1

FUNDAMENTALS OF RADIATIVE TRANSFER

1.1 THE ELECTROMAGNETIC SPECTRUM; ELEMENTARY PROPERTIES OF RADIATION

Electromagnetic radiation can be decomposed into a *spectrum* of constituent components by a prism, grating, or other devices, as was discovered quite early (Newton, 1672, with visible light). The spectrum corresponds to waves of various wavelengths and frequencies, related by $\lambda\nu = c$, where ν is the frequency of the wave, λ is its wavelength, and $c = 3.00 \times 10^{10}$ cm s^{-1} is the free space velocity of light. (For waves not traveling in a vacuum, c is replaced by the appropriate velocity of the wave in the medium.) We can divide the spectrum up into various regions, as is done in Figure 1.1. For convenience we have given the energy $E = h\nu$ and temperature $T = E/k$ associated with each wavelength. Here h is Planck's constant $= 6.625 \times 10^{-27}$ erg s, and k is Boltzmann's constant $= 1.38 \times 10^{-16}$ erg K^{-1}. This chart will prove to be quite useful in converting units or in getting a quick view of the relevant magnitude of quantities in a given portion of the spectrum. The boundaries between different regions are somewhat arbitrary, but conform to accepted usage.

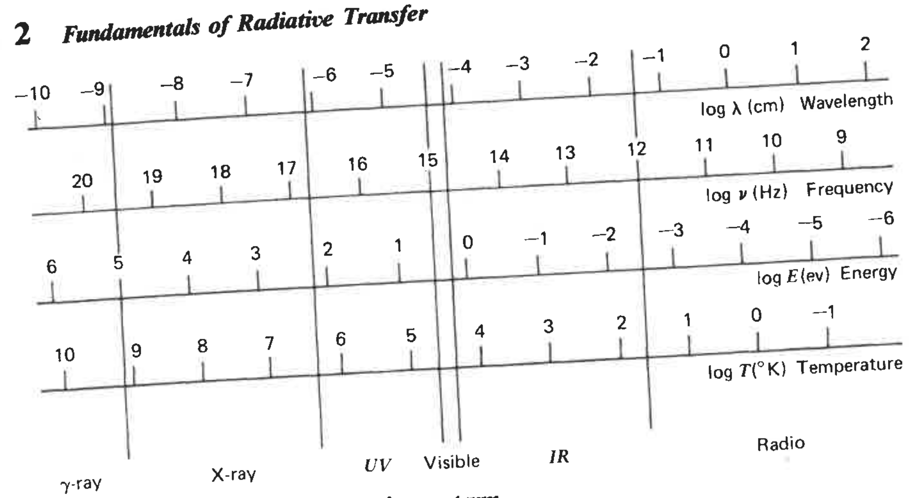

Figure 1.1 *The electromagnetic spectrum.*

1.2 RADIATIVE FLUX

Macroscopic Description of the Propagation of Radiation

When the scale of a system greatly exceeds the wavelength of radiation (e.g., light shining through a keyhole), we can consider radiation to travel in straight lines (called rays) in free space or homogeneous media—from this fact a substantial theory (transfer theory) can be erected. The detailed justification of this assumption is considered at the end of Chapter 2. One of the most primitive concepts is that of *energy flux*: consider an element of area dA exposed to radiation for a time dt. The amount of energy passing through the element should be proportional to $dA\,dt$, and we write it as $F\,dA\,dt$. The energy flux F is usually measured in erg s^{-1} cm^{-2}. Note that F can depend on the orientation of the element.

Flux from an Isotropic Source—the Inverse Square Law

A source of radiation is called *isotropic* if it emits energy equally in all directions. An example would be a spherically symmetric, isolated star. If we put imaginary spherical surfaces S_1 and S at radii r_1 and r, respectively, about the source, we know by conservation of energy that the total energy passing through S_1 must be the same as that passing through S. (We assume no energy losses or gains between S_1 and S.) Thus

$$F(r_1)\cdot 4\pi r_1^2 = F(r)\cdot 4\pi r^2,$$

or

$$F(r) = \frac{F(r_1)r_1^2}{r^2}.$$

If we regard the sphere S_1 as fixed, then

$$F = \frac{\text{constant}}{r^2}. \qquad (1.1)$$

This is merely a statement of conservation of energy.

