UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2022

Problem Set 5
Due: Monday, Mar 21, 2021, in class
This problem set is worth **30 points** (plus 10 potential bonus points).

As always, be sure to: type and print your solutions, show your work, circle your final answer, and use the appropriate number of significant figures.

1. **Journey to the Center of a Star [30 pts]**

(a) [5 pts] In lecture we discussed at some length how we can model the physical conditions in a star's interior. Assume the star has a slightly more realistic density profile (valid from $0 \leq r \leq R_*$) of

$$\rho(r) = \rho_c \left((r/R_*)^2 - 2r/R_* + 1 \right) = \rho_c \left(\frac{r}{R_*} - 1 \right)^2 \tag{1}$$

Plot $\rho(r)$ over the full range from $r = 0$ to $r = 2R_*$. Discuss why this density profile might be slightly more realistic than the constant-density model we assumed in class.

Solution:

Just by plugging in a few test values, we can see how the density behaves. For $r = 0$ we have $\rho = \rho_c$, and for $r = R_*$ we have $\rho = 0$, so the density appears to decrease from the center out to the surface, as we expect. For an intermediate value such as $r = R_*/2$, we see $\rho = \rho_c/4$; from this and more test points, or

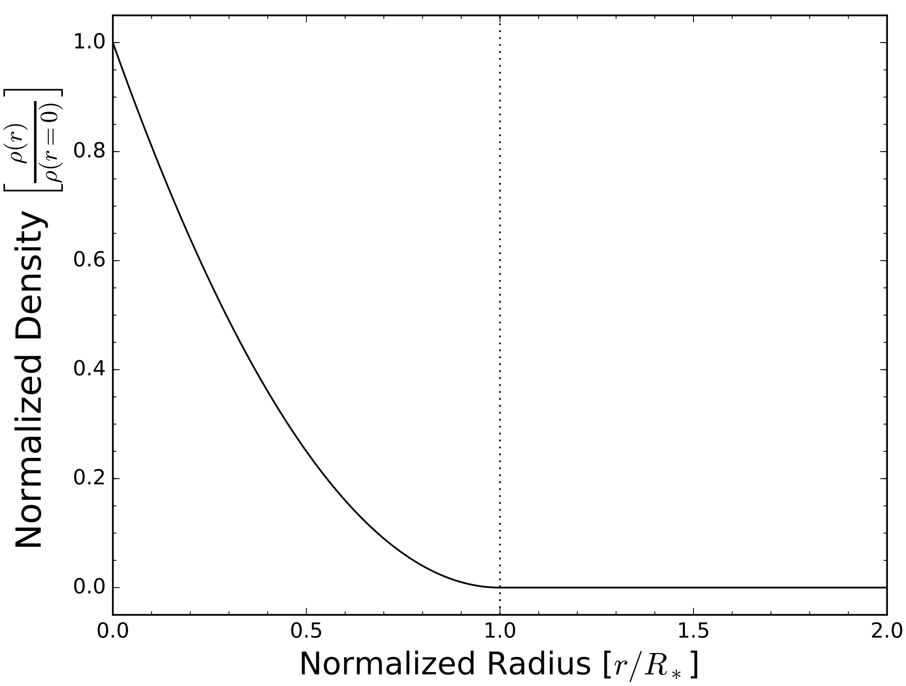

Figure 1: Density in the star for the functional form given. The vertical dotted line indicates the stellar surface.

from the quadratic functional form, we could see that $\rho(r)$ peaks at the core and decreases quadratically down to zero at $r = R_*$. The full plot is shown in Fig. 1.

This is a bit more realistic because we should expect density (along with pressure, temperature, and related quantities) to be highest in the core and smallest out near the surface.

(b) [5 pts] Show that the expression for the Enclosed Mass $M_{enc}(r)$ at an arbitrary radius r within this star is

$$4\pi\rho_c r^3 \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right) \tag{2}$$

Solution: To determine the enclosed mass within a radius r we use the recipe

$$M_{enc}(r) \equiv \int_{r=0}^{r=r} 4\pi r^2 \rho(r) dr. \tag{3}$$

Plugging in our expression given for $\rho(r)$, we have

$$M_{enc}(r) \equiv \int_{r=0}^{r=r} 4\pi r^2 \rho_c \left((r/R_*)^2 - 2r/R_* + 1\right) dr \tag{4}$$

$$= 4\pi\rho_c \int_{r=0}^{r=r} r^2 \left((r/R_*)^2 - 2r/R_* + 1\right) dr \tag{5}$$

$$= 4\pi\rho_c \int_{r=0}^{r=r} \left((r^4/R_*^2) - 2r^3/R_* + r^2\right) dr \tag{6}$$

$$= 4\pi\rho_c \left(\frac{r^5}{5R_*^2} - 2\frac{r^4}{4R_*} + \frac{r^3}{3}\right) \tag{7}$$

$$= \boxed{4\pi\rho_c r^3 \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)} \tag{8}$$

(9)

(c) [5 pts] Using the expression above for $M_{enc}(r)$, show that the gravitational acceleration $g_{in}(r)$ inside this star will be

$$g(r) = 4\pi\rho_c G r \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right). \tag{10}$$

.

