15 Stellar Structure

Questions you should be able to answer after these lectures:

- What equations, variables, and physics describe the structure of a star?
- What are the two main types of pressure in a star, and when is each expected to dominate?
- What is an equation of state, and what is the equation of state that is valid for the sun?

15.1 *Formalism*

One of our goals in this class is to be able to describe not just the observable, exterior properties of a star, but to understand all the layers of these cosmic onions — from the observable properties of their outermost layers to the physics that occurs in their cores. This next part will then be a switch from some of what we have done before, where we have focused on the "surface" properties of a star (like size, total mass, and luminosity), and considered many of these to be fixed and unchanging. Our objective is to be able to describe the entire internal structure of a star in terms of its fundamental physical properties, and to model how this structure will change over time as it evolves.

Before we define the equations that do this, there are two points that may be useful to understand all of the notation being used here, and the way in which these equations are expressed.

First, when describing the evolution of a star with a set of equations, we will use mass as the fundamental variable rather than radius (as we have mostly been doing up until this point.) It is possible to change variables in this way because mass, like radius, increases monotonically as you go outward in a star from its center. We thus will set up our equations so that they follow individual, moving shells of mass in the star. There are several benefits to this. For one, it makes the problem of following the evolution of our star a more well-bounded problem. Over a star's lifetime, its radius can change by orders of magnitude from its starting value, and so a radial coordinate must always be defined with respect to the hugely time-varying outer extent of the star. In contrast, as our star ages, assuming its mass loss is insignificant, its mass coordinate will always lie between zero and its starting value M — a value which can generally be assumed to stay constant for most stars over most of stellar evolution. Further, by following shells of mass that do not cross over each other, we implicitly assume conservation of mass at a given time, and the mass enclosed by any of these moving shells will stay constant as the star evolves, even as the radius changes. This property also makes it easier to follow compositional changes in our star.

In general, the choice to follow individual fluid parcels rather than reference a fixed positional grid is known as adopting Lagrangian coordinates instead of Eulerian coordinates. For a **Lagrangian** formulation of a problem:

- This is a particle-based description, following individual particles in a fluid over time
- Conservation of mass and Newton's laws apply directly to each particle being followed
- However, following each individual particle can be computationally expensive
- This expense can be somewhat avoided for spherically-symmetric (and thus essentially '1D') problems

In contrast, for a **Eulerian** formulation of a problem:

- This is a field-based description, recording changes in properties at each point on a fixed positional grid in space over time
- The grid of coordinates is not distorted by the fluid motion
- Problems approached in this way are generally less computationally expensive, and are generally easier for 2D and 3D problems

There are thus trade-offs for choosing each formulation. For stellar structure, Lagrangian coordinates are generally preferred, and we will rely heavily on equations expressed in terms of a stellar mass variable going forward.

Second, it might be useful to just recall the difference between the two types of derivatives that you may encounter in these equations. The first is a partial derivative, written as ∂f. The second is a total derivative, written as df. To illustrate the difference, let's assume that f is a function of a number of variables: $f(x,t)$. The partial derivative of f with respect to x is just $\frac{\partial f}{\partial x}$. Here, we have assumed in taking this derivative that x is held fixed with time and does not vary. However, most of the quantities that we will deal with in the equations of stellar structure *do* vary with time. The use of a partial derivative with respect to radius or mass indicates that we are considering the change in this space(like) coordinate for an instantaneous, fixed time value. In contrast, the total derivative does not hold any variables to be fixed, and considers how all of the dependent variables changes as a function of the variable considered. Note that when you see a quantity like $\dot{r}$ in an equation, this is actually the partial rather than total derivative with respect to time.

15.2 *Equations of Stellar Structure*

In this class, we will define four fundamental equations of stellar structure, and several additional relationships that, taken all together, will define the structure of a star and how it evolves with time. Depending on the textbook that you consult, you will find different versions of these equations using slightly different variables, or in a slightly different format.

