UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2024

Problem Set 1
Due: Wednesday, Jan 31, 2024, at the start of class (1100 Kansas Time)
This problem set is worth **36 points**.

1. **Astronomical Concepts [20 pts].**

   (a) In a galaxy far, far away, the gas giant Endor orbits a Sun-like star at a distance of a_E. Endor (mass m_E) is orbited by a Forest Moon (m_m) with the same separation as found in the Earth-Moon system ($a_☽$). What is the ratio (an algebraic expression, not just a number!) of the gravitational forces (i) between Endor and its star (mass m_*) and (ii) between Endor and its moon? Estimate which Force is stronger. [6 pts]

   Solution: Since
$$F_G = G\frac{m_1 m_2}{r^2},$$
The ratio of the two forces is then
$$\frac{F_{E,*}}{F_{m,E}} = \frac{m_*}{m_m}\left(\frac{a_☽}{1 \text{ au}}\right)^2.$$
I could look up $a_☽$, or I could remember that light takes ∼1 s to go from the Earth to the Moon and ∼8 min from the Sun to the Earth (1 au). So the ratio of distances is ∼1/500, and thus:
$$\boxed{\frac{F_{E,*}}{F_{m,E}} \approx \frac{m_*}{m_m}\frac{1}{2.5\times 10^5}}.$$
Without knowing the mass m_m of the Forest Moon, we can't calculate an exact number. If it's like Earth's Moon, then $m_* \approx 3\times 10^7 m_m$, but if it's a planet in its own right (say, twice the mass of the Earth) then $m_* \approx 1.5\times 10^5$. So probably the star-planet force is stronger, but we can't be certain.

   (b) You have invented a matter-antimatter reactor that converts physical material (matter) into energy with 100% efficiency. Congratulations, Zefram: you're a shoo-in for the Nobel Prize. (i) If you put 1 kg of matter (and an equal amount of antimatter) in your reactor, approximately how much energy (E_{reactor}) is released when the mass is converted directly into energy? (ii) If the reactor takes 2 s to use that fuel, what was its approximate power output, in Watts and in Solar Luminosities ($L_\odot$)? (iii) How does E_{reactor} compare to the total amount of energy used on Earth in a year? [7 pts]

   Solution: We know $E = mc^2$, so the total energy released in this reaction is the mass energy of the fuel:
$$\begin{aligned} E &\approx (2\times 1\text{ kg})(3\times 10^8 \text{ m s}^{-1})^2 \\ &\approx \boxed{2\times 10^{17}\text{ J}}. \end{aligned}$$
When the energy is released in just 2 s, then the luminosity of your reactor is
$$L_{\text{reactor}} = \frac{\Delta E}{\Delta t} = \frac{2\times 10^{17}\ J}{2\text{ s}} = 10^{17}\text{ W}.$$
Since $L_\odot \approx 4\times 10^{26}$ W, we have
$$\boxed{L_{\text{reactor}} \approx \frac{1}{4}\times 10^{-9} L_\odot}.$$
The world's annual energy consuption is roughly 500 million terajoules,
$$E_{\text{world}} \approx 500\times 10^6 \times 10^{12}\text{ J} \approx 5\times 10^{20}\text{ J}.$$
So your reactor produced an energy yield equal to about 0.0004 (0.04%) of the world's current energy output: still quite a feat!

(c) Write the astronomer's version of the Ideal Gas Law. Explain each term (including its physical units), and how it might be used [7 pts].

Solution: The astrophysicist's ideal gas law is

$$\boxed{P = nk_B T} \tag{1}$$

where:

i. P is the gas pressure (SI unit Pascals, Pa = N m^{-2}),

ii. n is the gas number density (SI unit of inverse volume, m^{-3}) – can also be calculated if you know the gas *density* ($\rho = M/V = n\langle m\rangle$, where $\langle m\rangle$ is the average gas particle mass),

iii. k_B is the Boltzmann constant, $\sim (1/7) \times 10^{-22}$ J/K, and

iv. T is the temperature of the gas.

It is used to convert between the pressure, number density, and/or temperature of a gas in order to constrain any of those quantities that might not yet be known.

2. **Order-of-Magnitude Estimation [16 pts].** Strive to do as many of these calculations in your head (or with pencil and paper) as possible, aside from looking up any necessary physical constants.

(a) **City on a Hill [5 pts.]** Roughly estimate the mass of Mount Oread, in kg and in $M_\oplus$ (Earth masses).

Solution: From the topographic maps at `https://ngmdb.usgs.gov/topoview/` I estimate that Mt. Oread is about 2.5 km long, 700 m wide at the base, and 150 ft $\approx$ 40 m higher than the surrounding land (Wikipedia tells me that is 60 m above downtown Lawrence, so your mileage may vary). I assume it has a triangular cross-sectional area, so

$$\begin{aligned} V_{\rm Oread} &\approx \frac{1}{2} w \times h \times \ell \\ &\approx (350\ m)(40\ m)(2500\ m) \\ &\approx 3.5 \times 10^8\ {\rm m}^3. \end{aligned}$$

Assuming a density the same as the Earth ($\sim$ 5000 kg m^{-3}), then we have

$$M_{\rm Oread} \sim (5000)(3.5 \times 10^8)\ {\rm kg} \sim \boxed{1.8 \times 10^{12}\ {\rm kg}}. \tag{2}$$

Compared to the Earth's mass of 6×10^{24} kg, KU's mountaintop perch is just $\boxed{\sim 3 \times 10^{-13} M_\oplus}$.

