UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2020

Problem Set 6
Due: Monday, April 27, 20222, 11am Kansas Time
This problem set is worth **53 points**.

As always, be sure to: show your work, circle your final answer, and use the appropriate number of significant figures.

1. **Our home, the Milky Way [15 pts]** Draw a rough sketch of our Milky Way galaxy as it might be viewed from an outside observer, as both a top-down and edge-on view. At a minimum, include and label the Galactic Center, bulge, halo, disk, spiral arms, and the Sun, as well as a scale bar.

   Solution: A search online can find dozens (if not hundreds) of Milky Way schematic diagrams. Go check them out!

2. **Other Galaxies [15 pts]**

   (a) [3 pts] List three ways astronomers might measure the distance to a distant Galaxy.
   Solution:

   i. Observe Cepheid variable stars in it and use the period-luminosity relation (along with the stars' apparent brightness) to estimate the distance.
   ii. Detect a type-Ia supernova in the galaxy, and use its light curve and apparent maximum brightness as a "standardizable candle" to estimate the distance.
   iii. Measure the redshift of the Galaxy's spectrum, and use the Hubble Law to estimate the distance.
   iv. Measure the Galaxy's rotation speed, and use the Tully-Fisher (or Faber-Jackson) relation to estimate its intrinsic luminosity; with its apparent brighness, estimate the distance.

   (b) [3 pts] A 2019 article in the journal *Nature* celebrated that year's Nobel Prize by writing that the award was given for "the first report of an exoplanet orbiting around a Sun-like star in another galaxy." Describe what is wrong with this statement, and how you would correct it.

   Solution: The poor author apparently didn't know the difference between a solar system (one star orbited by a few planets) and a galaxy (billions and billions of stars all orbiting their common center of mass). They should have said "the first report of an exoplanet orbiting around a Sun-like star in another solar system" (or "in our Milky Way galaxy").

   (c) [3 pts] The closest sizable galaxy to the Milky Way is Andromeda (also known as M31), which you can see from the Northern hemisphere on a dark night in the Autumn. If Andromeda is 770 kpc away, and is roughly the same physical size as the Milky Way, what is its approximate angular diameter on the sky?

   Solution: We know the angular size of an object with radius R and distance from the observer d will just be $2R/d$ radians, or roughly $100R/d$ degrees. The radius of the Milky Way's disk is roughly 20 kpc – if we assume Andromeda is the same size, then we have $R/d = 20/770 \approx 0.026$, and so Andromeda should be 2–3 degrees across on the sky. Lo, various websites also confirm that it is around 178 arcmin $\approx$ 3 degrees across!

   (d) [6 pts] Sketch the 'tuning-fork' diagram used for galaxy classification, and describe the properties of the main types of galaxies shown there.

   Solution: These include **elliptical galaxies** which can have the largest mass and are typically reddish-colored due to their lack of young stars and **spiral galaxies,** which are also fairly massive but have much more gas and dust, much higher rates of star formation, and so often have a more bluish color. There are both 'typical' spirals and 'barred' spirals, as well as 'lenticular' galaxies whose properties may be intermediate between those of ellipticals and spirals.

   There are also irregular and dwarf galaxies, but they don't appear on the usual 'tuning form' classification diagram.

3. **Space is Big. Really Big. [10 pts]** Plot the distance to the following objects on a (1D) logarithmic number line (e.g., with regularly-spaced intervals on the page indicating 1/10, 1, 10×, etc.):

   (a) Moon

   (b) Sun (closest star)

   (c) Neptune

   (d) Alpha Centauri (closest non-Sun star)

   (e) Sagittarius A*

   (f) Large Magellanic Cloud

   (g) Andromeda Galaxy

   (h) Distant galaxy with redshift $z = 1$ ($d = 4.3$ Gpc).

