UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2022

Problem Set 4
Due: Monday, Feb 21, 2022, at the start of class
This problem set is worth **51 points** (+10 bonus).

As always, be sure to: show your work, circle your final answer, and use the appropriate number of significant figures.

1. **Stellar Types [24 pts]**

   (a) Devise your own mnemonic for remembering the stellar spectral sequence. [1/2 pt per spectral type]

   Solution: For the Stellar types (if we include the brown dwarfs through L/T/Y), we might say:

   Orangutans build a fine grass kimono, merely lowering their yields.

   Onagers been angling for good Khartoum mincemeat, lightly toasted: yeah!

   (b) Estimate the approximate stellar type and luminosity class for the three stars in PSet3, Question 2 (i.e., the giant star, the small star, and the hot star). [6 pts]

   Solution: The first star: a star larger than the Sun but the same temperature would be a subgiant or giant, probably luminosity class IV or III. Since it has the same temperature as the Sun (which is G2V), this star would be G2III/IV.

   The second star: a small, relatively cool star would be a 'red dwarf.' It would have spectral type M and luminosity class V. If you want to get a bit more specific, this would be (roughly) an M2V star.

   The third star: Such a small, hot star would be a 'white dwarf,' the leftover core of a dead star. This is what our Sun will become in roughly 5 Gyr. (We haven't learned about white dwarfs yet — these stars really confused astronomers, too, when they were first discovered! If you guessed an "O dwarf" since the star is both hot and small, you'll get full credit).

   (c) For the next three questions, consider two stars with different apparent brightness in the B and R photometric bandpasses. From your photometry, you correctly infer than star one is redder, and star two is bluer.

      i. Which star is hotter and which cooler? Explain. If one can't tell from the information given, describe what additional information you would need. [5 pts]
      Solution: The brightness in the blue B and red R bands will tell us which star is hotter or cooler; hotter objects tend to have bluer spectra, while cooler objects tend to be redder. So
      $\boxed{\text{Star 2 has a bluer color and is likely hotter than star 1}}$.

      ii. Which star is (intrinsically) brighter and which fainter? Explain. If one can't tell from the information given, describe what additional information you would need. [5 pts]
      Solution: All we're given is the colors of the two stars. To know their intrinsic brightness (or absolute magnitude) we would need to know the particular B- or R-band brightness measured, along with the distance to the star. Stars that are hot or cool can both be either large or small depending on the luminosity class (dwarf, giant, etc.). So we can't tell from the information given.

      iii. Which star is more massive and which less? Explain. If one can't tell from the information given, describe what additional information you would need. [5 pts]
      Solution: We have just as little insight here as we did for the luminosity of the two stars. A hot star can be anywhere from $2M_\odot$ to $50M_\odot$ (or it could even be a tiny, but hot, white dwarf!), and a cool star could be anywhere from $0.1M_\odot$ to $10-20M_\odot$. So while star 1 might be slightly more likely to be lower-mass, it wouldn't be right to claim that with any certainty. Instead, we'd need some additional pieces of information: if the star were in a binary we could measure its mass via the Doppler shift; if we took a detailed spectra we might be able to derive a precise mass by comparing it to spectra of other, known stars.

2. **Angles & Distance in a Binary Star System [8 pts]**

   (a) You observe a binary star system (two stars orbiting each other) and from your observations you correctly infer that the stars are separated from each other by 100 au and are both 500 pc from the Earth. (i) What is the parallax to the binary system? (ii) What is the maximum angular separation of the two stars as viewed from Earth? [8 pts]

   Solution: For the parallax, we just use the same equation as before. We find

$$\boxed{\theta_{plx} = \frac{1''}{500} = 2 \text{ mas}}. \tag{1}$$

   The maximum angular separation of this binary is just the ratio of the distance scales involved:

$$\theta_{bin} = \frac{100 \text{ au}}{500 \text{ pc}}. \tag{2}$$

   Since the parallax equation tells us that 1 AU and 1 pc makes an angle of 1", the binary thus subtends an angle on the sky of:

$$\boxed{\theta_{bin} = \frac{100''}{500} = 200 \text{ mas}}. \tag{3}$$

   Often the astrometric (angular) motion resulting from stars orbiting each other is much larger than the parallax signal that gives us the distance to the system. But as we'll see in this class, both quantities are often of considerable importance.

