UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2022

Problem Set 1
Due: Wednesdays, Feb 02, 2022, at the start of class (1100 Kansas Time)
This problem set is worth **33 points**.

1. **Astronomical Concepts [20 pts].**

   (a) In a galaxy far, far away, the gas giant Endor orbits a Sun-like star at a distance of a_E. Endor (mass m_E) is orbited by a Forest Moon (m_m) with the same separation as found in the Earth-Moon system ($a_☽$). What is the ratio (an algebraic expression, not just a number!) of the gravitational forces (i) between Endor and its star (mass m_*) and (ii) between Endor and its moon? Estimate which Force is stronger. [6 pts]

   Solution: Since

$$F_G = G\frac{m_1 m_2}{r^2},$$

   The ratio of the two forces is then

$$\frac{F_{E,*}}{F_{m,E}} = \frac{m_*}{m_m}\left(\frac{a_☽}{1 \text{ au}}\right)^2.$$

   I could look up $a_☽$, or I could remember that light takes ∼1 s to go from the Earth to the Moon and ∼8 min from the Sun to the Earth (1 au). So the ratio of distances is ∼1/500, and thus:

$$\boxed{\frac{F_{E,*}}{F_{m,E}} \approx \frac{m_*}{m_m}\frac{1}{2.5\times10^5}}.$$

   Without knowing the mass m_m of the Forest Moon, we can't calculate an exact number. If it's like Earth's Moon, then $m_* \approx 3\times10^7 m_m$, but if it's a planet in its own right (say, twice the mass of the Earth) then $m_* \approx 1.5\times10^5$. So probably the star-planet force is stronger, but we can't be certain.

   (b) You have invented a matter-antimatter reactor that converts physical material into energy with 100% efficiency. Congratulations: you're a shoo-in for the Nobel Prize. (i) If you put 0.5 kg of matter (and an equal amount of antimatter) in your reactor, approximately how much energy (E_{reactor}) is released? (ii) If the reactor takes 0.5 s to use that fuel, what was its approximate power output in Solar Luminosities ($L_\odot$)? (iii) How does E_{reactor} compare to the total amount of energy used on Earth in a year? [7 pts]

   Solution: We know $E = mc^2$, so the total energy released in this reaction is the mass energy of the fuel:

$$\begin{aligned} E &\approx (1\text{ kg})(3\times10^8\text{ m s}^{-1})^2 \\ &\approx \boxed{10^{17}\text{ J}}. \end{aligned}$$

   When the energy is released in just 0.5 s, then the luminosity of your reactor is

$$L_{\text{reactor}} = \frac{\Delta E}{\Delta t} = \frac{10^{17}\ J}{0.5\text{ s}} = 4\times10^{17}\text{ W}.$$

   Since $L_\odot \approx 4\times10^{26}$ W, we have

$$\boxed{L_{\text{reactor}} \approx 10^{-9} L_\odot}.$$

   The world's annual energy consuption is roughly 500 million terajoules,

$$E_{\text{world}} \approx 500\times10^6\times10^{12}\text{ J} \approx 5\times10^{20}\text{ J}.$$

   So your reactor produced an energy yield equal to about 0.0002 (0.02%) of the world's current energy output: still quite a feat!

(c) Write the astronomer's version of the Ideal Gas Law. Explain each term (including its physical units), and how it might be used [7 pts].
Solution: The astrophysicist's ideal gas law is

$$\boxed{P = nk_BT} \tag{1}$$

where:

i. P is the gas pressure (SI unit Pascals, Pa = N m^{-2}),
ii. n is the gas number density (SI unit of inverse volume, m^{-3}) – can also be calculated if you know the gas *density* ($\rho = M/V = n\langle m\rangle$, where $\langle m\rangle$ is the average gas particle mass),
iii. k_B is the Boltzmann constant, $\sim (1/7) \times 10^{-22}$ J/K, and
iv. T is the temperature of the gas.

It is used to convert between the pressure, number density, and/or temperature of a gas in order to constrain any of those quantities that might not yet be known.

2. **Order-of-Magnitude Estimation [13 pts].** Strive to do as many of these calculations in your head (or with pencil and paper) as possible, aside from looking up any necessary physical constants.