   Solution: All of these distances can be looked up, and converted to parsecs. You can then plot them, e.g. as follows:

```
from pylab import *

distances = [1.3e-8, 4.8e-6, 1.4e-4, 1.3, 8500, 50000, 770000, 4.3e9]
names = ['Moon', 'Sun', 'Neptune', '$\\alpha$ Centauri', 'Sgr A*', 'LMC',
    'Andromeda', 'z=1']

figure(1, [2, 8])
gca().set_position([0.3, 0.03, .65, .9])
for dist,nam in zip(distances, names):
   semilogy(0, dist, 'ok')
   text(0, dist, ' '+nam, rotation=45, fontsize=12, verticalalignment='bottom')
   xticks([0], [])
   ylabel('Distance from Earth [pc]', fontsize=14)

yl = ylim()
xlim(-.25, .75)
plot([0,0], yl, ':k')
gca().set_yticks(10.**np.arange(-8, 11), minor=True)
ylim(yl)
```

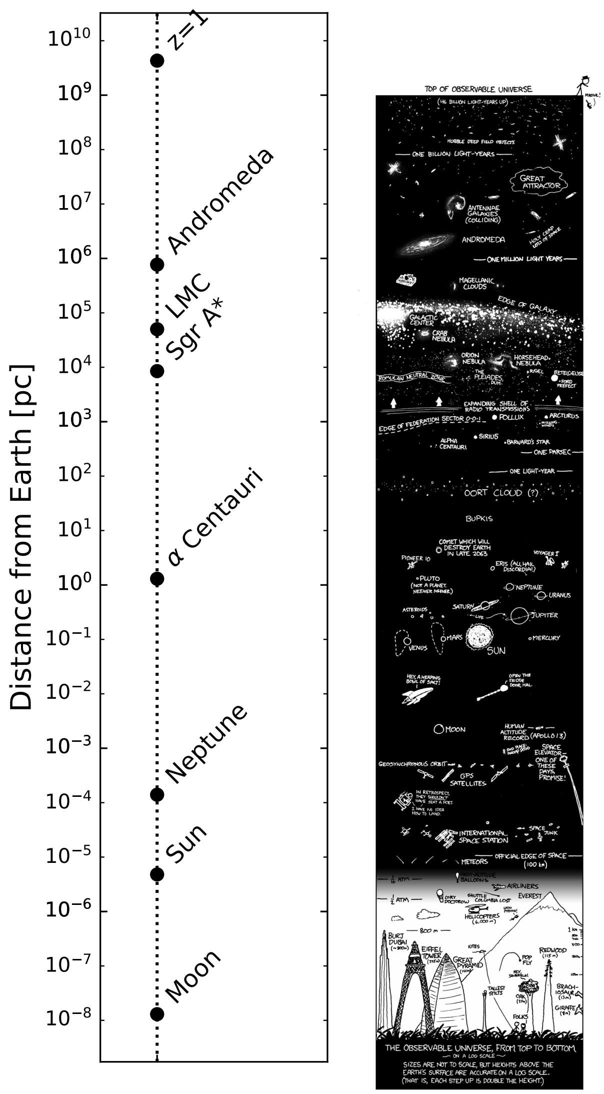

Figure 1: Left: Plotted distances. Right: from XKCD.

4. **Rotation Curves [13 pts]** A key insight in astronomy is that if we can see one object orbiting another, we can measure the total mass enclosed by the orbit using Kepler's Third Law. In Solar-System units, this is

$$\left(\frac{P}{\mathrm{yr}}\right)^2 \frac{M_{\mathrm{enclosed}}}{M_\odot} = \left(\frac{r}{\mathrm{AU}}\right)^3. \tag{1}$$

This applies to binary asteroids, planets with moons, stars with planets, and – as you will see here – even for distant galaxies.

(a) [5 pts] Using the expression above, show that for a small mass in a circular orbit a distance r from a much larger amount of mass M_{enclosed}, the orbital speed just depends on the mass interior to the orbit as

$$v_{\mathrm{orb}} \approx 30 \text{ km s}^{-1} \sqrt{\frac{M_{\mathrm{enc}}/M_\odot}{r/\mathrm{AU}}}. \tag{2}$$

(As an alternative approach, you can try equating the gravitational force between the objects to the expression for a centripetal force.)

