UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Physical Astronomy (ASTR 391) — Prof. Crossfield — Spring 2024

Problem Set 5 – REVISED
Due: Monday, March 29, 2024, in class
This problem set is worth **23 points** (plus 7 bonus points).

As always, be sure to: show your work, circle your final answer, and use the appropriate number of significant figures. Also, please **submit your PSet as a single PDF file** (not individual scanned images, which are tougher to keep track of), and **include your name in the PDF's filename**.

Journey to the Center of a Star

For this problem, you have two options: use a computer to perform numerical integration (via programming language or, as shown in lecture, a spreadsheet), or alternatively use mathematical integration (calculus). If you use numerical integration, be sure to use >20 layers so your results will be reasonably accurate. Regardless: remembering when making plots for the question below that any good plot has labeled axes!

1. [5 pts] In lecture we discussed at some length how we can model the physical conditions in a star's interior. Assume the star has a slightly more realistic density profile (valid from $0 \leq r \leq R_*$) of

$$\rho(r) = \rho_c \left((r/R_*)^2 - 2r/R_* + 1 \right) = \rho_c \left(\frac{r}{R_*} - 1 \right)^2 \quad (1)$$

Plot $\rho(r)$ over the full range from $r = 0$ to $r = 2R_*$. Discuss why this density profile might be slightly more reaslistic than the constant-density model we assumed in class.

Solution:

Just by plugging in a few test values, we can see how the density behaves. For $r = 0$ we have $\rho = \rho_c$, and for $r = R_*$ we have $\rho = 0$, so the density appears to decrease from the center out to the surface, as we expect. For an intermediate value such as $r = R_*/2$, we see $\rho = \rho_c/4$; from this and more test points, or from the quadratic

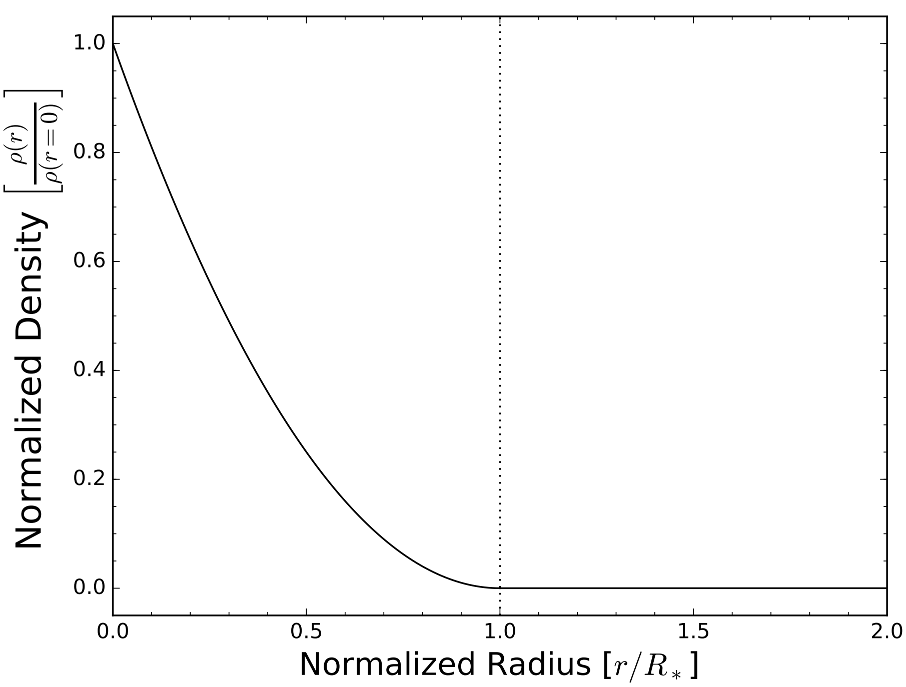

Figure 1: Density in the star for the functional form given. The vertical dotted line indicates the stellar surface.

functional form, we could see that $\rho(r)$ peaks at the core and decreases quadratically down to zero at $r = R_*$. The full plot is shown in Fig. 1.

This is a bit more realistic because we should expect density (along with pressure, temperature, and related quantities) to be highest in the core and smallest out near the surface.

2. [5 pts] Show that the Enclosed Mass profile $M_{enc}(r)$ within this star is equivalent to the expression

$$4\pi\rho_c r^3\left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right). \tag{2}$$

Then plot $M_{enc}(r)$.

