UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2022

Problem Set 6
Due: Monday, April 6, 2022, 11am Kansas Time
This problem set is worth **50 points**.

As always, be sure to: show your work, circle your final answer, and use the appropriate number of significant figures.

1. **Nuclear binding energies [15 pts]**

   For each of the following nuclear reactions, look up the mass of each of the following nuclei (e.g., on Wikipedia) and calculate ΔM, the change in mass from the ingredients to the products, in atomic mass units (amu). Then, via $\Delta E = \Delta M c^2$ (and/or by noting that $[1\text{amu } c^2 \approx 931.494 \text{ MeV}]$) compute the amount of energy released or absorbed by each reaction ($\Delta E > 0$ means energy is released).

   (a) $4\,p \rightarrow {}^4_2\text{He}$ (the total p-p chain)
   (b) $3\,{}^4_2\text{He} \rightarrow {}^{12}\text{C}$ (the triple-α reaction)
   (c) ${}^{12}\text{C} + {}^{12}\text{C} \rightarrow {}^{24}\text{Mg}$
   (d) ${}^{12}\text{C} + {}^{12}\text{C} \rightarrow {}^{16}\text{O} + 2{}^4\text{He}$
   (e) ${}^{19}\text{F} + {}^1\text{H} \rightarrow {}^{16}\text{O} + {}^4\text{He}$
   (f) ${}^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu$
   (g) $56\,p \rightarrow {}^{56}\text{Fe}$ (the full process, occurring only in the most massive stars)

   Solution:

   (a) $4\,p \rightarrow {}^4_2\text{He}$
   $\Delta M = 4 \times 1.0073 - 4.0026 = 0.027$ amu
   $\Delta E = 25$ MeV

   (b) $3\,{}^4_2\text{He} \rightarrow {}^{12}\text{C}$
   $\Delta M = 3 \times 4.0026 - 12 = 0.0078$ amu
   $\Delta E = 7.3$ MeV

   (c) ${}^{12}\text{C} + {}^{12}\text{C} \rightarrow {}^{24}\text{Mg}$ ΔE = 13.96 MeV
   (d) ${}^{12}\text{C} + {}^{12}\text{C} \rightarrow {}^{16}\text{O} + 2{}^4\text{He}$ ΔE = -0.112 MeV
   (e) ${}^{19}\text{F} + {}^1\text{H} \rightarrow {}^{16}\text{O} + {}^4\text{He}$ ΔE = 8.129 MeV
   (f) ${}^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{H} + e^+ + \nu$ ΔE = 1.445 MeV
   (g) $56p \rightarrow {}^{56}\text{Fe}$
   $\Delta M = 56 \times 1.0073 - 55.9349 = 0.474$ amu
   $\Delta E = 441$ MeV

   All the reactions are exothermic (producing energy) except for (d).

2. **Final Fates of Stars [20 pts].** Starting with its life on the main sequence, enumerate and describe the main stages in the life of a star with an initial (main-sequence) mass of:

   (a) $0.2 M_\odot$
   (b) $1 M_\odot$

(c) $2M_\odot$
(d) $10M_\odot$
(e) $20M_\odot$
(f) $40M_\odot$

Solution:

(a) $0.2M_\odot$: The star will slowly fuse H to He while on the Main Sequence throughout its entire life (which will be for many tens of Gyr). The star is fully convective and so fully and homogeneously mixed, and will continue fusing H until all is used up. Then it will just slowly fade away.

(b) $1M_\odot$: (i) Stars like the sun will fuse H to He in their core for their MS lifetime. (ii) Once the core H is used up, the star will become a red giant. (iii) The outer layers will be lost, and the core will remain as a hot He white dwarf (surrounded by a planetary nebula). The WD will then slowly cool and fade away.

(c) $2M_\odot$: Steps (i) and (ii) will be similar to the $1M_\odot$ star, but this star will likely subsequently (iii) fuse He to C in its core, perhaps while still fusing H to He in an intermediate shell. (iv) The star will still lose its outer layers and end up as a white dwarf, likely a Carbon white dwarf.