Solution: We know that an object's speed in an orbit of radius r and with period P is just $v = 2\pi r/P$. To get r/P from Kepler's third law, we just rearrange it to get

$$\left(\frac{r}{\mathrm{AU}}\right)^2 \left(\frac{P}{\mathrm{yr}}\right)^{-2} = \frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot} \tag{3}$$

and so

$$\frac{r}{\mathrm{AU}} \left(\frac{P}{\mathrm{yr}}\right)^{-1} = \left(\frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot}\right)^{1/2}, \tag{4}$$

which becomes

$$\frac{r}{P} = \left(\frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot}\right)^{1/2} \frac{\mathrm{AU}}{\mathrm{yr}} \approx 4.7 \left(\frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot}\right)^{1/2} \text{ km s}^{-1}. \tag{5}$$

We then plug this into $v = 2\pi r/P \approx 6.3 r/P$, to find

$$v_{orb} \approx 6.3 \times 4.7 \left(\frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot}\right)^{1/2} \text{ km s}^{-1} \approx \boxed{30 \left(\frac{\mathrm{AU}}{r} \frac{M_{\mathrm{enc}}}{M_\odot}\right)^{1/2} \text{ km s}^{-1}}. \tag{6}$$

(b) [3 pts] A wide range of objects orbit our Sun, from Mercury out to Neptune, Pluto, and beyond. Assuming all objects orbiting the Sun have circular orbits, plot their orbital speeds vs. semimajor axis — i.e., plot the above expression v_{orb}, the rotation curve for things orbiting around a single massive object.

Solution: Fig. 2 plots the orbital velocity, assuming a circular orbit, for the Solar System.

(c) [5 pts] Roughly sketch the rotation curve (v_{orb} vs. r) for a typical spiral galaxy. How is this similar to the previous plot you drew, for the Solar System? how is it different? What do the differences imply for what makes up most of the mass in galaxies, vs. what makes up most of the mass in a Solar system?

Solution: A typical rotation curve is shown in Fig. 3. The main similarities are just qualitative, in that both plots show orbital velocity vs. orbital semimajor axis (distance from the center). The key difference is that in the Solar System orbital velocity drops rapidly as r increases, but in a galaxy orbital velocity is (roughly) constant even to very large separations – the speed drops much more slowly than would be predicted. It turns out that this is because Dark Matter (not stars or gas or dust!) makes up most of the mass in a galaxy, while in a Solar System the star is by far the most significant mass present.

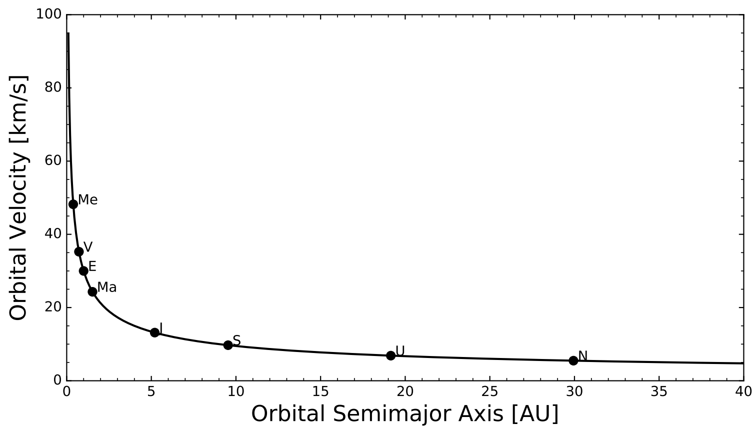

Figure 2: Orbital velocity vs. semimajor axis, for Solar System objects.

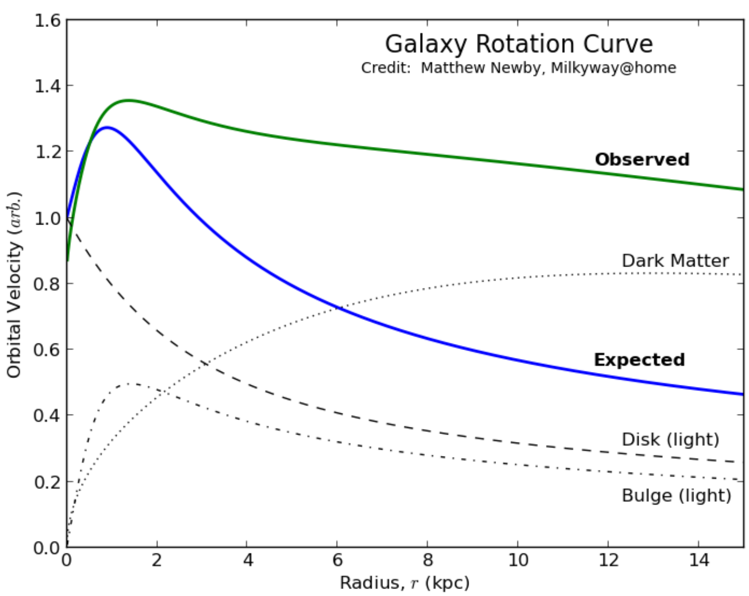

Figure 3: Rotational velocity for stars in a spiral galaxy (from `https://milkyway.cs.rpi.edu/download/images/gal_rotation_curve.png`). The observed rotation curve is in green: stars orbit much more rapidly than would be naively expected (blue, predicted rotation speeds) – this is because of the presence of dark matter (contribution indicated by dotted line).