1.3 THE SPECIFIC INTENSITY AND ITS MOMENTS

Definition of Specific Intensity or Brightness

The flux is a measure of the energy carried *by all rays* passing through a given area. A considerably more detailed description of radiation is to give the energy carried along by *individual rays*. The first point to realize, however, is that a single ray carries essentially no energy, so that we need to consider the energy carried by sets of rays, which differ infinitesimally from the given ray. The appropriate definition is the following: Construct an area dA normal to the direction of the given ray and consider all rays passing through dA whose direction is within a solid angle $d\Omega$ of the given ray (see Fig. 1.2). The energy crossing dA in time dt and in frequency range $d\nu$ is then defined by the relation

$$dE = I_\nu\, dA\, dt\, d\Omega\, d\nu, \qquad (1.2)$$

where I_ν is the *specific intensity* or *brightness*. The specific intensity has the

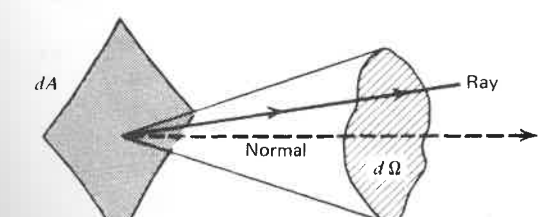

Figure 1.2 *Geometry for normally incident rays.*

dimensions

$$I_\nu(\nu,\Omega) = \text{energy (time)}^{-1}\ (\text{area})^{-1}\ (\text{solid angle})^{-1}\ (\text{frequency})^{-1}$$
$$= \text{ergs s}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}\ \text{Hz}^{-1}.$$

Note that I_ν depends on location in space, on direction, and on frequency.

Net Flux and Momentum Flux

Suppose now that we have a radiation field (rays in all directions) and construct a small element of area dA at some arbitrary orientation $\mathbf{n}$ (see Fig. 1.3). Then the differential amount of flux from the solid angle $d\Omega$ is (reduced by the lowered effective area $\cos\theta\, dA$)

$$dF_\nu(\text{erg s}^{-1}\ \text{cm}^{-2}\ \text{Hz}^{-1}) = I_\nu \cos\theta\, d\Omega. \tag{1.3a}$$

The *net flux* in the direction $\mathbf{n}$, $F_\nu(\mathbf{n})$ is obtained by integrating dF over all solid angles:

$$F_\nu = \int I_\nu \cos\theta\, d\Omega. \tag{1.3b}$$

Note that if I_ν is an isotropic radiation field (not a function of angle), then the net flux is *zero*, since $\int \cos\theta\, d\Omega = 0$. That is, there is just as much energy crossing dA in the $\mathbf{n}$ direction as the $-\mathbf{n}$ direction.

To get the flux of momentum normal to dA (momentum per unit time per unit area = pressure), remember that the momentum of a photon is E/c. Then the momentum flux along the ray at angle θ is dF_ν/c. To get

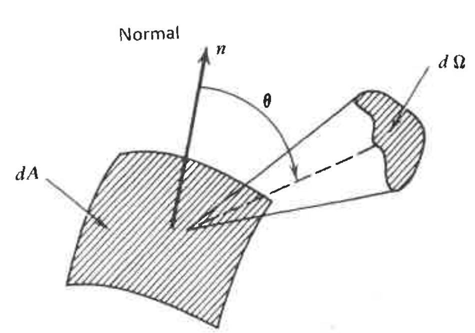

Figure 1.3 *Geometry for obliquely incident rays.*

the component of momentum flux normal to dA, we multiply by another factor of $\cos\theta$. Integrating, we then obtain

$$p_\nu(\text{dynes cm}^{-2}\ \text{Hz}^{-1}) = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega. \tag{1.4}$$

Note that F_ν and p_ν are *moments* (multiplications by powers of $\cos\theta$ and integration over $d\Omega$) of the intensity I_ν. Of course, we can always integrate over frequency to obtain the total (integrated) flux and the like.