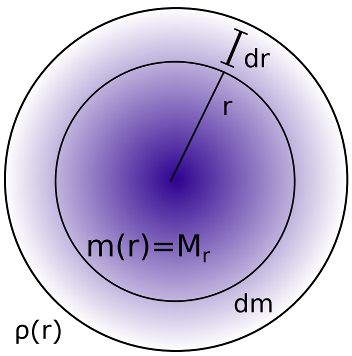

Figure 30: An illustration of a shell with mass dm and thickness dr. The mass enclosed inside of the shell is $m(r)$ (or M_r, depending on how you choose to write it). Assume that this object has a density structure $\rho(r)$

Mass continuity

The first two equations of stellar structure we have already seen before, as the conversion between the mass and radius coordinates

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{274}$$

and as the equation of hydrostatic equilibrium (Eq. 239), now recast in terms of mass:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{275}$$

Eq. 274 and its variant forms are known variously as the **Mass Continuity Equation** or the **Equation of Conservation of Mass**. Either way, this is the first of our four fundamental equations of stellar structure, and relates our mass coordinate m to the radius coordinate r, as shown in Fig. 30.

Note that up until now we have been generally either been assuming a uniform constant density in all of the objects we have considered, or have been making approximations based on the average density $\langle\rho\rangle$. However, to better and more realistically describe stars we will want to use density distributions that are more realistic (e.g., reaching their highest value in the center of the star, and decreasing outward to zero at the edge of the star). This means we should start trying to think about ρ as a function rather than a constant (even when it is not explicitly written as $\rho(r)$ or $\rho(m)$ in the following equations).

Hydrostatic equilibrium

The second equation of stellar structure (Eq. 275, the equation of hydrostatic equilibrium) concerns the motion of a star, and we derived it in Sec. 14.2. As we noted earlier, stars can change their radii by orders of magnitude over the course of their evolution. As a result, we must consider how the interiors of stars move due the forces of pressure and gravity. We have already seen a specific case for this equation: the case in which gravity and pressure are balanced such that there is no net acceleration, and the star is in hydrostatic equilibrium (Equation 239).

We want to first consider a more general form of Eq. 275 that allows for the forces to be out of balance and thus there to be a net acceleration, and second to change variables from a dependence on radius to a dependence on mass. We can begin by rewriting our condition of force balance in Equation 239 as

$$(276)\quad 0 = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}.$$

Each term in this equation has units of acceleration. Thus, this equation can be more generally written as

$$(277)\quad \ddot{r} = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}.$$

Using Equation 263 we can recast this expression in terms of a derivative with respect to m rather than r. This gives us the final form that we will use:

$$(278)\quad \ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2\,\frac{\partial P}{\partial m}.$$

This is the most general form of our second equation of stellar structure. When $\ddot{r}$ is zero we are in equilibrium and so we obtain Eq. 275, the equation of **hydrostatic equilibrium**. This more general form, Eq. 278, is sometimes referred to as the **Equation of Motion** or the **Equation of Momentum Conservation**.

The Thermal Transport Equation

We also need to know how the temperature profile of a star changes with depth. If we do that, we can directly connect the inferred profile of temperature vs. optical depth (Eq. 211) to a physical coordinate within the star.

Assume there is a luminosity profile (determined by the energy equation, to be discussed next), such that the flux at radius r is

$$(279)\quad F(r) = \frac{L(r)}{4\pi r^2}$$

In a plane-parallel atmosphere, we learned (Eq. 202) that the flux is related to the gradient of the radiation pressure. The assumptions we made then don't restrict the applicability of that relation only to the outer atmosphere, so we can apply it anywhere throughout the interior of our star. The only (minor) adjustment is that we replace dz with dr since we are now explicitly considering a spherical geometry, so we now have

$$(280)\quad F = -\frac{c}{\alpha}\frac{dP_{rad}}{dr}$$

Since we know that $P_{rad} = 4/3c\ \sigma_{SB} T^4$ (Eq. 305), we see that

$$\frac{dP_{rad}}{dr} = \frac{16\sigma_{SB}}{3c} T^3 \frac{dT}{dr}. \tag{281}$$

When combined with Eq. 279, we find the thermal profile equation,

$$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB} T^3 r^2} \tag{282}$$

The Energy Equation

Eq. 282 shows that we need to know the luminosity profile in order to determine the thermal profile. In the outer photosphere we earlier required that flux is conserved (Sec. 13.2), but go far enough in and all stars (until the ends of their lives) are liberating extra energy via fusion.

Thus the next equation of stellar structure concerns the generation of energy within a star. As with the equation of motion, we will first begin with a simple case of equilibrium. In this case, we are concerned with the thermodynamics of the star: this is the equation for Thermal Equilibrium, or a constant flow of heat with time for a static star (a situation in which there is no work being done on any of our mass shells).

