CHAPTER 3
The Two-Body Central Force Problem

In this chapter we shall discuss the problem of two bodies moving under the influence of a mutual central force as an application of the Lagrangian formulation. Not all the problems of central force motion are integrable in terms of well-known functions. However, we shall attempt to explore the problem as thoroughly as is possible with the tools already developed.

3–1 REDUCTION TO THE EQUIVALENT ONE-BODY PROBLEM

Consider a monogenic system of two mass points, m_1 and m_2, where the only forces are those due to an interaction potential U. It will be assumed at first that U is any function of the vector between the two particles, $\mathbf{r}_2 - \mathbf{r}_1$, or of their relative velocity, $\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1$, or of any higher derivatives of $\mathbf{r}_2 - \mathbf{r}_1$. Such a system has six degrees of freedom and hence six independent generalized coordinates. Let us choose these to be the three components of the radius vector to the center of mass, $\mathbf{R}$, plus the three components of the difference vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The Lagrangian will then have the form

$$L = T(\dot{\mathbf{R}}, \dot{\mathbf{r}}) - U(\mathbf{r}, \dot{\mathbf{r}}, \ldots). \tag{3–1}$$

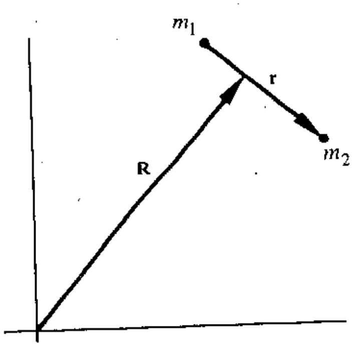

FIGURE 3–1
Coordinates for the two-body problem.

The kinetic energy T can be written as the sum of the kinetic energy of the motion of the center of mass, plus the kinetic energy of motion about the center of mass, T':

$$T = \tfrac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + T'$$

with

$$T' = \tfrac{1}{2}m_1\dot{\mathbf{r}}_1'^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2'^2.$$

Here $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are the radii vectors of the two particles relative to the center of mass and are related to $\mathbf{r}$ by

$$\mathbf{r}_1' = -\frac{m_2}{m_1 + m_2}\mathbf{r},$$
$$\mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{3–2}$$

Expressed in terms of $\mathbf{r}$ by means of Eq. (3–2), T' takes on the form

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2$$

and the total Lagrangian (3–1) is

$$L = \frac{m_1 + m_2}{2}\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, \ldots). \tag{3–3}$$

It is seen that the three coordinates $\mathbf{R}$ are cyclic, so that the center of mass is either at rest or moving uniformly. None of the equations of motion for $\mathbf{r}$ will contain terms involving $\mathbf{R}$ or $\dot{\mathbf{R}}$. Consequently the process of ignoration is particularly simple here. We merely drop the first term from the Lagrangian in all subsequent discussion.

The rest of the Lagrangian is exactly what would be expected if we had a fixed center of force with a single particle at a distance $\mathbf{r}$ from it, having a mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{3–4}$$

where μ is known as the *reduced mass*. Frequently Eq. (3–4) is written in the form

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{3–5}$$

Thus the central force motion of two bodies about their center of mass can always be reduced to an equivalent one-body problem.

3–2 THE EQUATIONS OF MOTION AND FIRST INTEGRALS

We now restrict ourselves to conservative central forces, where the potential is $V(r)$, a function of r only, so that the force is always along $\mathbf{r}$. By the results of the preceding section we need consider only the problem of a single particle of mass m moving about a fixed center of force, which will be taken as the origin of the coordinate system. Since potential energy involves only the radial distance, the problem has spherical symmetry, i.e., any rotation, about any fixed axis, can have no effect on the solution. Hence an angle coordinate representing rotation about a fixed axis must be cyclic. These symmetry properties result in a considerable simplification in the problem. Since the system is spherically symmetric, the total angular momentum vector,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

is conserved. It therefore follows that $\mathbf{r}$ is always perpendicular to the fixed direction of $\mathbf{L}$ in space. This can be true only if $\mathbf{r}$ always lies in a plane whose normal is parallel to $\mathbf{L}$. While the reasoning breaks down if $\mathbf{L}$ is zero, the motion in that case must be along a straight line going through the center of force, for $\mathbf{L} = 0$ requires $\mathbf{r}$ to be parallel to $\dot{\mathbf{r}}$, which can be satisfied only in straight line motion.* Thus, central force motion is always motion in a plane. Now, the motion of a single particle in space is described by three coordinates; in spherical polar coordinates these are the azimuth angle θ, the zenith angle (or colatitude) ψ, and the radial distance r. By choosing the polar axis to be in the direction of $\mathbf{L}$, the motion is always in the plane perpendicular to the polar axis. The coordinate ψ then has only the constant value $\pi/2$ and can be dropped from the subsequent discussion. The conservation of the angular momentum vector furnishes three independent constants of motion (corresponding to the three Cartesian components). In effect, two of these, expressing the constant *direction* of the angular momentum, have been used to reduce the problem from three to two degrees of freedom. The third of these constants, corresponding to the conservation of the magnitude of $\mathbf{L}$, remains still at our disposal in completing the solution.

Expressed now in plane polar coordinates the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \end{aligned} \tag{3–6}$$

As was foreseen θ is a cyclic coordinate, whose corresponding canonical momentum is the angular momentum of the system:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.$$

One of the two equations of motion is then simply

$$\dot{p}_\theta = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{3–7}$$

* Formally: $\dot{\mathbf{r}} = \dot{r}\mathbf{n}_r + r\dot{\theta}\mathbf{n}_\theta$, hence $\mathbf{r} \times \dot{\mathbf{r}} = 0$ requires $\dot{\theta} = 0$.

with the immediate integral

$$mr^2\dot{\theta} = l, \tag{3-8}$$

where l is the constant magnitude of the angular momentum. From (3–7) it also follows that

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0. \tag{3-9}$$

The factor $\frac{1}{2}$ is inserted because $\frac{1}{2}r^2\dot{\theta}$ is just the *areal velocity*—the area swept out by the radius vector per unit time. This interpretation follows from the diagram, Fig. 3–2, the differential area swept out in time dt being

$$dA = \tfrac{1}{2}r(r\,d\theta),$$

and hence

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

The conservation of angular momentum is thus equivalent to saying the areal velocity is constant. Here we have the proof of the well-known Kepler's second law of planetary motion: the radius vector sweeps out equal areas in equal times. It should be emphasized, however, that the conservation of the areal velocity is a general property of central force motion and is not restricted to an inverse square law of force.

The remaining Lagrange equation, for the coordinate r, is

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0. \tag{3-10}$$

Designating the value of the force along $\mathbf{r}$, $-\frac{\partial V}{\partial r}$, by $f(r)$ the equation can be rewritten as

$$m\ddot{r} - mr\dot{\theta}^2 = f(r). \tag{3-11}$$

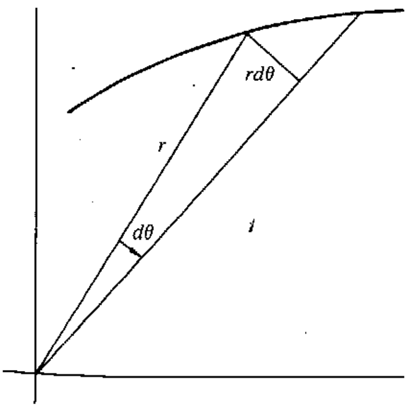

FIGURE 3–2
The area swept out by the radius vector in a time dt.

By making use of the first integral, Eq. (3–8), $\dot{\theta}$ can be eliminated from the equation of motion, yielding a second order differential equation involving r only:

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r). \tag{3–12}$$

There is another first integral of motion available, namely the total energy, since the forces are conservative. On the basis of the general energy conservation theorem we can immediately state that a constant of the motion is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r), \tag{3–13}$$

where E is the energy of the system. Alternatively, this first integral could be derived again directly from the equations of motion (3–7) and (3–12). The latter can be written as

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right). \tag{3–14}$$

If both sides of Eq. (3–14) are multiplied by $\dot{r}$ the left-hand side becomes

$$m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right).$$

The right-hand side similarly can be written as a total time derivative, for if $g(r)$ is any function of r, then the total time derivative of g has the form

$$\frac{d}{dt}g(r) = \frac{dg}{dr}\frac{dr}{dt}.$$

Hence Eq. (3–14) is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{l^2}{mr^2}\right)$$

or

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V\right) = 0,$$

and therefore

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V = \text{constant}. \tag{3–15}$$

Equation (3–15) is the statement of the conservation of total energy, for by using (3–8) for l the middle term can be written

$$\frac{1}{2}\frac{l^2}{mr^2} = \frac{1}{2mr^2}m^2r^4\dot{\theta}^2 = \frac{mr^2\dot{\theta}^2}{2},$$

and (3–15) reduces to (3–13).