$$F(\text{erg s}^{-1}\ \text{cm}^{-2}) = \int F_\nu\, d\nu \tag{1.5a}$$

$$p(\text{dynes cm}^{-2}) = \int p_\nu\, d\nu \tag{1.5b}$$

$$I(\text{erg s}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}) = \int I_\nu\, d\nu \tag{1.5c}$$

Radiative Energy Density

The specific energy density u_ν is defined as the energy per unit volume per unit frequency range. To determine this it is convenient to consider first the energy density per unit solid angle $u_\nu(\Omega)$ by $dE = u_\nu(\Omega)\, dV\, d\Omega\, d\nu$ where dV is a volume element. Consider a cylinder about a ray of length ct (Fig. 1.4). Since the volume of the cylinder is $dA\, c\, dt$,

$$dE = u_\nu(\Omega)\, dA\, c\, dt\, d\Omega\, d\nu.$$

Radiation travels at velocity c, so that in time dt all the radiation in the cylinder will pass out of it:

$$dE = I_\nu\, dA\, d\Omega\, dt\, d\nu.$$

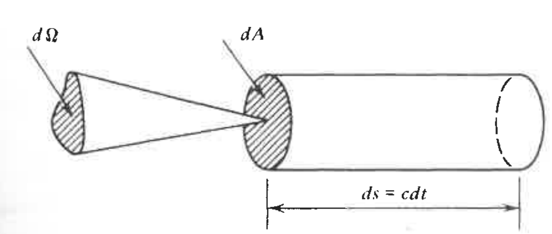

Figure 1.4 *Electromagnetic energy in a cylinder.*

Equating the above two expressions yields

$$u_\nu(\Omega)=\frac{I_\nu}{c}. \tag{1.6}$$

Integrating over all solid angles we have

$$u_\nu=\int u_\nu(\Omega)\,d\Omega=\frac{1}{c}\int I_\nu\,d\Omega,$$

or

$$u_\nu=\frac{4\pi}{c}J_\nu, \tag{1.7}$$

where we have defined the *mean intensity* J_ν:

$$J_\nu=\frac{1}{4\pi}\int I_\nu\,d\Omega. \tag{1.8}$$

The total radiation density (erg cm^{-3}) is simply obtained by integrating u_ν over all frequencies

$$u=\int u_\nu\,d\nu=\frac{4\pi}{c}\int J_\nu\,d\nu. \tag{1.9}$$

Radiation Pressure in an Enclosure Containing an Isotropic Radiation Field

Consider a reflecting enclosure containing an isotropic radiation field. Each photon transfers *twice* its normal component of momentum on reflection. Thus we have the relation

$$p_\nu=\frac{2}{c}\int I_\nu\cos^2\theta\,d\Omega.$$

This agrees with our previous formula, Eq. (1.4), since here we integrate only over 2π steradians. Now, by isotropy, $I_\nu=J_\nu$ so

$$p=\frac{2}{c}\int J_\nu\,d\nu\int\cos^2\theta\,d\Omega.$$

The angular integration yields

$$p=\tfrac{1}{3}u. \tag{1.10}$$

The radiation pressure of an isotropic radiation field is one-third the energy density. This result will be useful in discussing the thermodynamics of blackbody radiation.

Constancy of Specific Intensity Along Rays in Free Space

Consider any ray L and any two points along the ray. Construct areas dA_1 and dA_2 normal to the ray at these points. We now make use of the fact that energy is conserved. Consider the energy carried by that set of rays passing through both dA_1 and dA_2 (see Fig. 1.5). This can be expressed in two ways:

$$dE_1=I_{\nu_1}dA_1\,dt\,d\Omega_1\,d\nu_1=dE_2=I_{\nu_2}dA_2\,dt\,d\Omega_2\,d\nu_2.$$

Here $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 and so forth. Since $d\Omega_1=dA_2/R^2$, $d\Omega_2=dA_1/R^2$ and $d\nu_1=d\nu_2$, we have

$$I_{\nu_1}=I_{\nu_2}.$$

Thus the intensity is constant along a ray:

$$I_\nu=\text{constant}. \tag{1.11}$$

Another way of stating the above result is by the differential relation

$$\frac{dI_\nu}{ds}=0, \tag{1.12}$$

where ds is a differential element of length along the ray.