Consider the shell dm shown in Fig. 30. Inside of this shell we define a quantity ϵ_m that represents the net local gain of energy per time per unit mass (SI units of J s^{-1} kg^{-1}) due to local nuclear processes. Note that sometimes the volumetric power ϵ_r will also sometimes be used, but the power per unit mass ϵ_m is generally the more useful form. Regardless, we expect either ϵ to be very large deep in the stellar core and quickly go to zero in the outer layers where fusion is negligible – in those other regions, $\epsilon = 0$, L is constant, and we are back in the flux-conserving atmosphere of Sec. 13.2.

We then consider that the energy per time entering the shell is L_r (note that like M_r, this is now a local and internal rather than global or external property: it can be thought of as the luminosity of the star as measured at a radius r inside the star) and the energy per time that exits the shell is now $L_r + dL_r$ due to this local gain from nuclear burning in the shell. To conserve energy, we must then have (note that these are total rather than partial derivatives as there is no variation with time):

$$\frac{dL_r}{dm} = \epsilon_m. \tag{283}$$

This is the equation for **Thermal Equilibrium** in a star. While Thermal Equilibrium and Hydrostatic Equilibrium are separate conditions, it is generally unlikely that a star will be in Thermal Equilibrium without already being in Hydrostatic equilibrium, thus guaranteeing that there is no change in the energy flow in the star with time or with work being done. In general, Thermal Equilibrium and Eq. 283 require that any local energy losses in the shell (typically from energy propagating outward in the star) are exactly balanced by the rate of energy production in that shell due to nuclear burning. On a macroscopic scale, it means that the rate at which energy is produced in the center of the star is exactly equal to the star's luminosity: the rate at which that energy exits the surface.

How likely is it that a star satisfies this requirement? While a star may spend most of its life near Thermal Equilibrium while it is on the main sequence, most of the evolutionary stages it goes through do not satisfy Eq. 283: for example, pre-main sequence evolution (protostars) and post-main sequence evolution (red giants). How can we describe conservation of energy for an object that is not in Thermal equilibrium?

Following standard texts (e.g., Prialnik), we can make use of u, the internal energy density in a shell in our star. We can change u either by doing work on the shell, or by having it absorb or emit heat. We have already described how the heat in the shell can change with L_r and ϵ_m. Similarly, the incremental work done on the shell can be defined as a function of pressure and the incremental change in volume:

(284)
$$dW = -PdV$$
(285)
$$= -P\left(\frac{dV}{dm}dm\right)$$
(286)
$$= -P\ d\left(\frac{1}{\rho}\right)dm$$

The change in internal energy per unit mass (du) is equal to the work done per unit mass ($\frac{dW}{dm}$), so finally we can rewrite Eq. 286 as:

(287) $$du = -Pd\left(\frac{1}{\rho}\right)$$

Taking the time derivative of each side,

(288) $$\frac{du}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

Compression of the shell will decrease dV, and thus require energy to be added to the shell, while expansion increases dV and is a way to release energy in the shell.

Changes in the internal energy of the shell u with time can then be described in terms of the both the work done on the shell and the changes in heat:

(289) $$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

The general form of Eq. 289 is the next equation of stellar structure, known either as the **Energy Equation** or the **Equation of Conservation of Energy**.

You may also sometimes see this equation written in various other forms, such as in terms of the temperature T and entropy S of the star. In this form, you then have

$$\frac{\partial L_r}{\partial m} = \epsilon_m - T\frac{dS}{dt} \tag{290}$$

Chemical Composition

An additional relationship that is useful for determining stellar evolution is the change in a star's composition.This relation will be less of an 'equation' for the purposes of this class, and more a rough depiction of how the composition of a star can vary with time.

We can define the composition of a star using a quantity called the mass fraction of a species:

$$X_i = \frac{\rho_i}{\rho}. \tag{291}$$

Here, ρ_i is the partial density of the i^{th} species.

Particles in a star are defined by two properties: their baryon number $\mathcal{A}$ (or the number of total protons and neutrons they contain) and their charge $\mathcal{Z}$. Using the new notation of baryon number, we can rewrite

$$n = \frac{\rho}{\bar{m}}, \tag{292}$$

as the corresponding partial number density of the i^{th} species:

$$n_i = \frac{\rho_i}{\mathcal{A}_i\, m_H}. \tag{293}$$

We can then slightly rewrite our expression for the composition as

$$X_i = n_i \frac{\mathcal{A}_i}{\rho} m_H. \tag{294}$$

Changes in composition must obey (at least) two conservation laws. Conservation of charge:

$$\mathcal{Z}_i + \mathcal{Z}_j = \mathcal{Z}_k + \mathcal{Z}_l. \tag{295}$$

and conservation of baryon number:

$$\mathcal{A}_i + \mathcal{A}_j = \mathcal{A}_k + \mathcal{A}_l. \tag{296}$$

If you also consider electrons, there must also be a conservation of lepton number.