These two first integrals give us in effect two of the quadratures necessary to complete the problem. As there are two variables, r and θ, a total of four integrations are needed to solve the equations of motion. The first two integrations have left the Lagrange equations as two first order equations (3–8) and (3–15); the two remaining integrations can be accomplished (formally) in a variety of ways. Perhaps the simplest procedure starts from Eq. (3–15). Solving for $\dot{r}$ we have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}, \tag{3–16}$$

or

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}. \tag{3–17}$$

At time $t = 0$ let r have the initial value r_0. Then the integral of both sides of the equation from the initial state to the state at time t takes the form

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}. \tag{3–18}$$

As it stands Eq. (3–18) gives t as a function of r and the constants of integration E, l, and r_0. However, it may be inverted, at least formally, to give r as a function of t and the constants. Once the solution for r is thus found, the solution θ follows immediately from Eq. (3–8), which can be written as

$$d\theta = \frac{l\,dt}{mr^2}. \tag{3–19}$$

If the initial value of θ is θ_0, then the integral of (3–19) is simply

$$\theta = l\int_0^t \frac{dt}{mr^2(t)} + \theta_0. \tag{3–20}$$

Equations (3–18) and (3–20) are the two remaining integrations, and formally the problem has been reduced to quadratures, with four constants of integration E, l, r_0, $\dot{\theta}_0$. These constants are not the only ones that can be considered. We might equally as well have taken r_0, θ_0, $\dot{r}_0$, $\dot{\theta}_0$, but of course E and l can always be determined in terms of this set. For many applications, however, the set containing the energy and angular momentum is the natural one. In quantum mechanics such constants as the initial values of r and θ, or of $\dot{r}$ and $\dot{\theta}$, become meaningless, but we can still talk in terms of the system energy or of the system angular momentum. Indeed, the salient differences between classical and quantum mechanics appear in the properties of E and l in the two theories. In order to discuss the transition to quantum theories it is important therefore that the classical description of the system be in terms of its energy and angular momentum.

3–3 THE EQUIVALENT ONE-DIMENSIONAL PROBLEM, AND CLASSIFICATION OF ORBITS

While the problem has thus been solved formally, practically speaking the integrals (3–18) and (3–20) are usually quite unmanageable, and in any specific case it is often more convenient to perform the integration in some other fashion. But before obtaining the solution for specific force laws, let us see what can be learned about the motion in the general case, using only the equations of motion and the conservation theorems, without requiring explicit solutions.

For example, with a system of known energy and angular momentum, the magnitude and direction of the velocity of the particle can be immediately determined in terms of the distance r. The magnitude v follows at once from the conservation of energy in the form

$$E = \frac{1}{2}mv^2 + V(r)$$

or

$$v = \sqrt{\frac{2}{m}(E - V(r))}. \tag{3–21}$$

The radial velocity—the component of $\dot{\mathbf{r}}$ along the radius vector—has already been given in Eq. (3–16). Combined with the magnitude v this is sufficient information to furnish the direction of the velocity.* These results, and much more, can also be obtained from consideration of an equivalent one-dimensional problem.

The equation of motion in r, with $\dot{\theta}$ expressed in terms of l, Eq. (3–12), involves only r and its derivatives. It is the same equation as would be obtained for a fictitious one-dimensional problem in which a particle of mass m is subject to a force

$$f' = f + \frac{l^2}{mr^3}. \tag{3–22}$$

The significance of the additional term is clear if it is written as $mr\dot{\theta}^2 = mv_\theta^2/r$, which is the familiar centrifugal force. An equivalent statement can be obtained from the conservation theorem for energy. By Eq. (3–15) the motion of the particle in r is that of a one-dimensional problem with a fictitious potential energy:

$$V' = V + \frac{1}{2}\frac{l^2}{mr^2}. \tag{3–22'}$$

As a check we note that

$$f' = -\frac{\partial V'}{\partial r} = f(r) + \frac{l^2}{mr^3},$$

* Alternatively, the conservation of angular momentum furnishes $\dot{\theta}$, the angular velocity, which together with $\dot{r}$ gives both the magnitude and direction of $\dot{\mathbf{r}}$.

which agrees with Eq. (3–22). The energy conservation theorem (3–15) can thus also be written as

$$E = V' + \frac{1}{2}m\dot{r}^2. \tag{3–15′}$$

As an illustration of this method of examining the motion, consider a plot of V' against r for the specific case of an attractive inverse square law of force:

$$f = -\frac{k}{r^2}.$$

((For positive k the minus sign ensures that the force is *toward* the center of force.) The potential energy for this force is

$$V = -\frac{k}{r}$$

and the corresponding fictitious potential is

$$V' = -\frac{k}{r} + \frac{l^2}{2mr^2}.$$

Such a plot is shown in Fig. 3–3; the two dotted lines represent the separate components

$$-\frac{k}{r} \quad \text{and} \quad \frac{l^2}{2mr^2},$$

and the solid line is the sum V'.

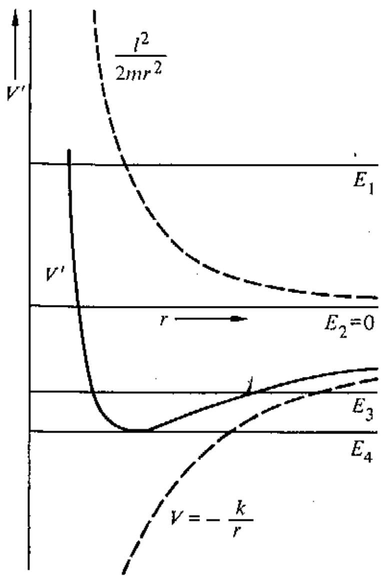

FIGURE 3–3
The equivalent one-dimensional potential for attractive inverse square law of force.

Let us consider now the motion of a particle having the energy E_1, as shown in Figs. 3–3 and 3–4. Clearly this particle can never come closer than r_1 (cf. Fig. 3–4). Otherwise with $r < r_1$, V' exceeds E_1 and by Eq. (3–15′) the kinetic energy would have to be negative, corresponding to an imaginary velocity! On the other hand, there is no upper limit to the possible value of r, so that the orbit is not bounded. A particle will come in from infinity, strike the "repulsive centrifugal barrier," be repelled, and travel back out to infinity (cf. Fig. 3–5). The distance between E and V' is $\frac{1}{2}m\dot{r}^2$, that is, proportional to the square of the radial velocity, and becomes zero, naturally, at the *turning point* r_1. At the same time the distance between E and V' is $\frac{1}{2}m\dot{r}^2$, that is, proportional to the square of the radial velocity, Hence the distance between the V and V' curves is $\frac{1}{2}mr^2\dot{\theta}^2$. These curves therefore supply the magnitude of the particle velocity and its components for any distance r, at the given energy and angular momentum. This information is sufficient to provide an approximate picture of the form of the orbit.

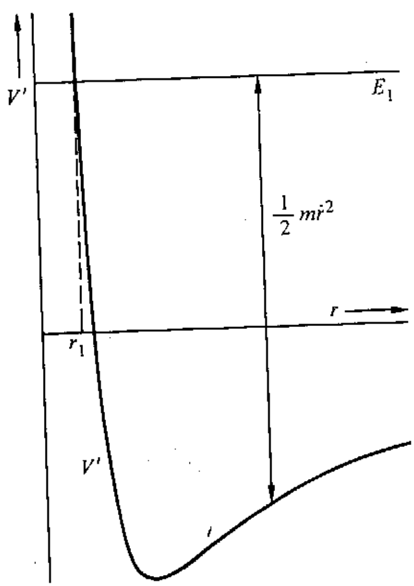

FIGURE 3–4
Unbounded motion at positive energies for inverse square law of force.

For the energy $E_2 = 0$ (cf. Fig. 3–3) a roughly similar picture of the orbit behavior is obtained. But for any lower energy, such as E_3 indicated in Fig. 3–6, we have a different story. In addition to a lower bound r_1, there is also a maximum value r_2 that cannot be exceeded by r with positive kinetic energy. The motion is then "bounded," and there are two turning points, r_1 and r_2, also known as *apsidal distances*. This does not necessarily mean that the orbits are closed. All that can be said is that they are bounded, contained between two circles of radius r_1 and r_2 with turning points always lying on the circles (cf. Fig. 3–7).

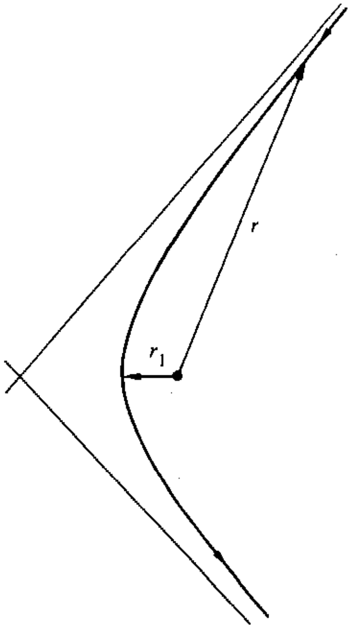

FIGURE 3–5
Schematic picture of the orbit for E_1 corresponding to unbounded motion.