(d) $10M_\odot$: Steps (i), (ii), and (iii) will be similar to the $2M_\odot$ star, but further fusion in the core (and additional shell-burning layers) will result. (iv) Eventually an Fe core will be left which cannot support itself, and (v) the core will collapse and the star will explode in a "core-collapse supernova." (vi) Only a neutron star will be left (surrounded by a hot, thin expanding cloud of debris).

(e) $20M_\odot$: This star will likely follow the same steps as the $10M_\odot$ star above, but its overall lifetime will be shorter.

(f) $40M_\odot$: This star will follow most of the same steps as the $10M_\odot$ star, but (vi) its collapse will be so unstoppable that the central regions will fall inside an event horizon, leaving a Black Hole behind.

3. **Compact Objects [15 pts].**

(a) What are the typical mass and size of a **white dwarf**? From these numbers, calculate (in SI units) a typical white dwarf's average density, surface gravity, and roughly estimate (via $dP/dr \sim P_c/R = \rho g$) its central pressure P_c.

Solution: The typical mass and radius of a white dwarf are something like $\sim M_\odot$ and $\sim R_\oplus$, respectively. Plugging in these values, we find

$$\rho_{avg} = \frac{M}{\frac{4}{3}\pi R^3} = \boxed{1.8 \times 10^9 \text{ kg m}^{-3}} \tag{1}$$

and

$$g_{surf} = \frac{GM}{R^2} = \boxed{3.3 \times 10^6 \text{ m s}^{-2}} \tag{2}$$

which gives

$$P_{surf} \approx \rho g R = \boxed{3.8 \times 10^{22} \text{ Pa}}. \tag{3}$$

(b) What are the typical mass and size of a **neutron star**? From these numbers, calculate (in SI units) a typical neutron star's average density, surface gravity, and roughly estimate (via $dP/dr \sim P_c/R = \rho g$) its central pressure P_c.

Solution: The typical mass and radius of a neutron star are something like $\sim M_\odot$ and 2 km, respectively. Plugging in these values, we find

$$\rho_{avg} = \frac{M}{\frac{4}{3}\pi R^3} = \boxed{5.9 \times 10^{19} \text{ kg m}^{-3}} \tag{4}$$

and

$$g_{surf} = \frac{GM}{R^2} = \boxed{3.3 \times 10^{13} \text{ m s}^{-2}} \tag{5}$$

which gives

$$P_{surf} \approx \rho g R = \boxed{3.9 \times 10^{36}\ \mathrm{Pa}}. \tag{6}$$

Pretty extreme!

(c) What is a typical mass of a stellar-remnant **black hole**? What is the size of its event horizon? From these numbers, calculate (in SI units) the surface gravity at the event horizon. Describe how the surface gravity would change if the black hole were more massive.

Solution: A typical stellar-remnant black hole has a mass of a few solar masses; maybe even $\sim 10M_\odot$, given the black hole discoveries from LIGO via gravitational waves.

The size of a black hole's event horizon is

$$R_S \approx (3\ \mathrm{km})\frac{M_{BH}}{M_\odot} \approx \boxed{30\ \mathrm{km}}. \tag{7}$$

The surface gravity at the event horizon will be

$$g_{EH} = \frac{GM_{BH}}{R_S^2} = \frac{GM_{BH}}{\left[(3\ \mathrm{km})\frac{M_{BH}}{M_\odot}\right]^2} = \frac{GM_\odot^2}{(3\ \mathrm{km})^2 M_{BH}}. \tag{8}$$

So the gravity at the event horizon actually gets weaker for more massive black holes — not necessarily an intuitive result!

For a black hole with mass $10M_\odot$, its surface gravity will be $\boxed{g_{EH} \approx 1.5 \times 10^{12}\ \mathrm{m\ s^{-2}}}$. Yow!