Without attempting to go into a detailed formulation of an equation for the rate of change of X we can see that it must depend on the starting composition and the density, and (though it does not explicitly appear in these equations) the temperature, as this will also govern the rate of the nuclear reactions responsible for the composition changes (analogous to the collision timescale $t_{col} = \frac{v}{nA}$ as shown in Figure 37, in which the velocity of particles is set by the gas temperature). This leads us to our last 'equation' of stellar structure, which for us will just be a placeholder function $\mathbf{f}$ representing that the change in composition is a function of these variables:

$$\dot{\mathbf{X}} = \mathbf{f}(\rho, T, \mathbf{X}). \tag{297}$$

Technically, this $\mathbf{X}$ is a vector representing a series of equations for the change of each X_i.

The final fundamental relation we need in order to derive the structure of a star is an expression for the temperature gradient, which will be derived a bit later on.

15.3 *Pressure*

We have already seen a relationship for the gas pressure for an ideal gas, $P = nkT$. However, now that we have begun talking more about the microscopic composition of the gas we can actually be more specific in our description of the pressure. Assuming the interior of a star to be largely ionized, the gas will be composed of ions (e.g., H^+) and electrons. Their main interactions ('collisions') that are responsible for pressure in the star will be just between like particles, which repel each other due to their electromagnetic interaction. As a result, we can actually separate the gas pressure into the contribution from the ion pressure and the electron pressure:

$$P_{gas} = P_e + P_{ion} \tag{298}$$

For a pure hydrogen star, these pressures will be equivalent, however as the metallicity of a star increases, the electron pressure will be greater than the ion pressure, as the number of free electrons per nucleon will go up (for example, for helium, the number of ions is half the number of electrons).

Assuming that both the ions and electrons constitute an ideal gas, we can rewrite the ideal gas equation for each species:

$$P_e = n_e kT \tag{299}$$

and

$$P_{ion} = n_{ion} kT \tag{300}$$

However, this is not the full story: there is still another source of pressure in addition to the gas pressure that we have not been considering: the pressure from radiation.

Considering this pressure then at last gives us the total pressure in a star:

$$P = P_{ion} + P_e + P_{rad} \tag{301}$$

We can determine the radiation pressure using an expression for pressure that involves the momentum of particles:

$$P = \frac{1}{3}\int_0^\infty v\ p\ n(p)\ dp \tag{302}$$

Here v is the velocity of the particles responsible for the pressure, p is their typical momentum, and $n(p)$ is the number density of particles in the momentum range $(p, p + dp)$. We first substitute in values appropriate for photons ($v = c$, $p = \frac{h\nu}{c}$). What is $n(p)$? Well, we know that the Blackbody (Planck) function (Equation 16) has units of energy per volume per interval of frequency per steradian. So, we can turn this into number of particles per volume per interval of momentum by (1) dividing by the typical energy of a particle (for a photon, this is $h\nu$), then (3) multiplying by the solid angle 4π, and finally (4) using $p = \frac{E}{c}$ to convert from energy density to momentum density.

$$P_{rad} = \frac{1}{3}\,4\pi\int_0^\infty c\left(\frac{h\nu}{c}\right)\left(\frac{1}{h\nu}\right)\left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}}-1\right]^{-1}d\nu \tag{303}$$

Putting this all together,

$$P_{rad} = \frac{1}{3}\left(\frac{4}{c}\right)\left[\pi\int_0^\infty\left(\frac{1}{c}\right)\frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}}-1\right]^{-1}d\nu\right] \tag{304}$$

Here, the quantity in brackets is the same integral that is performed in order to yield the Stefan-Boltzmann law (Equation 126). The result is then

$$P_{rad} = \frac{1}{3}\left(\frac{4}{c}\right)\sigma T^4 \tag{305}$$

The quantity $\frac{4\sigma}{c}$ is generally defined as a new constant, a.