3. **A Circumbinary Planet [13 pts].** A star system has two similar stars orbiting each other; the orbit has an *apparent* (i.e., angular) semimajor axis of 2", and the parallax to the system is 100 mas (milli-arcsec).

   (a) Estimate the physical semimajor axis of the orbit in AU. [4 pts]

   Solution: The apparent (angular) semimajor axis is ϕ=2". If we knew the distance we could calculate the physical semimajor axis a. Instead we're only given the parallax, which we can use to find the distance.

   If the parallax were 1" then the distance would be 10 pc; the parallax is actually $10\times$ less so the distance is greater by the same amount. Thus, $d = 10$ pc.

   If the distance were just 1 pc, then a 2" semimajor axis would imply $a = 2$ au. Instead the system is $10\times$ futher away, so we must have $\boxed{a = 20 \text{ au}}$.

   (b) If the orbital period of the binary is estimated to be 100 yr, estimate the total mass in the binary and the approximate mass of each component, in $M_\odot$. What kind of stars are these likely to be? [5 pts]

   Solution: We know from Kepler's Laws that

$$(P/1 \text{ yr})^2 (M_{tot}/M_\odot) = (a/1 \text{ au})^3. \tag{4}$$

   In this case $P \approx 100$ yr and $a = 20$ au, so we have

$$M_{tot} \approx 20^3 \times 100^{-2} \text{ yr} \rightarrow \boxed{M_{tot} \approx 0.8 M_\odot}. \tag{5}$$

   Since we're told that the stars are similar, we can safely assume $\boxed{m_1 \approx m_2 \approx M_{tot}/2 = 0.4 M_\odot}$. With this mass, the system must be an M-dwarf binary.

   (c) The binary is discovered to host a circumbinary planet — i.e., a planet in a wide orbit going around both of the two stars — at an average angular separation from the binary of 20 arcsec on the sky. Assuming the planet is on a circular orbit, estimate the planet's semimajor axis in AU, orbital period in years, and orbital velocity in SI units. [4 pts]

   Solution: We know the orbital velocity (assuming a circular orbit) will just be $v = 2\pi a/P$, but we don't know a or P. We can estimate both of these as in the preceding parts.

Since the angular semimajor axis is 20" and we know $d = 10$ pc, we must have $a = 200$ au. Then from Kepler's Laws, we can estimate the period as

$$P/(1 \text{ yr}) \approx 200^{3/2} \times 0.8^{-1/2} \approx 2800 \times 1.1 \rightarrow P \approx 3000 \text{ yr}. \tag{6}$$

(Note that we have assumed M_{tot} is just the mass of the stars – this is a reasonable assumption if we know that this new object is a planet, since in general $M_{planet} \ll M_*$).
This means that the planet's orbital velocity is just

$$v = \frac{2\pi \times 200 \text{ au}}{3000 \text{ yr}} \approx \frac{1200 \times 1.5 \times 10^{11}}{3000 \times 10^7} \text{m s}^{-1} \approx \boxed{3 \times 10^3 \text{ m s}^{-1}}. \tag{7}$$

4. **Order of Magnitude [6 pts]**. In SI units:

   (a) Estimate the luminosity of an ice cube.

   (b) Estimate the luminosity of your body.

   (c) Estimate the luminosity of the Earth.

   Solution: For all of these, the general solution will take the form $L = A\sigma T^4$, where A is the surface area of the object and $\sigma \approx 6 \times 10^{-8}$ in SI units. Let's take an ice cube that's s=2 cm across, so $A_{ice} = 6s^2 \approx 6(0.02\text{m})^2 \sim 0.002 \text{ m}^2$. I'll assume a tall, narrow, cylindrical human body with height of 2 m and radius 10 cm, so $A_{me} \approx 2\pi(2 \times 0.1) \text{ m}^2 \sim 1 \text{ m}^2$. Finally, the Earth is nearly a sphere and so $A_\oplus = 4\pi R_\oplus^2 \approx (12)(6,000,000\text{m})^2 \sim 4 \times 10^{14}\text{m}^2$.

