium in which our system is embedded. We also assume that there are not any other sources of internal support against gravity in the system apart from the its internal, kinetic energy (there is no magnetic field in the source, or rotation). Below, we introduce some of the many ways we can apply this tool.

Virial Theorem applied to a Star

For stars, the Virial Theorem relates the internal (i.e. thermal) energy to the gravitational potential energy. We can begin with the equation of hydrostatic equilibrium, Eq. 239. We multiply both sides by $4\pi r^3$ and integrate as follows

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R \left(\frac{GM(r)}{r}\right)\left(4\pi r^2 \rho(r)\right) dr \tag{258}$$

The left-hand side can be integrated by parts,

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = 4\pi r^3 P|_0^R - 3\int_0^R P 4\pi r^2 dr \tag{259}$$

and since $r(0) =$ and $P(R) = 0$, the first term equals zero. We can deal with the second term by assuming that the star is an ideal gas, replacing $P = nkT$, and using the thermal energy density

$$u = \frac{3}{2} nkT = \frac{3}{2} P \tag{260}$$

This means that the left-hand side of Eq. 258 becomes

$$-2\int_0^R u(4\pi r^2 dr) = -2E_{th} \tag{261}$$

Where E_{th} is the total thermal energy of the star.

As for the right-hand side of Eq. 258, we can simplify it considerably by recalling that

$$\Phi_g = -\frac{GM(r)}{r} \tag{262}$$

and

$$dM = 4\pi r^2 \rho(r) dr. \tag{263}$$

Thus the right-hand side of Eq. 258 becomes simply

$$\int_0^R \Phi_g(M') dM' = E_{\text{grav}} \tag{264}$$

And so merely from the assumptions of hydrostatic equilibrium and an ideal gas, it turns out that

(265) $E_{\text{grav}} = -2E_{th}$

or alternatively,

(266) $E_{tot} = -E_{th} = E_{\text{grav}}/2$

The consequence is that the total energy of the bound system is negative, and that it has negative heat capacity – a star heats up as it loses energy! Eq. 266 shows that if the star radiates a bit of energy so that E_{tot} decreases, E_{th} increases while E_{grav} decreases by even more. So energy was lost from the star, causing its thermal energy to increase while it also becomes more strongly gravitationally bound. This behavior shows up in all gravitational systems with a thermal description — from stars to globular clusters to Hawking radiation near a black hole to the gravitational collapse of a gas cloud into a star.

Virial Theorem applied to Gravitational Collapse

We can begin by restating the Virial Theorem in terms of the average total energy of a system $\langle E \rangle$:

(267) $\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle$

A classic application of this relationship is then to ask, if the sun were powered only by energy from its gravitational contraction, how long could it live? To answer this, we need to build an expression for the gravitational potential energy of a uniform sphere: our model for the gravitational potential felt at each point inside of the sun. We can begin to put this into equation form by considering what the gravitational potential is for an infinitesimally thin shell of mass at the surface of a uniformly-dense sphere.

Using dM as defined previously, the differential change in gravitational potential energy that this shell adds to the sun is

(268) $dU = -\frac{GM(r)dM}{r}.$

The simplest form for $M(r)$ is to assume a constant density. In this case, we can define

(269) $M(r) = \frac{4}{3}\pi r^3 \rho$

To determine the total gravitational potential from shells at all radii, we must integrate Equation 268 over the entire size of the sphere from 0 to R, substituting our expressions for dM and $M(r)$ from Equations 263 and 269:

$$(270)\quad U = -\frac{G(4\pi\rho)^2}{3}\int_0^R r^4 dr.$$

Note that if this were not a uniform sphere, we would have to also consider ρ as a function of radius: $\rho(r)$ and include it in our integral as well. That would be a more realistic situation for a star like our sun, but we will keep it simple for now.

Performing this integral, and replacing the average density ρ with the quantity $\frac{3M}{4\pi R^3}$, we then find

$$(271)\quad U = -\frac{G(3M)^2}{R^6}\frac{R^5}{5} = -\frac{3}{5}\frac{GM^2}{R}$$

which is the gravitational potential (or binding energy) of a uniform sphere. All together, this is equivalent to the energy it would take to disassemble this sphere, piece by piece, and move each piece out to a distance of infinity (at which point it would have zero potential energy and zero kinetic energy).

To understand how this relates to the energy available for an object like the sun to radiate as a function of its gravitational collapse, we have to perform one more trick, and that is to realize that Equation 267 doesn't just tell us about the average energy of a system, but how that energy has evolved. That is to say,

$$(272)\quad \Delta E = \frac{1}{2}\Delta U$$

So, the change in energy of our sun as it collapsed from an initial cloud to its current size is half of the binding energy that we just calculated. How does our star just lose half of its energy as it collapses, and where does it go? The Virial Theorem says that as a cloud collapses it turns half of its potential energy into kinetic energy (Equation 256). The other half then goes into terms that are not accounted for in the Virial Theorem: radiation, internal excitation of atoms and molecules and ionization (see the Saha Equation, Equation 171).

Making the simplistic assumption that all of the energy released by the collapse goes into radiation, then we can calculate the energy available purely from gravitational collapse and contraction to power the luminosity of the sun. Assuming that the initial radius of the cloud from which our sun formed is not infinity, but is still large enough that the initial gravitational potential energy is effectively zero, the energy which is radiated from the collapse is half the current gravitational potential energy of the sun, or

$$(273)\quad E_{radiated} = -\frac{3}{10}\frac{GM_\odot^2}{R_\odot}$$

Eq. 273 therefore links the Virial Theorem back to the Kelvin-Helmholtz timescale of Sec. 14.5. For the sun, this is a total radiated energy of $\sim 10^{41}$ J. If we assume that the sun radiates this energy at a rate equal to its current luminosity ($\sim 10^{26}$ W) then we can calculate that the sun could be powered at its current luminosity just by this collapse energy for 10^{15} s, or 3×10^{7} years. While this is a long time, it does not compare to our current best estimates for the age of the earth and sun: ~4.5 billion years. As an interesting historical footnote, it was Lord Kelvin who first did this calculation to estimate the age of the sun (back before we knew that the sun must be powered by nuclear fusion). He used this calculation to argue that the Earth must only be a few million years old, he attacked Charles Darwin's estimate of hundreds of millions of years for the age of the earth, and he argued that the theory of evolution and natural selection must be bunk. In the end of course, history has shown who was actually correct on this point.