Solution: Remember that the gravitational acceleration in a star is just

$$g_{\text{inside}}(r) \equiv \frac{GM_{enc}(r)}{r^2}. \tag{11}$$

This means the gravitational acceleration inside a star is just

$$g(r) = \frac{G}{r^2} 4\pi\rho_c r^3 \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right) \tag{12}$$

$$= \boxed{4\pi\rho_c G r \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)} \tag{13}$$

(14)

(d) [6 pts] Plot $M_{enc}(r)$ and $g_{in}(r)$ over the range from $r = 0$ to $2R_*$.

Solution: The plots are shown in Fig. 2.

(e) [4 pts] Starting with the equation of hydrostatic equilibrium ($dP/dr = -\rho(r)g(r)$), we could continue in this vein and calculate the internal pressure and temperature of the star – but things would quickly get really messy.

Instead, we can make a rough approximation to get a sense of the conditions inside the star, by assuming that $dP/dR \approx P_c/R_*$ (i.e., the pressure at the center of the star divided by the star's radius), and further assuming density and gravity are constant: $\rho(r) = \rho_{\text{avg}}$ and $g(r) = g_{\text{surface}}$.

Under these simplifying assumptions, the pressure at the center of the star is just $P_c \approx \rho_{\text{avg}} g_{\text{surf}} R_*$. Calculate a numerical value of P_c (in SI units) for the Sun, and for a red dwarf with $M_*/M_\odot = R_*/R_\odot = 0.3$. How do these compare to the atmospheric pressure here on Earth?

Solution: As directed, we make the greatly-simplifying assumption that the central pressure in the star is just $P_c \approx \rho_{\rm avg} g_{\rm surf} R_*$. We're asked to estimate this quantity for both the Sun and an M dwarf. We'll need to calculate their average density ($M_*/[\frac{4}{3}\pi R_*^3]$) and surface gravity ($GM_*/R_*^2$). So the final expression we want is

$$
\begin{aligned}
P_c &\approx \rho_{\rm avg} g_{\rm surf} && (15)\\
&\approx \frac{M_*}{\frac{4}{3}\pi R_*^3}\frac{GM_*}{R_*^2} R_* && (16)\\
&\approx \frac{3GM_*^2}{4\pi R_*^4}. && (17)
\end{aligned}
$$

Star	$M_*/M_\odot$	$R_*/R_\odot$	$\rho_{\rm avg}$ [kg/m^3]	$g_{\rm surf}$ [m/s^2]	P_c [Pa]
Sun	1	1	1400	270	2.7×10^{14}
M dwarf	0.3	0.3	15700	910	3.0×10^{15}

Pressure at sea level is 1 bar$\approx 10^5$ Pa, so the pressure at the center of these stars is roughly 10 billion times greater!

(f) [5 pts] Assume that our star is an ideal gas made entirely of hydrogen atoms. In this case, derive a symbolic expression for the temperature T_c at the center of a star in terms of its pressure and density.

Then, calculate a numerical value for the central temperature of both the Sun and the M dwarf described above. How do these compare to the surface temperatures of these stars?

Solution: Since we're told to assume that the star is an ideal gas, it must obey

$$P = nk_BT \quad (18)$$

everywhere (including at its core, where we'd specifically be considering P_c, n_c, and T_c).

Since we're asked to give the answer in terms of density (not number density), we'll have to remember that $n = \rho/m_{\rm avg}$, where $m_{\rm avg}$ is the average mass of particles under consideration – in this case, just hydrogen atoms. So our expression is then

$$\boxed{T_c \approx \frac{m_{\rm avg}}{k_B}\frac{P_c}{\rho_{\rm avg}}}. \quad (19)$$

$$
\begin{aligned}
T_c &\approx \frac{P_c m_{\rm avg}}{\rho k_B} && (20)\\
&\approx \frac{3Gm_{\rm avg}}{4\pi k_B}\frac{M_*^2}{R_*^4\rho_{\rm avg}} && (21)\\
&\approx \frac{Gm_{\rm avg}}{k_B}\frac{M_*}{R_*} && (22)\\
& && (23)
\end{aligned}
$$