If the energy is E_4 just at the minimum of the fictitious potential as shown in Fig. 3–8, then the two bounds coincide. In such case motion is possible at only one radius; $\dot{r} = 0$, and the orbit is a circle. Remembering that the effective "force" is the negative of the slope of the V' curve, the requirement for circular orbits is simply that f' be zero, or

$$f(r) = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2.$$

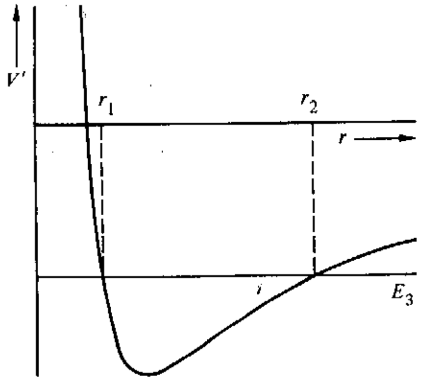

FIGURE 3–6
The equivalent one-dimensional potential for inverse square law of force, illustrating bounded motion at negative energies.

We have here the familiar elementary condition for a circular orbit, that the applied force be equal and opposite to the "reversed effective force" of centripetal

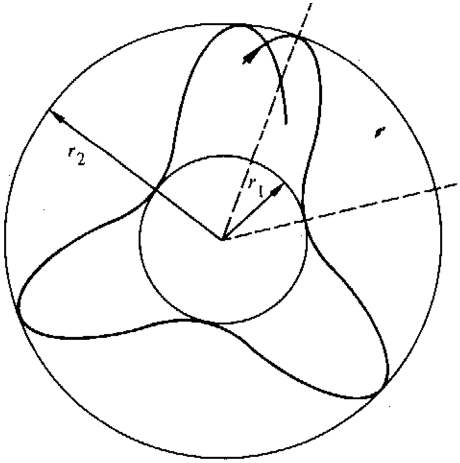

FIGURE 3–7
Schematic illustration of the nature of the orbits for bounded motion.

acceleration.* The properties of circular orbits and the conditions for them will be studied in greater detail below in Section 3–6.

It is to be emphasized that all of this discussion of the orbits for various energies has been at one value of the angular momentum. Changing l will change the quantitative details of the V' curve but it will not affect the general classification of the types of orbits.

For the attractive inverse square law of force discussed above, we shall see that the orbit for E_1 is a hyperbola, for E_2 a parabola, and for E_3 an ellipse. With other forces the orbits may not have such simple forms. However, the same general qualitative division into open, bounded, and circular orbits will be true for any

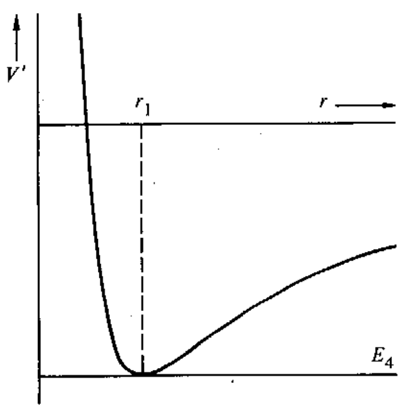

FIGURE 3–8
The equivalent one-dimensional potential of inverse square law of force, illustrating the condition for circular orbits.

* The case $E < E_4$ does not correspond to physically possible motion, for then $\dot{r}^2$ would have to be negative, or $\dot{r}$ imaginary.

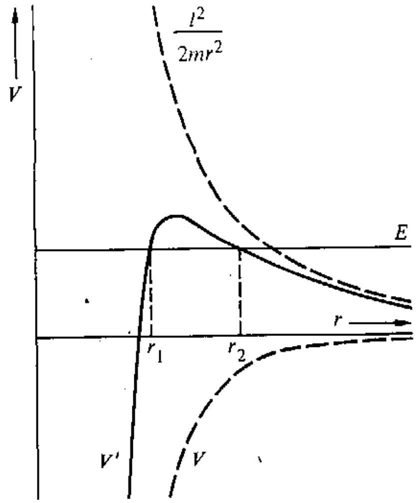

FIGURE 3–9
The equivalent one-dimensional potential for an attractive inverse fourth law of force.

attractive potential that (1) falls off slower than $1/r^2$ as $r \to \infty$; (2) becomes infinite slower than $1/r^2$ as $r \to 0$. The first condition ensures that the potential predominates over the centrifugal term for large r, while the second condition is such that for small r it is the centrifugal term that is the important one.

The qualitative nature of the motion will be altered if the potential does not satisfy these requirements, but we may still use the method of the equivalent potential to examine features of the orbits. As an example, consider the attractive potential

$$V(r) = -\frac{a}{r^3}, \qquad \text{with} \quad f = -\frac{3a}{r^4}.$$

The energy diagram then is as shown in Fig. 3–9. For an energy E there are two possible types of motion, depending upon the initial value of r. If r_0 is less than r_1 the motion will be bounded, r will always remain less than r_1, and the particle will eventually pass through the center of force. If r is initially greater than r_2, then it will always remain so; the motion is unbounded, and the particle can never get inside the "potential" hole. The initial condition $r_1 < r_0 < r_2$ is again not physically possible.

Another interesting example of the method occurs for a linear restoring force (isotropic harmonic oscillator):

$$f = -kr, \qquad V = \frac{1}{2}kr^2.$$

For zero angular momentum, corresponding to motion along a straight line, $V' = V$ and the situation is as shown in Fig. 3–10. For any positive energy the motion is bounded and, as we know, simple harmonic. If $l \neq 0$ we have the state of affairs shown in Fig. 3–11. The motion then is always bounded for all physically

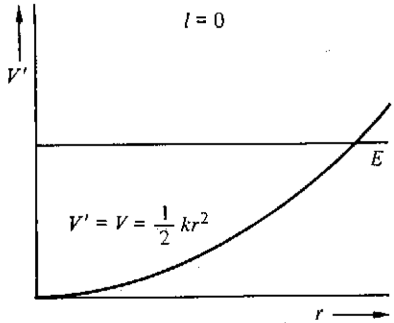

FIGURE 3–10

possible energies and does not pass through the center of force. In this particular case it is easily seen that the orbit is elliptic, for if $\mathbf{f} = -k\mathbf{r}$, the x and y components of the force are

$$f_x = -kx, \qquad f_y = -ky.$$

The total motion is thus the resultant of two simple harmonic oscillations at right angles, and of the same frequency, which in general leads to an elliptic orbit. A well-known example is the spherical pendulum for small amplitudes. The familiar Lissajous figures are obtained as the composition of two sinusoidal oscillations at right angles where the ratio of the frequencies is a rational number.* Central force motion under a linear restoring force therefore constitutes the simplest of the Lissajous figures.

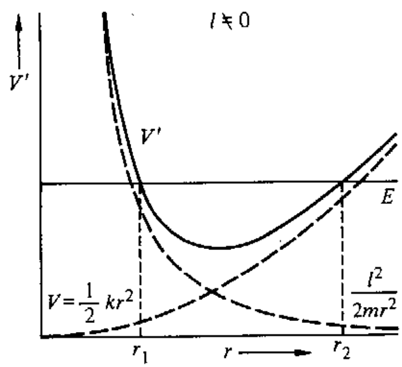

FIGURE 3–11

3–4 THE VIRIAL THEOREM

Another property of central force motion can be derived as a special case of a general theorem valid for a large variety of systems—the *virial theorem.* It differs

* See, for example, K. R. Symon, *Mechanics*, 3rd ed., (Reading, Massachusetts: Addison-Wesley, 1971), Section 3–10.

in character from the theorems previously discussed in being *statistical* in nature, i.e., it is concerned with the time averages of various mechanical quantities.

Consider a general system of mass points with position vectors $\mathbf{r}_i$ and applied forces $\mathbf{F}_i$ (including any forces of constraint). The fundamental equations of motion are then

$$\dot{\mathbf{p}}_i = \mathbf{F}_i. \tag{1-1}$$

We shall be interested in the quantity

$$G = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i,$$

where the summation is over all particles in the system. The total time derivative of this quantity is

$$\frac{dG}{dt} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i. \tag{3-23}$$

The first term can be transformed to

$$\sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T,$$

while the second by Eq. (1–1) is

$$\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i.$$

Equations (3–23) therefore reduces to

$$\frac{d}{dt}\sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i. \tag{3-24}$$

The time average of Eq. (3–24) over a time interval τ is obtained by integrating both sides with respect to t from 0 to τ, and dividing by τ:

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt \equiv \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$$

or

$$\overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau}[G(\tau) - G(0)]. \tag{3-25}$$

If the motion is periodic, i.e., all coordinates repeat after a certain time, and if τ is chosen to be the period, then the right-hand side of (3–25) vanishes. A similar conclusion can be reached even if the motion is not periodic, provided that the coordinates and velocities for all particles remain finite so that there is an upper bound to G. By choosing τ sufficiently long, the right-hand side of Eq. (3–25) can be made as small as desired. In both cases it then follows that

$$\bar{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}. \tag{3-26}$$

Equation (3–26) is known as the *virial theorem*, and the right-hand side is called the *virial of Clausius*. In this form the theorem is very useful in the kinetic theory of gases. Thus the virial theorem can be used to derive Boyle's Law for perfect gases by means of the following brief argument.