A nice independent 'check' that a previous student pointed out is that the Great Pyramid of Giza, in Egypt, is estimated to weigh roughly six millions tons, $\approx 6 \times 10^9$ kg. I've never been to Egypt, but it at least makes sense to me that Mt. Oread is bigger than a human-built pyramid: both have about the same height, but I estimated a base area much larger than that of the pyramid.

(b) **How Big? [5 pts]**. The French revolutionaries of the late 18th century defined the meter by setting the Earth's equator-to-pole distance to be 10,000 km. Estimate the radius ($R_\oplus$), volume ($V_\oplus$), and mass ($M_\oplus$) of the Earth, in SI units.

Solution: Since the Earth is approximately a sphere, its circumference $c_\oplus = 40,000\,{\rm km} \approx 2\pi R_\oplus$. Thus,

$$R_\oplus \approx \frac{40,000\,{\rm km}}{2\pi} \approx \frac{40,000\,{\rm km}}{6} \boxed{6,500\,{\rm km}} \tag{3}$$

which compares reasonably well with the modern value of $R_\oplus = 6,370$ km.

Still approximating the Earth as a sphere, its volume is then just

$$V_\oplus = \frac{4}{3}\pi R_\oplus^3 \tag{4}$$

$$\approx 4(6,500\,{\rm km})^3 \tag{5}$$

$$\approx 4 \times (6.5)^3 \left(10^6\,{\rm m}\right)^3 \tag{6}$$

$$\approx (26 \times 40)\left(10^{18}\,{\rm m}^3\right) \tag{7}$$

$$\approx (100)\left(10^{18}\,{\rm m}^3\right) \tag{8}$$

$$\approx \boxed{\left(10^{20}\,{\rm m}^3\right)} \tag{9}$$

(10)

... which compares well with the Earth's known volume of $V_\oplus = 1.083 \times 10^{20}$ m^3.

Mass is volume times density, i.e. $M_\oplus = V_\oplus \langle \rho_\oplus \rangle$. We know that water has $\rho_{\mathrm{H_2O}} = 1.0$ g/cc, and the Earth must be denser than this. We could look up the density of metals (7–10 g/cc) and assume that the Earth isn't solid metal and so must be *less* dense than these. (Of course, we could also just look up the average density of the Earth directly, but where's the fun in that?). If we assume $\langle \rho_\oplus \rangle \approx 4\text{g cm}^{-3} = 4000 \text{ kg m}^{-3}$, then we have

$$M_\oplus = V_\oplus \langle \rho_\oplus \rangle \quad (11)$$

$$\approx \left(10^{20}\text{ m}^3\right)\left(4000 \text{ kg m}^{-3}\right) \quad (12)$$

$$\boxed{\approx 4 \times 10^{24}\text{ kg}} \quad (13)$$

... which is a bit lower than the known value of 5.972×10^{24}kg. We underestimated a bit because the Earth's actual density is 5.5 g/cc (not the 4.0 assumed above). But still: not too bad!

(c) **How Big?! [3 pts]** Jupiter is roughly $10\times$ larger (in physical size) than the Earth (i.e., $R_{Jup} \approx 10R_\oplus$), and the Sun is roughly $10\times$ larger than Jupiter ($R_\odot \approx 10R_{Jup}$). Roughly estimate the volume of both of these objects, *relative to the volume of the Earth* (i.e., in units of $V_\oplus$).

Solution: Since all these objects are approximately spherical and $V \propto R^3$, we know

$$\frac{V_{Jup}}{V_\oplus} = \left(\frac{R_{Jup}}{R_\oplus}\right)^3 \approx 1000 \quad (14)$$

and so

$$\boxed{V_{Jup} \approx 1000 V_\oplus}. \quad (15)$$

The actual value is 1322 or so, because actually Jupiter is a bit more than $10\times$ the size of the Earth.

For the Sun, we know

$$\frac{V_\odot}{V_\oplus} = \left(\frac{R_\odot}{R_\oplus}\right)^3 \approx 10^6 \quad (16)$$

and so

$$\boxed{V_\odot \approx 10^6 V_\oplus}. \quad (17)$$

The actual value is around 1.31×10^6: our estimate is off again because we just scaled from the Jovian value above, and Jupiter is actually bit more than $10\times$ the size of the Earth.

(d) **In an Age Before Spotify... [3 pts].** Pick your favorite over-the-air radio station. What is the frequency at which it broadcasts its signals? Estimate the approximate wavelength of the station's radio wave signals.

Solution: I don't listen to much radio these days, though I listen to more radio than Spotify! Still, let's pick the FM station 91.5. In the USA, FM station frequencies are given in MHz while AM stations are listed in kHz. So

$$\boxed{\nu = 91.5\text{ MHz.}} \quad (18)$$

To find the wavelength, I remember that for any kind of light waves (which includes radio signals!) the speed of light equals the lights frequency times its wavelength: $c = \nu\lambda$. So we then have

$$\lambda = \frac{c}{\nu} \approx \frac{3 \times 10^8 \text{ m/s}}{91.5 \text{ MHz}} \sim \frac{3 \times 10^8 \text{ m/s}}{9 \times 10^7 \text{ Hz}} \approx \frac{1}{3} \times 10 \text{ m} \approx \boxed{3 \text{ m}}. \quad (19)$$