Proof of the Inverse Square Law for a Uniformly Bright Sphere

To show that there is no conflict between the constancy of specific intensity and the inverse square law, let us calculate the flux at an arbitrary

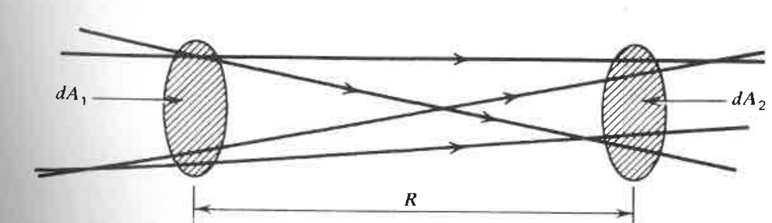

Figure 1.5 Constancy of intensity along rays.

***Figure 1.6** Flux from a uniformly bright sphere.*

distance from a sphere of uniform brightness B (that is, all rays leaving the sphere have the same brightness). Such a sphere is clearly an isotropic source. At P, the specific intensity is B if the ray intersects the sphere and zero otherwise (see Fig. 1.6). Then,

$$F=\int I\cos\theta\, d\Omega = B\int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta\cos\theta\, d\theta,$$

where $\theta_c = \sin^{-1} R/r$ is the angle at which a ray from P is tangent to the sphere. It follows that

$$F=\pi B(1-\cos^2\theta_c)=\pi B\sin^2\theta_c$$

or

$$F=\pi B\left(\frac{R}{r}\right)^2. \qquad (1.13)$$

$= \frac{L}{4\pi r^2} \Rightarrow B = \frac{L}{4\pi^2 r^2}$

Thus the specific intensity is constant, but the solid angle subtended by the given object decreases in such a way that the inverse square law is recovered.

A useful result is obtained by setting $r=R$:

$$F=\pi B. \qquad (1.14)$$

That is, the flux at a surface of uniform brightness B is simply πB.

1.4 RADIATIVE TRANSFER

If a ray passes through matter, energy may be added or subtracted from it by emission or absorption, and the specific intensity will not in general remain constant. "Scattering" of photons into and out of the beam can also affect the intensity, and is treated later in §1.7 and 1.8.

Emission

The spontaneous *emission coefficient* j is defined as the energy emitted per unit time per unit solid angle and per unit volume:

$$dE = j\, dV\, d\Omega\, dt.$$

A monochromatic emission coefficient can be similarly defined so that

$$dE = j_\nu\, dV\, d\Omega\, dt\, d\nu, \qquad (1.15)$$

where j_ν has units of erg cm^{-3} s^{-1} ster^{-1} Hz^{-1}.

In general, the emission coefficient depends on the direction into which emission takes place. For an *isotropic* emitter, or for a distribution of randomly oriented emitters, we can write

$$j_\nu = \frac{1}{4\pi} P_\nu, \qquad (1.16)$$

where P_ν is the radiated power per unit volume per unit frequency. Sometimes the spontaneous emission is defined by the (angle integrated) *emissivity* ϵ_ν, defined as the energy emitted spontaneously per unit frequency per unit time per unit mass, with units of erg gm^{-1} s^{-1} Hz^{-1}. If the emission is isotropic, then

$$dE = \epsilon_\nu \rho\, dV\, dt\, d\nu \frac{d\Omega}{4\pi}, \qquad (1.17)$$

where ρ is the mass density of the emitting medium and the last factor takes into account the *fraction* of energy radiated into $d\Omega$. Comparing the above two expressions for dE, we have the relation between ϵ_ν and j_ν:

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}, \qquad (1.18)$$

holding for isotropic emission. In going a distance ds, a beam of cross section dA travels through a volume $dV = dA\, ds$. Thus the intensity added to the beam by spontaneous emission is:

$$dI_\nu = j_\nu\, ds. \qquad (1.19)$$

Absorption

We define the *absorption coefficient*, α_ν(cm^{-1}) by the following equation, representing the loss of intensity in a beam as it travels a distance ds (by