(a) **City on a Hill [5 pts.]** Roughly estimate the mass of Mount Oread, in kg and in $M_\oplus$ (Earth masses).
Solution: From the topographic maps at `https://ngmdb.usgs.gov/topoview/` I estimate that Mt. Oread is about 2.5 km long, 700 m wide at the base, and 150 ft $\approx$ 40 m higher than the surrounding land (Wikipedia tells me that is 60 m above downtown Lawrence, so your mileage may vary). I assume it has a triangular cross-sectional area, so

$$\begin{aligned} V_{\rm Oread} &\approx \tfrac{1}{2} w \times h \times \ell \\ &\approx (350\ m)(40\ m)(2500\ m) \\ &\approx 3.5 \times 10^8\ \mathrm{m}^3. \end{aligned}$$

Assuming a density the same as the Earth ($\sim$ 5000 kg m^{-3}), then we have

$$M_{\rm Oread} \sim (5000)(3.5 \times 10^8)\ \mathrm{kg} \sim \boxed{1.8 \times 10^{12}\ \mathrm{kg}}. \tag{2}$$

Compared to the Earth's mass of 6×10^{24} kg, KU's mountaintop perch is just $\boxed{\sim 3 \times 10^{-13} M_\oplus}$.
A nice independent 'check' that a previous student pointed out is that the Great Pyramid of Giza, in Egypt, is estimated to weigh roughly six millions tons, $\approx 6 \times 10^9$ kg. I've never been to Egypt, but it at least makes sense to me that Mt. Oread is bigger than a human-built pyramid: both have about the same height, but I estimated a base area much larger than that of the pyramid.

(b) **How Big? [5 pts]**. The French revolutionaries of the late 18th century defined the meter by setting the Earth's equator-to-pole distance to be 10,000 km. Estimate the radius ($R_\oplus$), volume ($V_\oplus$), and mass ($M_\oplus$) of the Earth, in SI units.
Solution: Since the Earth is approximately a sphere, its circumference $c_\oplus = 40,000\,\mathrm{km} \approx 2\pi R_\oplus$. Thus,

$$R_\oplus \approx \frac{40,000\,\mathrm{km}}{2\pi} \approx \frac{40,000\,\mathrm{km}}{6} \boxed{6,500\,\mathrm{km}} \tag{3}$$

which compares reasonably well with the modern value of $R_\oplus = 6,370$ km.
Still approximating the Earth as a sphere, its volume is then just

$$\begin{aligned} V_\oplus &= \tfrac{4}{3}\pi R_\oplus^3 && (4) \\ &\approx 4(6,500\,\mathrm{km})^3 && (5) \\ &\approx 4 \times (6.5)^3 \left(10^6\,\mathrm{m}\right)^3 && (6) \\ &\approx (26 \times 40)\left(10^{18}\,\mathrm{m}^3\right) && (7) \\ &\approx (100)\left(10^{18}\,\mathrm{m}^3\right) && (8) \\ &\approx \boxed{\left(10^{20}\,\mathrm{m}^3\right)} && (9) \\ & && (10) \end{aligned}$$

... which compares well with the Earth's known volume of $V_\oplus = 1.083 \times 10^{20}$ m^3.

Mass is volume times density, i.e. $M_\oplus = V_\oplus \langle \rho_\oplus \rangle$. We know that water has $\rho_{\mathrm{H_2O}} = 1.0$ g/cc, and the Earth must be denser than this. We could look up the density of metals (7–10 g/cc) and assume that the Earth isn't solid metal and so must be *less* dense than these. (Of course, we could also just look up the average density of the Earth directly, but where's the fun in that?). If we assume $\langle \rho_\oplus \rangle \approx 4\mathrm{g\ cm^{-3}} = 4000\ \mathrm{kg\ m^{-3}}$, then we have

$$M_\oplus = V_\oplus \langle \rho_\oplus \rangle \quad (11)$$

$$\approx \left(10^{20}\ \mathrm{m^3}\right)\left(4000\ \mathrm{kg\ m^{-3}}\right) \quad (12)$$

$$\boxed{\approx 4 \times 10^{24}\ \mathrm{kg\ m^{-3}}} \quad (13)$$

... which is a bit lower than the known value of 5.972×10^{24}kg m^{-3}. We underestimated a bit because the Earth's actual density is 5.5 g/cc (not the 4.0 assumed above). But still: not too bad!

(c) **How Big?! [3 pts]** Jupiter is roughly 10× larger (in physical size) than the Earth (i.e., $R_{Jup} \approx 10R_\oplus$), and the Sun is roughly 10× larger than Jupiter ($R_\odot \approx 10R_{Jup}$). Roughly estimate the volume of both of these objects, *relative to the volume of the Earth* (i.e., in units of $V_\oplus$).

Solution: Since all these objects are approximately spherical and $V \propto R^3$, we know

$$\frac{V_{Jup}}{V_\oplus} = \left(\frac{R_{Jup}}{R_\oplus}\right)^3 \approx 1000 \quad (14)$$

and so

$$\boxed{V_{Jup} \approx 1000 V_\oplus}. \quad (15)$$

The actual value is 1322 or so, because actually Jupiter is a bit more than 10× the size of the Earth.

For the Sun, we know

$$\frac{V_\odot}{V_\oplus} = \left(\frac{R_\odot}{R_\oplus}\right)^3 \approx 10^6 \quad (16)$$

and so

$$\boxed{V_\odot \approx 10^6 V_\oplus}. \quad (17)$$

The actual value is around 1.31×10^6: our estimate is off again because we just scaled from the Jovian value above, and Jupiter is actually bit more than 10× the size of the Earth.