Consider a gas consisting of N atoms confined within a container of volume V. The gas is further assumed to be at a temperature T (not to be confused with the symbol for kinetic energy). Then by the equipartition theorem of kinetic theory, the average kinetic energy of each atom is given by $\frac{3}{2}kT$, k being the Boltzmann constant, a relation that in effect is the definition of temperature. The left-hand side of Eq. (3–26) is therefore

$$\frac{3}{2}NkT.$$

On the right-hand side of Eq. (3–26), the forces $\mathbf{F}_i$ include both the forces of interaction between atoms and the forces of constraint on the system. A perfect gas is defined as one for which the forces of interaction contribute negligibly to the virial. This occurs, e.g., if the gas is so tenuous that collisions between atoms occur rarely, compared to collisions with the walls of the container. It is these walls that constitute the constraint on the system, and the forces of constraint, $\mathbf{F}_c$, are localized at the wall and come into existence whenever a gas atom collides with the wall. The sum on the right-hand side of Eq. (3–26) can therefore be replaced in the average by an integral over the surface of the container. The force of constraint represents the reaction of the wall to the collision forces exerted by the atoms on the wall, i.e., to the pressure P. With the usual outward convention for the unit vector $\mathbf{n}$ in the direction of the normal to the surface, we can write therefore

$$d\mathbf{F}_i = -P\mathbf{n}\,dA,$$

or

$$\frac{1}{2}\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{P}{2}\int \mathbf{n}\cdot\mathbf{r}\,dA.$$

But, by Gauss' theorem,

$$\int \mathbf{n}\cdot\mathbf{r}\,dA = \int \nabla\cdot\mathbf{r}\,dV = 3V.$$

The virial theorem, Eq. (3–26), for the system representing a perfect gas therefore can be written

$$\frac{3}{2}NkT = \frac{3}{2}PV,$$

which, cancelling the common factor of $\frac{3}{2}$ on both sides, is the familiar Boyle's Law. Where the interparticle forces contribute to the virial, the perfect gas law of course no longer holds. The virial theorem is then the principal tool, in classical kinetic theory, for calculating the equation-of-state corresponding to such imperfect gases.

One can further show that if the forces $\mathbf{F}_i$ are the sum of nonfrictional forces $\mathbf{F}_i'$ and frictional forces $\mathbf{f}_i$ proportional to the velocity, then the virial depends only on the $\mathbf{F}_i'$; there is no contribution from the $\mathbf{f}_i$. Of course, the motion of the system must not be allowed to die down as a result of the frictional forces. Energy must constantly be pumped into the system to maintain the motion; otherwise *all* time averages would vanish as τ increases indefinitely. (See Exercise 4.)

If the forces are derivable from a potential, then the theorem becomes

$$\bar{T} = \frac{1}{2}\overline{\sum_i \nabla V \cdot \mathbf{r}_i}, \tag{3-27}$$

and for a single particle moving under a central force it reduces to

$$\bar{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r} r}. \tag{3-28}$$

If V is a power-law function of r,

$$V = ar^{n+1},$$

where the exponent is chosen so that the force law goes as r^n, then

$$\frac{\partial V}{\partial r} r = (n+1)V,$$

and Eq. (3-28) becomes

$$\bar{T} = \frac{n+1}{2}\bar{V}. \tag{3-29}$$

By an application of Euler's theorem for homogeneous functions (cf. p. 61) it is clear that Eq. (3-29) holds also whenever V is a homogeneous function in r of degree $n+1$. For the further special case of inverse square law forces n is -2 and the virial theorem takes on a well-known form:

$$\bar{T} = -\frac{1}{2}\bar{V}. \tag{3-30}$$

3-5 THE DIFFERENTIAL EQUATION FOR THE ORBIT, AND INTEGRABLE POWER-LAW POTENTIALS

In treating specific details of actual central force problems a change in the orientation of our discussion is desirable. Hitherto solving a problem has meant finding r and θ as functions of time with E, l, and so on, as constants of integration. But most often what we really seek is the equation of the orbit, i.e., the dependence of r upon θ, eliminating the parameter t. For central force problems the elimination is particularly simple, since t occurs in the equations of motion only as a variable of differentiation. Indeed one equation of motion, (3-8), simply provides a definite relation between a differential change dt and the corresponding change $d\theta$:

$$l\,dt = mr^2\,d\theta. \tag{3–31}$$

The corresponding relation between derivatives with respect to t and θ is

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}. \tag{3–32}$$

These relations may be used to convert the equation of motion (3–12) into a different equation for the orbit. Alternatively they can be applied to the formal solution of the equations of motion, given in Eq. (3–17), to furnish the equation of the orbit directly. For the moment we shall follow the former of these possibilities.

From (3–32) a second derivative with respect to t can be written

$$\frac{d^2}{dt^2} = \frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}\right),$$

and the Lagrange equation for r, (3–12), becomes

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r). \tag{3–33}$$

Now, to simplify (3–33) we notice that

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta};$$

hence if the variable is changed to $u = 1/r$, we have

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f\left(\frac{1}{u}\right). \tag{3–34a}$$

Since

$$\frac{d}{du} = \frac{dr}{du}\frac{d}{dr} = -\frac{1}{u^2}\frac{d}{dr},$$

Eq. (3–34a) can be written alternatively in terms of the potential as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right). \tag{3–34b}$$

Either form of Eq. (3–34) is thus a differential equation for the orbit if the force law f, or the potential V, is known. Conversely if the equation of the orbit is known, that is, r is given as a function of θ, then one can work back and obtain the force law $f(r)$.

Here, however, we want to obtain some rather general results. For example, it can be shown from (3–34) that the orbit is symmetrical about the turning points. To prove this statement it will be noted that if the orbit is symmetrical it should be possible to reflect it about the direction of the turning angle without producing any change. If the coordinates are so chosen that the turning point occurs for $\theta = 0$, then the reflection can be effected mathematically by the substitution of $-\theta$ for θ. The differential equation for the orbit, (3–34), is obviously invariant under such a substitution. Further the initial conditions, here

$$u = u(0), \qquad \left(\frac{du}{d\theta}\right)_0 = 0, \qquad \text{for } \theta = 0,$$

will likewise be unaffected. Hence the orbit equation must be the same whether expressed in terms of θ or $-\theta$, which is the desired conclusion. *The orbit is therefore invariant under reflection about the apsidal vectors.* In effect this means that the complete orbit can be traced if the portion of the orbit between any two turning points is known. Reflection of the given portion about one of the apsidal vectors produces a neighboring stretch of the orbit, and this process can be repeated indefinitely until the rest of the orbit is completed, as illustrated in Fig. 3–12.

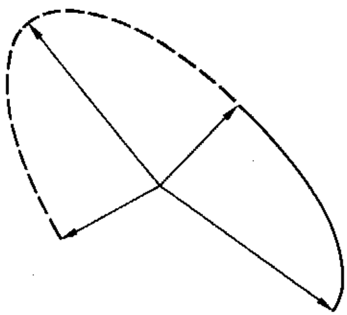

FIGURE 3–12
Extension of the orbit by reflection of a portion about the apsidal vectors.

For any particular force law the actual equation of the orbit must be obtained by integrating the differential equation Eq. (3–34), in either of its forms. However it is not necessary to go through all the details of the integration, as most of the work has already been done in solving the equation of motion (3–12). All that remains is to eliminate t from the solution (3–17) by means of (3–31), resulting in

$$d\theta = \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}}. \qquad (3\text{–}35)$$

With slight rearrangements the integral of (3–35) is

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} + \theta_0 \qquad (3\text{–}36)$$

or, if the variable of integration is changed to $u = 1/r$,

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}. \tag{3–37}$$

As in the case of the equation of motion, Eq. (3–37), while solving the problem formally, is not always a practicable solution, because the integral often cannot be expressed in terms of well-known functions. In fact, only certain types of force laws have been investigated. The most important are the power-law functions of r,

$$V = ar^{n+1} \tag{3–38}$$

so that the force varies as the nth power of r.* With this potential (3–37) becomes

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2}u^{-n-1} - u^2}}. \tag{3–39}$$

This again will be integrable in terms of simple functions only in certain cases. If the quantity in the radical is of no higher power in u than quadratic, the denominator has the form $\sqrt{\alpha u^2 + \beta u + \gamma}$ and the integration can be directly effected in terms of circular functions. This restriction is equivalent to requiring that

$$-n - 1 = 0, 1, 2,$$

or, excluding the $n = -1$ case, for

$$n = -2, -3,$$

corresponding to inverse square or inverse cube force laws. One further easily integrable case is for $n = 1$, i.e., the linear force; for then Eq. (3–39) can be written as

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2}\frac{1}{u^2} - u^2}}. \tag{3–39'}$$

If now we make the substitution

$$u^2 = x, \qquad du = \frac{dx}{2\sqrt{x}},$$

* The case $n = -1$ is to be excluded from the following discussion. In the potential (3–38) it corresponds to a constant potential, i.e., no force at all. It is an equally anomalous case if the exponent is used in the force law directly, since a force varying as r^{-1} corresponds to a logarithmic potential, which is not a power law at all. A logarithmic potential is unusual for motion about a point; it is more characteristic of a *line* source.