We can also define the specific energy (the energy per unit mass) for radiation, using the relation

$$u_{rad} = 3\frac{P_{rad}}{\rho} \quad (306)$$

When solving problems using the Virial theorem, we have encountered a similar expression for the internal energy of an ideal gas:

$$KE_{gas} = \frac{3}{2}NkT \quad (307)$$

From the ideal gas law for the gas pressure ($P = nkT$), we can see that the specific internal energy $\frac{KE}{\bar{m}}$ then can be rewritten in a similar form:

$$u_{gas} = \frac{P_{gas}}{\rho} \quad (308)$$

15.4 *The Equation of State*

In a star, an equation of state relates the pressure, density, and temperature of the gas. These quantities are generally dependent on the composition of the gas as well. An **equation of state** then has the general dependence $P = P(\rho, T, X)$. The simplest example of this is the ideal gas equation. Inside some stars radiation pressure will actually dominate over the gas pressure, so perhaps our simplest plausible (yet still general) equation of state would be

$$P = P_{gas} + P_{rad} \quad (309)$$

$$= nkT + \frac{4F}{3c} \quad (310)$$

$$= \frac{\rho k T}{\mu m_p} + \frac{4\sigma_{SB}}{3c}T^4 \quad (311)$$

where μ is now the **mean molecular weight per particle** – e.g., $\mu = 1/2$ for fully ionized H.

But a more general and generally applicable equation of state is often that of an adiabatic equation of state. As you might have encountered before in a physics class, an adiabatic process is one that occurs in a system without any exchange of heat with its environment. In such a thermally-isolated system, the change in internal energy is due only to the work done on or by a system. Unlike an isothermal process, an adiabatic process will by definition change the temperature of the system. As an aside, we have encountered both adiabatic and isothermal processes before, in our description of the early stages of star formation. The initial collapse of a star (on a free-fall time scale) is a roughly isothermal process: the optically thin cloud is able to essentially radiate all of the collapse energy into space unchecked, and the temperature does not substantially increase. However, once the initial collapse is halted when the star becomes optically thick, the star can only now radiate a small fraction of its collapse energy into space at a time. It then proceeds to contract nearly adiabatically.

Adiabatic processes follow an equation of state that is derived from the first law of thermodynamics: for a closed system, the internal energy is equal to the amount of heat supplied, minus the amount of work done.

As no heat is supplied, the change in the specific internal energy (energy per unit mass) u comes from the work done by the system. We basically already derived this in Equation 287:

$$du = -Pd\left(\frac{1}{\rho}\right) \tag{312}$$

As we have seen both for an ideal gas and from our expression for the radiation pressure, the specific internal energy is proportional to $\frac{P}{\rho}$:

$$u = \phi\frac{P}{\rho} \tag{313}$$

Where ϕ is an arbitrary constant of proportionality. If we take a function of that form and put it into Equation 312 we recover an expression for P in terms of ρ for an adiabatic process:

$$P \propto \rho^{\frac{\phi+1}{\phi}} \tag{314}$$

We can rewrite this in terms of an adiabatic constant K_a and an adiabatic exponent γ_a:

$$P = K_a\rho^{\gamma_a} \tag{315}$$

For an ideal gas, $\gamma_a = \frac{5}{3}$.

This adiabatic relation can also be written in terms of volume:

$$PV^{\gamma_a} = K_a \tag{316}$$

This can be compared to the corresponding relationship for an ideal gas, in which PV = constant.

15.5 *Summary*

In summary, we have a set of coupled stellar structure equations (Eq. 274, Eq. 278, Eq. 282, Eq. 289, and Eq. 315):

(317) $$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

(318) $$\ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}.$$

(319) $$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB} T^3 r^2}$$

(320) $$\frac{du}{dt} = \epsilon_m - \frac{\partial L_r}{\partial m} - P\frac{d}{dt}\left(\frac{1}{\rho}\right)$$

(321) $$P = K_a \rho^{\gamma_a}$$

If we can solve these together in a self-consistent way, we have good hope of revealing the unplumbed depths of many stars. To do this we will also need appropriate boundary conditions. Most of these are relatively self-explanatory:

(322) $$M(0) = 0$$

(323) $$M(R) = M_{tot}$$

(324) $$L(0) = 0$$

(325) $$L(R) = 4\pi R^2 \sigma_{SB} T_{\rm eff}^4$$

(326) $$\rho(R) = 0$$

(327) $$P(R) \approx 0$$

(328) $$T(R) \approx T_{\rm eff}$$

(329)

To explicitly solve the equations of stellar structure even with all these constraints in hand is still a beast of a task. In practice one integrates numerically, given some basic models (or tabulations) of opacity and energy generation.