(d) [13 pts] **BONUS POINTS** (optional): Assume you have a spherically symmetric cloud of material, with a density profile $\rho(r) = \ell_0 r^{-2}$, where ℓ_0 is some reference density with units kg/m. (Obviously this functional form is nonphysical since $\rho(r=0) \to \infty$, but set that aside for now – it's only a model).

i. Calculate M_{enclosed} as a function of r and ℓ_0 (as necessary). [3 pts]
Solution: As we recall from modeling stellar interiors, we know that

$$M_{\text{enc}}(r) = \int_0^r 4\pi r^2 \rho(r) dr. \tag{7}$$

In this case, we'll then have

$$\begin{aligned} M_{\text{enc}}(r) &= \int_0^r 4\pi r^2 \ell_0 r^{-2} dr & (8)\\ &= 4\pi \ell_0 \int_0^r dr & (9)\\ &= \boxed{4\pi \ell_0 r} & (10) \end{aligned}$$

ii. Use M_{enclosed} to calculate the orbital velocity v_{orb} for objects orbiting in this cloud as a function of r and ℓ_0 (as necessary). [4 pts]
Solution: As for the orbital velocity, we know

$$v_{\text{orb}} \approx 30 \text{ km s}^{-1} \sqrt{\frac{M/M_\odot}{r/\text{AU}}}. \tag{11}$$

For the situation described above, this will become

$$\begin{aligned} v_{\text{orb}} &\approx 30 \text{ km s}^{-1} \sqrt{4\pi} \sqrt{\tfrac{\ell_0 r}{M_\odot} \tfrac{\text{AU}}{r}} & (12)\\ &\approx \boxed{100 \text{ km s}^{-1} \sqrt{\frac{\ell_0 \times (1 \text{ AU})}{M_\odot}}}. & (13) \end{aligned}$$

iii. If this object is a galaxy and the rotation speed of stars in its outer reaches ($r_{\max} \approx 10$ kpc) is observed to be roughly 200 km/s, what is ℓ_0 (in kg/m)? Then (the final goal!) use ℓ_0 to calculate the total mass involved, $M_{\text{enc}}(r = r_{\max})$, in Solar masses $M_\odot$. [6 pts]
Solution: If the rotation speed is given by

$$v_{\text{orb}} = 200 \text{ km s}^{-1} \approx 100 \text{ km s}^{-1} \sqrt{\frac{\ell_0 \times (1 \text{ AU})}{M_\odot}} \tag{14}$$

then we must have

$$\frac{\ell_0 \times (1 \text{ AU})}{M_\odot} = 4 \tag{15}$$

and so

$$\boxed{\ell_0 = 4 \frac{M_\odot}{1 \text{ AU}} \approx 5.3 \times 10^{19} \text{ kg m}^{-1}}. \tag{16}$$

Based on the relation for enclosed mass, we then have:

$$\begin{aligned} M_{\text{enc}} &= 4\pi \ell_0 r & (17)\\ &= 4\pi \left(5.3 \times 10^{19} \text{ kg m}^{-1}\right) (10 \text{ kpc}) & (18)\\ &\approx 12 \left(5.3 \times 10^{19} \text{ kg m}^{-1}\right) \left(10^4 \times 3 \times 10^{16} \text{ m}\right) & (19)\\ &\approx 12 \times 5.3 \times 3 \left(\times 10^{19} \text{ kg m}^{-1}\right) \left(10^{20} \text{ m}\right) & (20)\\ &\approx 200 \left(\times 10^{39} \text{ kg}\right) & (21)\\ &\approx 100 \left(\times 10^{9} M_\odot\right) & (22)\\ &\approx \boxed{10^{11} M_\odot} & (23) \end{aligned}$$

So this rotation curve and galaxy mass describe a system not so very different from the Milky Way!