Eq. (3–39′) becomes

$$\theta = \theta_0 - \frac{1}{2}\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2mE}{l^2}x - \frac{2ma}{l^2} - x^2}}, \tag{3–40}$$

which again is in the desired form. Thus, a solution in terms of simple functions is obtained for the exponents.

$$n = 1, -2, -3.$$

This does not mean other powers are not integrable, merely that they lead to functions not as well known. For example, there is a range of exponents for which Eq. (3–39) involves *elliptic integrals*, with the solution expressed in terms of *elliptic functions*.

By definition an elliptic integral is

$$\int R(x, \omega)\, dx,$$

where R is any rational function of x and ω, and ω is defined as

$$\omega = \sqrt{\alpha x^4 + \beta x^2 + \gamma x^2 + \delta x + \eta}.$$

Of course α and β cannot simultaneously be zero, for then the integral could be evaluated in terms of circular functions. It can be shown (Whittaker and Watson, *Modern Analysis*, 4th ed., p. 512) that any such integral can be reduced to forms involving circular functions and the Legendre elliptic integrals of the first, second, and third kind. There exist complete and detailed tables of these standard elliptic integrals, and their properties and connections with elliptic functions have been discussed exhaustively in the literature. Intrinsically they do not require any higher logical concept for their use than do circular functions; they are just not as familiar. From the definition it is seen that the integral in (3–39) can be evaluated in terms of elliptic functions if

$$n = -4, -5.$$

We can attempt to put the integral in another form also by leading to elliptic integrals by multiplying numerator and denominator by u^ρ where ρ is some undetermined exponent. The integral then becomes

$$\int \frac{u^\rho\, du}{\sqrt{\frac{2mE}{l^2}u^{2\rho} - \frac{2ma}{l^2}u^{-n-1+2\rho} - u^{2(\rho+1)}}},$$

where the expression in the radical will be a polynomial of higher order than a quartic except if $\rho = 1$. The integral will therefore be no worse than elliptic only if

$$-n - 1 + 2 = 0, 1, 2, 3, 4$$

or

$$n = +1, 0, -1, -2, -3.$$

For $n = +1, -2, -3$ the solutions reduce to circular functions, the case $n = -1$ has already been eliminated, so that this procedure leads to elliptic functions only for $n = 0$.

We again can obtain integrals of the elliptic type in certain cases by changing the variable to $u^2 = x$. The integral in question then appears as

$$\frac{1}{2}\int \frac{dx}{\sqrt{\frac{2mE}{l^2}x - \frac{2ma}{l^2}x^{(1-n)/2} - x^2}},$$

which reduces to the elliptic for for

$$\frac{1-n}{2} = 3, 4$$

leading to the exponents

$$n = -5, -7.$$

Finally we again can perform the trick of multiplying numerator and denominator by x, and the condition for obtaining elliptic integrals or simpler is then

$$\frac{1-n}{2} + 2 = 0, 1, 2, 3, 4$$

or

$$n = +5, +3, +1, -1, -3,$$

which leads to new possibilities only for $n = +5, +3$. The total number of integral exponents resulting in elliptic functions is thus

$$n = +5, +3, 0, -4, -5, -7.$$

Although this exhausts the possibilities for integral exponents, with suitable transformations some fractional exponents can also be shown to lead to elliptic integrals.

3–6 CONDITIONS FOR CLOSED ORBITS (BERTRAND'S THEOREM)

We have not yet extracted all the information that can be obtained from the equivalent one-dimensional problem or from the orbit equation without explicitly solving for the motion. In particular, it is possible to derive a powerful and thought-provoking theorem on the types of attractive central forces that lead to *closed orbits*, i.e., orbits in which the particle eventually retraces its own footsteps.

Conditions have already been described for one kind of closed orbit, namely a circle about the center of force. For any given l, this will occur if the equivalent potential $V'(r)$ has a minimum or maximum at some distance r_0 and if the energy E is just equal to $V'(r_0)$. The requirement that V' have an extremum is equivalent to the vanishing of f' at r_0, leading to the condition derived previously (cf. p. 79),

$$f(r_0) = -\frac{l^2}{mr_0^3}, \tag{3-41}$$

which says the force must be attractive for circular orbits to be possible. In addition, the energy of the particle must be given by

$$E = V(r_0) + \frac{l^2}{2mr_0^2}, \tag{3-42}$$

which, by Eq. (3–15), corresponds to the requirement that for a circular orbit $\dot{r}$ is zero. Equations (3–41) and (3–42) are both elementary and familiar. Between them they imply that for any attractive central force it is possible to have a circular orbit at some arbitrary radius r_0, provided the angular momentum l is given by Eq. (3–41) and the particle energy by Eq. (3–42).

The character of the circular orbit depends on whether the extremum of V' is a minimum, as in Fig. 3–8, or a maximum, as in Fig. 3–9. If the energy is slightly above that required for a circular orbit at the given value of l, then for a minimum in V' the motion, though no longer circular, will still be bounded. However if V' exhibits a maximum, then the slightest raising of E above the circular value, Eq. (3–34), results in motion that is unbounded, with the particle moving both through the center of force and out to infinity for the potential shown in Fig. 3–9. Borrowing the terminology from the case of static equilibrium the circular orbit arising in Fig. 3–8 is said to be *stable*; that in Fig. 3–9 is *unstable*. The stability of the circular orbit is thus determined by the sign of the second derivative of V' at the radius of the circle, being stable for positive second derivative (V' concave up) and unstable for V' concave down. A stable orbit therefore occurs if

$$\left.\frac{\partial^2 V'}{\partial r^2}\right|_{r=r_0} = -\left.\frac{\partial f}{\partial r}\right|_{r=r_0} + \frac{3l^2}{mr_0^4} > 0.$$

Using Eq. (3–41) this condition can be written

$$\left.\frac{\partial f}{\partial r}\right|_{r=r_0} < -\frac{3f(r_0)}{r_0}, \tag{3-43}$$

or

$$\left.\frac{d\ln f}{d\ln r}\right|_{r=r_0} < -3. \tag{3-43'}$$

If the force behaves like a power law of r in the vicinity of the circular radius r_0,

$$f = -\frac{k}{r^{n+1}},$$

then the stability condition, Eq. (3–43), becomes

$$-\frac{(n+1)k}{r_0^{n+2}} < -\frac{3k}{r_0^{n+2}}$$

or

$$n < 2. \tag{3–44}$$

A power-law attractive potential varying more slowly than $1/r^2$ is thus capable of stable circular orbits for all values of r_0.

If the circular orbit is stable, then a small increase in the particle energy above the value for a circular orbit results in only a slight variation of r about r_0. It can be shown (cf. Appendix A) that for such small deviations from the circularity conditions, the particle executes a simple harmonic motion in $u(\equiv 1/r)$ about u_0:

$$u = u_0 + a\cos\beta\theta. \tag{3–45}$$

Here a is an amplitude that depends on the deviation of the energy from the value for circular orbits, and β is a quantity arising from a Taylor series expansion of the force law $f(r)$ about the circular orbit radius r_0. It is shown in Appendix A that β is given by

$$\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\bigg|_{r=r_0}. \tag{3–46}$$

As the radius vector of the particle sweeps completely around the plane, u goes through β cycles of its oscillation (cf. Fig. 3–13). If β is a rational number, the ratio of two integers, p/q, then after q revolutions of the radius vector the orbit would begin to retrace itself, i.e., the orbit would be *closed*.

At each r_0 such that the inequality in Eq. (3–43) is satisfied, it is possible to establish a stable circular orbit by giving the particle an initial energy and angular momentum prescribed by Eqs. (3–41) and (3–42). The question naturally arises as to what form must the force law take in order that the slightly perturbed orbit

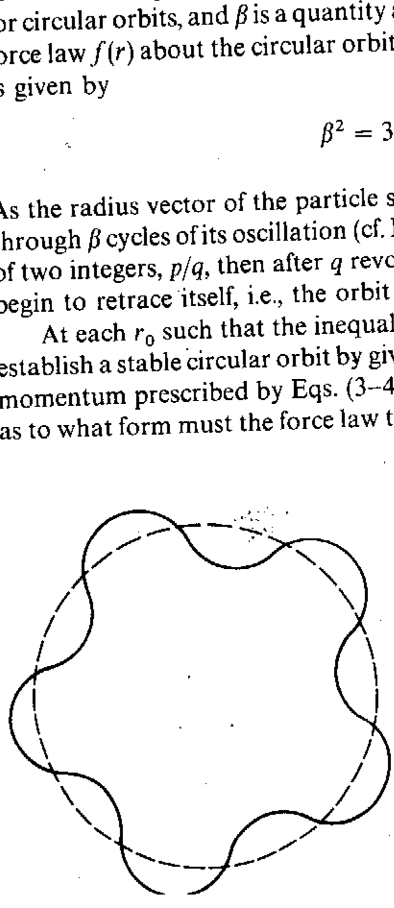

FIGURE 3–13
Orbit for motion in a central force deviating slightly from a circular orbit.