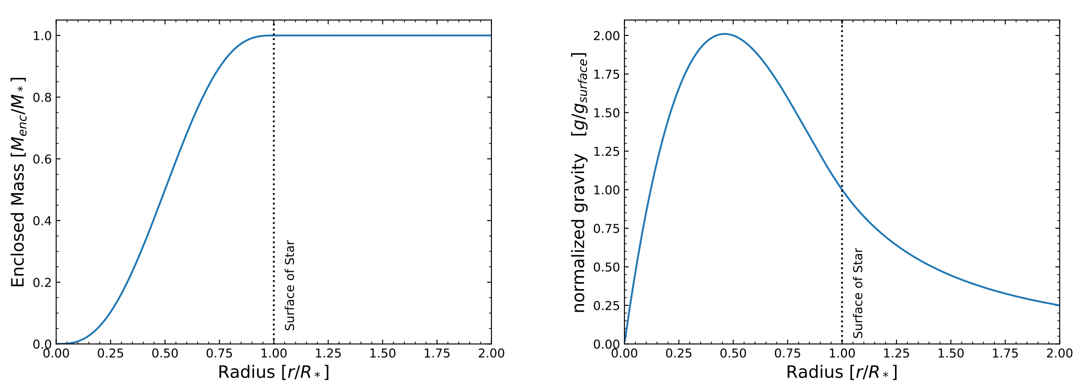

Figure 2: Enclosed mass (*left*) and gravity *right*) in the star. The vertical dotted line indicates the stellar surface.

Taking $m_{\text{avg}} = m_H \approx (1/6) \times 10^{-26}$ kg, $k_B \approx (1/7) \times 10^{22}$ J/K, and $G \approx (2/3) \times 10^{-11}$ (in SI units), we then find

$$\boxed{T_c \approx 2.3 \times 10^7 \text{ K}} \tag{24}$$

for both the Sun and for the M dwarf.

2. BONUS: Under Pressure [10 pts]

   (a) [6 pts] Using Equations 1, 2, and 10, calculate an analytic expression (i.e., a formula) for the pressure inside the star.

   Solution: As we discussed in class, the equation of hydrostatic equilibrium can be rewritten in integral form to get the pressure profile, $P(r)$, throughout the interior of a star. This is:

$$P(r) = \int_r^{R_*} \rho(r) g(r) dr \tag{25}$$

   Since Eqs. 1 and 10 give $\rho(r)$ and $g(r)$, respectively, we can plug them in and then find:

$$P(r) = \int_r^{R_*} \left[\rho_c \left((r/R_*)^2 - 2r/R_* + 1\right)\right] \left[4\pi\rho_c G r \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)\right] dr \tag{26}$$

$$= 4\pi\rho_c^2 G \int_r^{R_*} \left[\left((r/R_*)^2 - 2r/R_* + 1\right)\right] \left[r \left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)\right] dr \tag{27}$$

(28)

   This will quickly become a morass of polynomial terms — if you don't use your favorite tool (e.g., Mathematica or something similar) or just calculate P numerically then trouble is likely near.

   (b) [4 pts] Plot $P(r)$ over the range from $r = 0$ to $2R_*$.

   Solution: I chose a numerical solution, and the result is shown in Fig. 3.

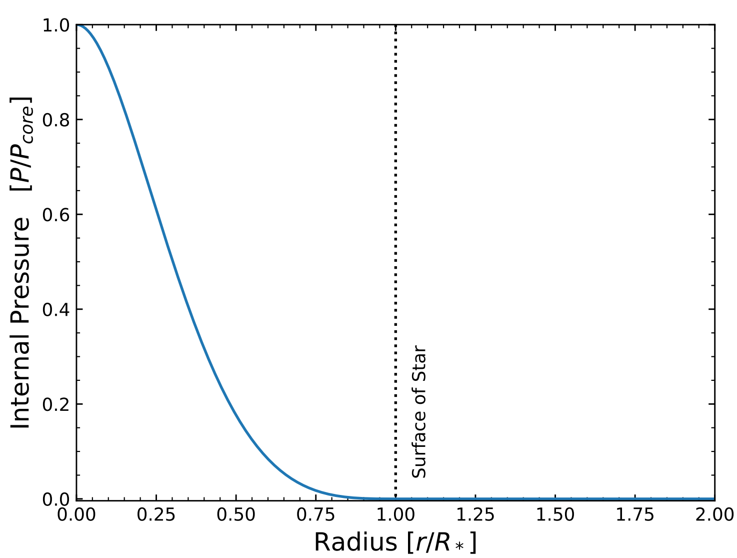

Figure 3: Pressure in the star for the quadratic density profile given in Eq. 1. The vertical dotted line indicates the stellar surface.