   We then have:

$$L_{ice} \approx (0.002)(6 \times 10^{-8})(270 \text{ K})^4 \approx 81 \times 10^{-2} \sim \boxed{1 \text{ W}}. \tag{8}$$

   An ice cube emits roughly 1 W of power! More than I would have guessed...

$$L_{me} \approx (1)(6 \times 10^{-8})(300 \text{ K})^4 \approx 6 \times 81 \sim \boxed{500 \text{ W}}. \tag{9}$$

   ... but the ice cube still emits a lot less than a human body...

$$L_\oplus \approx (4 \times 10^{14})(6 \times 10^{-8})(300 \text{ K})^4 \approx 24 \times 10^6 \times 81 \times 10^8 \sim \boxed{2 \times 10^{17} \text{ W}}. \tag{10}$$

   ... and both ice cubes and humans emit *a lot* less than the Earth (which is itself a lot less than the sun, $L_\odot \approx 4 \times 10^{26}$ W).

5. **Bonus: Orbital Energy [10 pts]**. Assume a planet of mass m is in a circular orbit with semimajor axis a around a star of mass M_*.

   (a) What is the gravitational potential energy of this two-body binary system? [1 pt]

   Solution: From introductory physics, this is just

$$\boxed{U_G = -\frac{GmM_*}{a}}. \tag{11}$$

   (b) What is the kinetic energy of the system in terms of the planet's orbital velocity v? [1 pt]

   Solution: From introductory physics, this is just

$$\boxed{K = \frac{1}{2}mv^2}. \tag{12}$$

(c) Describe (or draw a diagram) indicating the direction and magnitude of the astronomically relevant forces acting on the planet. Remember, things like 'centripetal forces' aren't real *physical* forces – they're essentially just (useful!) descriptions of motion. [2 pts]

Solution: The only force acting on the planet is the force of gravity from the star it orbits. This is an attractive force, pulling the planet toward the star (the force is perpendicular to the planet's circular orbital motion). The magnitude of the force is given by

$$\boxed{F_{p,\mathrm{tot}} = \frac{GmM_*}{a^2}}. \tag{13}$$

(d) Using Newton's Second Law, derive an expression for the planet's orbital velocity v^2 in terms of the other quantities given. [3 pts]

Solution: Newton's second law is just $F_{tot} = ma = m\ddot{r}$ (writing the acceleration as the second time derivative of r, lest we confuse acceleration and semimajor axis).. We already calculated the force, immediately above. For our planet, the right-hand side of Newton's second law means that since it is on a circular orbit it is undergoing uniform circular motion — thus it undergoes centripetal acceleration, $\ddot{r} = a_r = v^2/r$. Equating the force (physics) with the mass times acceleration (dynamics), we have:

$$\frac{GmM_*}{a^2} = \frac{mv^2}{a}. \tag{14}$$

Rearranging for v^2, we have

$$\boxed{v^2 = \frac{GM_*}{a}}. \tag{15}$$

(e) Write an expression for the total energy in the system in terms of m, M_*, and a but not v. Simplify your answer as much as possible. [3 pts]

Solution: Since the total mechanical energy of a system is just $E_{tot} = U + K$, we have

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GmM_*}{a}. \tag{16}$$

Plugging in our expression for v^2 above, we find that

$$E_{tot} = \tfrac{1}{2}\tfrac{GmM_*}{a} - \tfrac{GmM_*}{a} \tag{17}$$

$$= \boxed{-\frac{1}{2}\frac{GmM_*}{a}}. \tag{18}$$

This result indicates that the total energy of a gravitational system is typically just $E_{tot} = U_g/2$, a result known as the **virial theorem** that is of great interest to dynamicists.