about any of these circular orbits should be closed. It is clear that under these conditions β must not only be a rational number, but it must be the *same* rational number at all distances that a circular orbit is possible. Otherwise, since β can take on only discrete values, the number of oscillatory periods would change discontinuously with r_0, and indeed the orbits could not be closed at the discontinuity. With β^2 everywhere constant, the defining equation for β^2, Eq. (3–46), becomes in effect a differential equation for the force law f in terms of the independent variable r_0. We can indeed consider Eq. (3–46) to be written in terms of r if we keep in mind that the equation is valid only over the ranges in r for which stable circular orbits are possible. A slight rearrangement of Eq. (3–46) leads to the equation

$$\frac{d \ln f}{d \ln r} = \beta^2 - 3, \qquad (3\text{–}47)$$

which can be immediately integrated to give a force law:

$$f(r) = -\frac{k}{r^{3-\beta^2}}. \qquad (3\text{–}48)$$

All force laws of this form, with β a rational number, lead to closed stable orbits for initial conditions that differ only *slightly* from conditions defining a circular orbit. Included within the possibilities allowed by Eq. (3–48) are some familiar forces such as the inverse square law ($\beta \equiv 1$), but of course many other behaviors, such as $f = -kr^{-2/9}$, ($\beta = \frac{5}{3}$) are also permitted.

Suppose the initial conditions deviate more than slightly from the requirements for circular orbits; will these same force laws still give circular orbits? The question can be answered directly by keeping an additional term in the Taylor series expansion of the force law and solving the resultant orbit equation. While the calculations involved are elementary they are somewhat lengthy. Details are given in Appendix A. What is found is that for more than first-order deviations from circularity, the orbits are closed only for $\beta^2 = 1$ and $\beta^2 = 4$. The first of these values of β^2, by Eq. (3–48), leads to the familiar attractive inverse square law; the second is an attractive force proportional to the radial distance—Hooke's law! These force laws, and only these, could possibly produce closed orbits for any arbitrary combination of l and $E (E < 0)$, and in fact we know from direct solution of the orbit equation that they do. Hence, *the only central forces that result in closed orbits for all bound particles are the inverse square law and Hooke's law.**

This is a remarkable result, well worth the tedious algebra required. It is a commonplace of astronomical observation that celestial objects that are bound move in orbits that are in first approximation closed. For the most part, the small

* This conclusion was apparently first derived by J. Bertrand, *Comptes Rendus* 77, 849–853 (1873), and is frequently referred to as Bertrand's theorem. See other pertinent literature referenced at the end of this chapter.

deviations from a closed orbit are traceable to perturbations such as the presence of other bodies. The prevalence of closed orbits holds true whether we consider only the solar system, or look out to the many examples of true binary stars that have been examined. Now, Hooke's law is a most unrealistic force law to hold at all distances, for it implies a force increasing indefinitely to infinity. Thus, the existence of closed orbits for a wide range of initial conditions by itself leads to the conclusion that the gravitational force varies as the inverse square of the distance. It is not necessary, for example, to use the elliptic character of the orbits to arrive at the gravitational force law.

We can phrase this conclusion in a slightly different manner, one that is of somewhat more significance in modern physics. The orbital motion in a plane can be looked on as compounded of two oscillatory motions, one in r and one in θ. That the orbit is closed is equivalent to saying that the periods of the two oscillations are commensurate—that they are *degenerate*. Hence, *the degenerate character of orbits in a gravitational field fixes the form of the force law.* Later on we shall encounter other formulations of the relation between degeneracy and the nature of the potential.

3–7 THE KEPLER PROBLEM: INVERSE SQUARE LAW OF FORCE

The inverse square law is the most important of all the central force laws and it deserves detailed treatment. For this case the force and potential can be written as

$$f = -\frac{k}{r^2}, \qquad V = -\frac{k}{r}. \tag{3–49}$$

There are several ways to integrate the equation for the orbit, the simplest being to substitute (3–49) in the differential equation for the orbit (3–34):

$$\frac{d^2u}{d\theta^2} + u = \frac{-mf\left(\frac{1}{u}\right)}{l^2u^2} = \frac{mk}{l^2}. \tag{3–50}$$

Changing the variable to $y = u - \frac{mk}{l^2}$, the differential equation becomes

$$\frac{d^2y}{d\theta^2} + y = 0,$$

which has the immediate solution

$$y = B\cos(\theta - \theta'),$$

B and θ' being the two constants of integration. In terms of r the solution is

$$\frac{1}{r} = \frac{mk}{l^2}(1 + e\cos(\theta - \theta')), \tag{3–51}$$

where

$$e = B\frac{l^2}{mk}. \tag{3–52}$$

It is instructive to obtain the orbit equation also from the formal solution (3–39). While this procedure is longer than the simple integration of the differential equation (3–50), it has the advantage that the significant constant of integration e is automatically evaluated in terms of the energy E and the angular momentum l of the system. We write (3–39) in the form

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mku}{l^2} - u}}, \tag{3–53}$$

where the integral is now taken as indefinite. The quantity θ' appearing in (3–53) is a constant of integration determined by the initial conditions and will not necessarily be the same as the initial angle θ_0 at time $t = 0$. The indefinite integral is of the standard form,*

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos - \frac{\beta + 2\gamma x}{\sqrt{q}}, \tag{3–54}$$

where

$$q = \beta^2 - 4\alpha\gamma.$$

To apply this to (3–53) we must set

$$\alpha = \frac{2mE}{l^2}, \qquad \beta = \frac{2mk}{l^2}, \qquad \gamma = -1,$$

and the discriminant q is therefore

$$q = \left(\frac{2mk}{l^2}\right)^2 \left(1 + \frac{2El^2}{mk^2}\right).$$

With these substitutions, Eq. (3–53) becomes

$$\theta = \theta' - \arccos \frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}.$$

* Cf., for example, B. O. Pierce, *A Short Table of Integrals*, 3d ed. no 161; 4th ed. no. 166. See also I.S. Gradshtein and I. W. Ryzhik, *Table of Integrals*, no. 2.261, or M. R. Spiegel, *Mathematical Handbook* no. 14.280. (For full description of these books, see the section on 'Works of Reference' in the Bibliography, p. 630). A constant, $-\pi/2$, is to be added to the result as given in all of these tables of integrals in order to obtain (3–45), a procedure that is permissible since the integral is indefinite.

Finally, by solving for $u, \equiv \frac{1}{r}$, the equation of the orbit is found to be

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta')\right), \tag{3–55}$$

which agrees with (3–51), except that here e is evaluated in terms of E and l. The constant of integration θ' can now be identified from Eq. (3–55) as one of the turning angles of the orbit. It will be noted that only three of the four constants of integration appear in the orbit equation, and this is always a characteristic property of the orbit. In effect, the fourth constant locates the initial position of the particle on the orbit. If we are interested solely in the orbit equation this information is clearly irrelevant and hence does not appear in the answer. Of course, the missing constant has to be supplied if it is desired to complete the solution by finding r and θ as functions of time. Thus, if one chooses to integrate the conservation theorem for angular momentum,

$$mr^2\, d\theta = l\, dt,$$

by means of (3–55), one must specify in addition the initial angle θ_0.

Now, the general equation of a conic with one focus at the origin is

$$\frac{1}{r} = C(1 + e\cos(\theta - \theta')), \tag{3–56}$$

where e is the eccentricity of the conic section. By comparison with Eq. (3–55) it follows that the orbit is always a conic section, with the eccentricity

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}. \tag{3–57}$$

The nature of the orbit depends on the magnitude of e according to the following scheme:

$$\begin{aligned} &e > 1, \quad && E > 0: && \text{hyperbola,} \\ &e = 1, && E = 0: && \text{parabola,} \\ &e < 1, && E < 0: && \text{ellipse,} \\ &e = 0, && E = -\frac{mk^2}{2l^2}: && \text{circle.} \end{aligned}$$

This classification agrees with the qualitative discussion of the orbits based on the energy diagram of the equivalent one-dimensional potential V'. The condition for circular motion appears here in a somewhat different form, but it can easily be derived as a consequence of the previous conditions for circularity. For a circular orbit, T and V are constant in time, and from the virial theorem

$$E \equiv T + V = -\frac{V}{2} + V = \frac{V}{2}.$$

Hence

$$E = -\frac{k}{2r_0}. \tag{3-58}$$

But from Eq. (3–41), the statement of equilibrium between the central force and the "effective force," we can write

$$-\frac{k}{r_0^2} = -\frac{l^2}{mr_0^3},$$

or

$$r_0 = \frac{l^2}{mk}. \tag{3-59}$$

With this formula for the orbital radius, Eq. (3–58) becomes

$$E = -\frac{mk^2}{2l^2},$$

the above condition for circular motion.

In the case of elliptic orbits it can be shown the major axis depends solely on the energy, a theorem of considerable importance in the Bohr theory of the atom. The semimajor axis is one half the sum of the two apsidal distances r_1 and r_2 (cf. Fig. 3–6). By definition the radial velocity is zero at these points, and the conservation of energy implies that the apsidal distances are therefore the roots of the equation

$$E - \frac{l^2}{2mr^2} + \frac{k}{r} = 0,$$

or

$$r^2 + \frac{k}{E}r - \frac{l^2}{2mE} = 0. \tag{3-60}$$

Now, the coefficient of the linear term in a quadratic equation is the negative of the sum of the roots. Hence the semimajor axis is given by

$$a = \frac{r_1 + r_2}{2} = -\frac{k}{2E}. \tag{3-61}$$

Note that in the circular limit, Eq. (3–61) agrees with Eq. (3–58). In terms of the semimajor axis, the eccentricity of the ellipse can be written

$$e = \sqrt{1 - \frac{l^2}{mka}}, \tag{3-62}$$

(a relation we will have use for in a later chapter). Further, from Eq. (3–62) we have the expression

$$\frac{l^2}{mk} = a(1 - e^2), \tag{3-63}$$

in terms of which the elliptical orbit equation (3–51) can be written

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta')}. \tag{3–64}$$

From Eq. (3–64) it follows that the two apsidal distances (which occur when $\theta - \theta'$ is 0 and π, respectively) are equal to $a(1-e)$ and $a(1+e)$, as is to be expected from the properties of an ellipse.

3–8 THE MOTION IN TIME IN THE KEPLER PROBLEM

The orbital equation for motion in a central inverse-square force law can thus be solved in a fairly straightforward manner with results that can be stated in simple closed expressions. To describe the motion of the particle in time as it traverses the orbit is, however, a much more involved matter. In principle the relation between the radial distance of the particle r and the time (relative to some starting point) is given by Eq. (3–18), which here takes on the form

$$t = \sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{\dfrac{k}{r}-\dfrac{l^2}{2mr^2}+E}}. \tag{3–65}$$

Similarly, the polar angle θ and the time are connected through the conservation of angular momentum,

$$dt = \frac{mr^2}{l}\,d\theta,$$

which combined with the orbit equation (3–51) leads to

$$t = \frac{l^3}{mk^2}\int_{\theta_0}^{\theta}\frac{d\theta}{[1+e\cos(\theta-\theta')]^2}. \tag{3–66}$$

Either of these integrals can be carried out in terms of elementary functions. (For Eq. (3–66) see, for example, formula 14.391 in *Mathematical Handbook of Formulas and Tables*, ed. by M. R. Spiegel). But the relations are very complex, and their inversion to give r or θ as functions of t pose formidable problems, especially when one wants the high precision needed for astronomical observations.

To illustrate some of these involvements let us consider the situation for parabolic motion ($e = 1$), where the integrations can be most simply carried out. It is customary to measure the plane polar angle from the radius vector at the point of closest approach—a point most usually designated as the *perihelion.** This convention corresponds to setting θ' in the orbit equation (3–51) equal to

* Literally, the term should be restricted to orbits around the sun, while the more general term should be *periapsis*. However, it has become customary to use perihelion no matter where the center of force is. Even for space craft orbiting the moon, official descriptions of the orbital parameters refer to perihelion where pericynthion would be the pedantic term.

zero. Correspondingly, time is measured from the moment, T, of perihelion passage. Using the trigonometric identity

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2},$$

Eq. (3–66) then reduces for parabolic motion to the form

$$t = \frac{l^3}{4mk^2}\int_0^\theta \sec^4\frac{\theta}{2}\,d\theta.$$

The integration is easily performed by a change of variable to $x = \tan(\theta/2)$, leading to the integral

$$t = \frac{l^3}{2mk^2}\int_0^{\tan(\theta/2)} (1 + x^2)\,dx,$$

or

$$t = \frac{l^3}{2mk^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right). \tag{3–67}$$

This is a straightforward relation for t as a function of θ; inversion to obtain θ at a given time requires solving a cubic equation for $\tan(\theta/2)$, and then finding the corresponding arctan. The radial distance at the given time is then given through the orbital equation.

For elliptic motion, Eq. (3–65) is most conveniently integrated through an auxiliary variable ψ, denoted as the *eccentric anomaly*,* and defined by the relation

$$r = a(1 - e\cos\psi). \tag{3–68}$$

By comparison with the orbit equation, (3–64), it is clear that ψ also covers the interval 0 to 2π as θ goes through a complete revolution, and that the perihelion occurs at $\psi = 0$ (where $\theta = 0$ by convention) and the aphelion at $\psi = \pi = \theta$. A geometrical interpretation can be given to ψ, but it is of historical interest only (see, e.g., McCuskey, *Introduction to Celestial Mechanics*, p. 45). Expressing E and l in terms of a, e, and k, Eq. (3–65) can be rewritten for elliptic motion as

$$t = \sqrt{\frac{m}{2k}}\int_{r_0}^{r} \frac{r\,dr}{\sqrt{r - \dfrac{r^2}{2a} - \dfrac{a(1 - e^2)}{2}}}, \tag{3–69}$$

where, by the convention on the starting time, r_0 is the perihelion distance. Substitution of r in terms of ψ from Eq. (3–68) reduces this integral, after some

* The name connects with the terminology of medieval astronomy in which θ was called the *true anomaly*.

algebra, to the simple form

$$t = \sqrt{\frac{ma^3}{n}} \int_0^\psi (1 - e\cos\psi)\, d\psi. \tag{3–70}$$

First, we may note that Eq. (3–70) provides an expression for the period, τ, of elliptical motion, if the integral is carried over the full range in ψ of 2π:

$$\tau = 2\pi a^{3/2}\sqrt{\frac{m}{k}}. \tag{3–71}$$

This important result can also be obtained directly from the properties of an ellipse. From the conservation of angular momentum the areal velocity is constant and is given by

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}. \tag{3–72}$$

The area of the orbit, A, is to be found by integrating (3–72) over a complete period τ:

$$\int_0^\tau \frac{dA}{dt}\,dt = A = \frac{l\tau}{2m}.$$

Now, the area of an ellipse is

$$A = \pi ab,$$

where, by the definition of eccentricity, the semiminor axis b is related to a according to the formula

$$b = a\sqrt{1 - e^2}.$$

By (3–62) it is seen that the semiminor axis can also be written as

$$b = a^{1/2}\sqrt{\frac{l^2}{mk}},$$

and the period is therefore

$$\tau = \frac{2m}{l}\pi a^{3/2}\sqrt{\frac{l^2}{mk}} = 2\pi a^{3/2}\sqrt{\frac{m}{k}},$$

as was found previously. Equation (3–71) states that, other things being equal, the square of the period is proportional to the cube of the major axis, and this conclusion is often referred to as the third of Kepler's laws.* Actually, Kepler was

* Kepler's three laws of planetary motion, published around 1610, were the result of his pioneering analysis of planetary observations and laid the groundwork for Newton's great advances. The second law, the conservation of areal velocity, is a general theorem for central force motion, as has been noted previously. However, the first—that the planets move in elliptical orbits about the sun at one focus—and the third are restricted

concerned with the specific problem of planetary motion in the gravitational field of the sun. A more precise statement of his law would therefore be: The square of the periods of the various planets are proportional to the cube of their major axes. In this form the law is only approximately true. It must be remembered that the motion of a planet about the sun is a two-body problem and m in (3–71) must be replaced by the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

where m_1 may be taken as referring to the planet and m_2 to the sun. Further, the gravitational law of attraction is

$$f = -G\frac{m_1 m_2}{r^2},$$

so that the constant k is

$$k = Gm_1 m_2. \tag{3–73}$$

Under these conditions (3–71) becomes

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_1 + m_2)}} \approx \frac{2\pi a^{3/2}}{\sqrt{Gm_2}}, \tag{3–74}$$

if we neglect the mass of the planet compared to the sun. It is the approximate version of Eq. (3–74) that is Kepler's third law, for it states that τ is proportional to $a^{3/2}$, with the same constant of proportionality for all planets. However, the planetary mass m_1 is not always completely negligible compared to the sun's; for example, Jupiter has a mass about 0.1 % of the mass of the sun. On the other hand Kepler's third law is rigorously true for the electron orbits in the Bohr atom, since μ and k are then the same for all orbits in a given atom.

To return to the general problem of the position in time for an elliptic orbit, we may rewrite Eq. (3–70) slightly by introducing the frequency of revolution ω as

$$\omega = \frac{2\pi}{\tau} = \sqrt{\frac{k}{ma^3}}. \tag{3–75}$$

The integration in Eq. (3–70) is of course easily performed, resulting in the relation

$$\omega t = \psi - e\sin\psi, \tag{3–76}$$

known as *Kepler's equation.* The quantity ωt goes through the range 0 to 2π, along with ψ and θ, in the course of a complete orbital revolution and is therefore also denoted as an anomaly, specifically the *mean anomaly*.

To find the position in orbit at a given time t, Kepler's equation, (3–76), would first be inverted to obtain the corresponding eccentric anomaly ψ. Equation (3–68) then yields the radial distance, while the polar angle θ can be expressed in terms of ψ by comparing the defining equation (3–68) with the orbit equation (3–64):

$$1 + e\cos\theta = \frac{1 - e^2}{1 - e\cos\psi}.$$

With a little algebraic manipulation this can be simplified to

$$\cos\theta = \frac{\cos\psi - e}{1 - e\cos\psi}. \tag{3–77}$$

By successively adding and subtracting both sides of Eq. (3–77) from unity and taking the ratio of the resulting two equations, one is led to the alternate form

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}. \tag{3–78}$$

Either Eq. (3–77) or (3–78) thus provides θ, once ψ is known. The solution of the transcendental Kepler's equation (3–76) to give the value of ψ corresponding to a given time is a problem that has attracted the attention of many famous mathematicians ever since Kepler posed the question early in the seventeenth century. Newton, for example, contributed what today would be called an analog solution. Indeed, it can be claimed that the practical need to solve Kepler's equation to accuracies of a second of arc over the whole range of eccentricity fathered many of the developments in numerical mathematics in the eighteenth and nineteenth centuries. A few of the more than 100 methods of solution developed in the pre-computer era are considered in the exercises to this chapter.

3–9 THE LAPLACE–RUNGE–LENZ VECTOR

The Kepler problem is also distinguished by the existence of an additional conserved vector besides the angular momentum. For a general central force, Newton's second law of motion can be written vectorially as

$$\dot{\mathbf{p}} = f(r)\,\frac{\mathbf{r}}{r}. \tag{3–79}$$

The cross product of $\dot{\mathbf{p}}$ with the constant angular momentum vector $\mathbf{L}$ therefore can be expanded as

$$\dot{\mathbf{p}} \times \mathbf{L} = \frac{mf(r)}{r}[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})]$$

$$= \frac{mf(r)}{r}[\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}]. \tag{3–80}$$

Equation (3–80) can be further simplified by noting that

$$\mathbf{r}\cdot\dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = r\dot{r}$$

(or, in less forml terms, the component of the velocity in the radial direction is $\dot{r}$). As **L** is constant, Eq. (3–80) can then be rewritten, after a little manipulation, as

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\left[\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right],$$

or

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = -mf(r)r^2\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right). \tag{3–81}$$

Without specifying the form of $f(r)$ we can go no further. But Eq. (3–81) can be immediately integrated if $f(r)$ is inversely proportional to r^2 —the Kepler problem. Writing then $f(r)$ in the form prescribed by Eq. (3–49), Eq. (3–81) becomes

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \frac{d}{dt}\left(\frac{mk\mathbf{r}}{r}\right),$$

which says that for the Kepler problem there exists a *conserved vector* **A** defined by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\frac{\mathbf{r}}{r}. \tag{3–82}$$

In recent times, the vector **A** has become known amongst physicists as the Runge–Lenz vector, but priority belongs to Laplace.*

From the definition of **A**, one can easily see that

$$\mathbf{A} \cdot \mathbf{L} = 0, \tag{3–83}$$

since **L** is perpendicular to $\mathbf{p} \times \mathbf{L}$ and **r** is perpendicular to $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. It follows from this orthogonality of **A** to **L** that **A** must be some fixed vector in the plane of the orbit. If θ is used to denote the angle between **r** and the fixed direction of **A**, then the dot product of **r** and **A** is given by

$$\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - mkr. \tag{3–84}$$

Now, by permutation of the terms in the triple dot product, we have

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p}) = l^2,$$

---

* Laplace explicitly exhibited the components of **A** in the first part of his "*Traite de Mecanique Celeste*," which appeared in 1799. The designation as the Laplace vector, common in a number of treatises on celestial mechanics, is therefore probably the proper eponym. W. R. Hamilton apparently discovered **A** as a conserved quantity independently in 1845. The first derivation in vector language, substantially as given here, was that of J. W. Gibbs about 1900. C. Runge repeated the derivation in a popular German text on vector analysis (1919) and was quoted as a reference by W. Lenz in a 1924 paper on quantum mechanical treatment of the perturbed hydrogen atom. Since then the literature on the Laplace–Runge–Lenz vector and its uses has become enormous. For further historical details see H. Goldstein, *American Journal of Physics*, **43**, 735 (1975) and **44**, 1123 (1976).

so that Eq. (3–84) becomes

$$Ar\cos\theta = l^2 - mkr,$$

or

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right). \tag{3–85}$$

The Laplace–Runge–Lenz vector thus provides still another way of deriving the orbit equation for the Kepler problem! Comparison of Eq. (3–85) with the orbit equation in the form of Eq. (3–51) shows that **A** is in the direction of the radius vector to the perihelion point on the orbit, and has a magnitude

$$A = mke. \tag{3–86}$$

For the Kepler problem we thus have identified two vector constants of the motion **L** and **A**, and a scalar E. Since a vector must have three independent components, this corresponds to seven conserved quantities in all. Now, a system such as this with three degrees of freedom has six independent constants of the motion, corresponding, say, to the three components each of the initial position and velocity of the particle. Further, the constants of the motion we have found are all algebraic functions of **r** and **p** that describe the orbit as a whole (orientation in space, eccentricity, etc.); none of these seven conserved quantities relate to where the particle is located in the orbit at the initial time. Since one of the constants of the motion must relate to this information, say in the form of T, the time of the perihelion passage, there can be only five independent constants of the motion describing the size, shape, and orientation of the orbit. It can therefore be concluded that not all of the quantities making up **L**, **A**, and E can be independent; there must in fact be two relations connecting these quantities. One such relation has already been obtained as the orthogonality of **A** and **L**, Eq. (3–83). The other follows from Eq. (3–86) when the eccentricity is expressed in terms of E and l from Eq. (3–57), leading to

$$A^2 = m^2k^2 + 2mEl^2, \tag{3–87}$$

thus confirming that there are only five *independent* constants out of the seven.*

The angular momentum vector and the energy alone contain only four independent constants of the motion: the Laplace–Runge–Lenz vector thus adds one more. It is natural to ask why there should not exist for any general central force law some conserved quantity that together with **L** and E serves to define the orbit in a manner similar to the Laplace–Runge–Lenz vector for the special case of the Kepler problem. The answer seems to be that such conserved quantities can in fact be constructed,† but that they are in general rather peculiar functions of the

* The arguments in the above paragraph were apparently first presented by Laplace in 1799. He also then explicitly demonstrated the relation between the magnitude of **A** and the eccentricity, Eq. (3–86).

†See, for example, D. M. Fradkin, *Progress of Theoretical Physics* **37**, 798, May 1967.

motion. The constants of the motion relating to the orbit between them define the orbit, i.e., lead to the orbit equation giving r as a function of θ. We have seen that in general orbits for central force motion are not closed; the arguments of Section 3–6 showed that closed orbits implied rather stringent conditions on the form of the force law. It is a property of nonclosed orbits that the curve will eventually pass through any arbitrary (r, θ) point that lies between the bounds of the turning points of r. Intuitively this can be seen from the nonclosed nature of the orbit; as θ goes around a full cycle the particle must never retrace its footsteps on any previous orbit. Thus the orbit equation is such that r is a multivalued function of θ (modulo 2π), in fact it is an *infinite-valued function* of θ. The corresponding conserved quantity additional to L and E defining the orbit must similarly involve an infinite-valued function of the particle motion. Only where the orbits are closed, or more generally where the motion is *degenerate*, as in the Kepler problem, can we expect the additional conserved quantity to be a simple algebraic function of $\mathbf{r}$ and $\mathbf{p}$ such as the Laplace–Runge–Lenz vector. From these arguments we would expect a simple analog of such a vector to exist for the case of a Hooke's Law force, where, as we have seen, the orbits are also degenerate. This is indeed the case, except that the natural way to formulate the constant of the motion leads not to a vector but to a tensor of the second rank (see Section 9–7). Thus, the existence of an additional constant or integral of the motion, beyond $\mathbf{E}$ and $\mathbf{L}$, that is a simple algebraic function of the motion is sufficient to indicate that the motion is degenerate and the bounded orbits are closed.

3–10 SCATTERING IN A CENTRAL FORCE FIELD

Historically, the interest in central forces arose out of the astronomical problems of planetary motion. There is no reason, however, why central force motion must be thought of only in terms of such problems; mention has already been made of the orbits in the Bohr atom. Another field that can be investigated in terms of classical mechanics is the *scattering* of particles by central force fields. Of course, if the particles are on the atomic scale it must be expected that the specific results of a classical treatment will often be incorrect physically, for quantum effects are usually large in such regions. Nevertheless there are many classical predictions that remain valid to a good approximation. More important, the procedures for *describing* scattering phenomena are the same whether the mechanics is classical or quantum; one can learn to speak the language equally as well on the basis of classical physics.

In its one-body formulation the scattering problem is concerned with the scattering of particles by a *center of force*. We consider a uniform beam of particles—whether electrons, or α-particles, or planets is irrelevant—all of the same mass and energy incident upon a center of force. It will be assumed that the force falls off to zero for very large distances. The incident beam is characterized by specifying its *intensity* I (also called flux density), which gives the number of particles crossing unit area normal to the beam in unit time. As a particle