Solution: To determine the enclosed mass within a radius r via integration (calculus!) we use the recipe

$$M_{enc}(r) \equiv \int_{r=0}^{r=r} 4\pi r^2\rho(r)dr. \tag{3}$$

Plugging in our expression given for $\rho(r)$, we have

$$\begin{aligned} M_{enc}(r) \equiv& \int_{r=0}^{r=r} 4\pi r^2\rho_c\left((r/R_*)^2 - 2r/R_* + 1\right)dr & (4)\\ =& 4\pi\rho_c\int_{r=0}^{r=r} r^2\left((r/R_*)^2 - 2r/R_* + 1\right)dr & (5)\\ =& 4\pi\rho_c\int_{r=0}^{r=r}\left((r^4/R_*^2) - 2r^3/R_* + r^2\right)dr & (6)\\ =& 4\pi\rho_c\left(\frac{r^5}{5R_*^2} - 2\frac{r^4}{4R_*} + \frac{r^3}{3}\right) & (7)\\ =& \boxed{4\pi\rho_c r^3\left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)} & (8)\\ & & (9)\end{aligned}$$

3. [5 pts] Calculate the gravitational acceleration profile $g(r)$ inside the star, using the expression above for $M_{enc}(r)$. Then plot $g(r)$.

Solution: Remember that the gravitational acceleration in a star is just

$$g_{\text{inside}}(r) \equiv \frac{GM_{enc}(r)}{r^2}. \tag{10}$$

This means the gravitational acceleration inside a star is just

$$\begin{aligned} g(r) =& \frac{G}{r^2}4\pi\rho_c r^3\left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right) & (11)\\ =& \boxed{4\pi\rho_c Gr\left(\frac{r^2}{5R_*^2} - \frac{r}{2R_*} + \frac{1}{3}\right)} & (12)\\ & & (13)\end{aligned}$$

4. [3 pts] Starting with the equation of hydrostatic equilibrium ($dP/dr = -\rho(r)g(r)$), we could continue in this vein and calculate the internal pressure and temperature of the star – but things would quickly get really messy.

Instead, we can make a rough approximation to get a sense of the conditions inside the star, by assuming that $dP/dR \approx P_c/R_*$ (i.e., the pressure at the center of the star divided by the star's radius), and further assuming density and gravity are constant: $\rho(r) = \rho_{\text{avg}}$ and $g(r) = g_{\text{surface}}$.

Under these simplifying assumptions, the pressure at the center of the star is just $P_c \approx \rho_{\text{avg}}g_{\text{surf}}R_*$. Calculate a numerical value of P_c (in SI units) for the Sun, and for a red dwarf with $M_*/M_\odot = R_*/R_\odot = 0.3$. How do these compare to the atmospheric pressure here on Earth?

Solution: As directed, we make the greatly-simplifying assumption that the central pressure in the star is just $P_c \approx \rho_{\text{avg}}g_{\text{surf}}R_*$. We're asked to estimate this quantity for both the Sun and an M dwarf. We'll need to calculate their average density ($M_*/[\frac{4}{3}\pi R_*^3]$) and surface gravity ($GM_*/R_*^2$). So the final expression we want is

$$\begin{aligned} P_c \approx& \rho_{\text{avg}}g_{\text{surf}} & (14)\\ \approx& \frac{M_*}{\frac{4}{3}\pi R_*^3}\frac{GM_*}{R_*^2}R_* & (15)\\ \approx& \frac{3GM_*^2}{4\pi R_*^4}. & (16)\end{aligned}$$

Star	$M_*/M_\odot$	$R_*/R_\odot$	$\rho_{\rm avg}$ [kg/m^3]	$g_{\rm surf}$ [m/s^2]	P_c [Pa]
Sun	1	1	1400	270	$\boxed{2.7 \times 10^{14}}$
M dwarf	0.3	0.3	15700	910	$\boxed{3.0 \times 10^{15}}$

Pressure at sea level is 1 bar$\approx 10^5$ Pa, so the pressure at the center of these stars is roughly 10 billion times greater!

5. [5 pts] Assume that our star is an ideal gas made entirely of hydrogen atoms. In this case, derive a symbolic expression for the temperature T_c at the center of a star in terms of its pressure and mass density.

   Then, calculate a numerical value for the central temperature of both the Sun and the M dwarf described above. How do these compare to the surface temperatures of these stars?

   Solution: Since we're told to assume that the star is an ideal gas, it must obey

$$P = nk_BT \tag{17}$$

   everywhere (including at its core, where we'd specifically be considering P_c, n_c, and T_c).

   Since we're asked to give the answer in terms of density (not number density), we'll have to remember that $n = \rho/m_{\rm avg}$, where $m_{\rm avg}$ is the average mass of particles under consideration – in this case, just hydrogen atoms. So our expression is then

$$\boxed{T_c \approx \frac{m_{\rm avg}}{k_B}\frac{P_c}{\rho_{\rm avg}}}. \tag{18}$$

$$\begin{aligned} T_c &\approx \frac{P_c m_{\rm avg}}{\rho k_B} && (19)\\ &\approx \frac{3Gm_{\rm avg}}{4\pi k_B}\frac{M_*^2}{R_*^4\rho_{\rm avg}} && (20)\\ &\approx \frac{Gm_{\rm avg}}{k_B}\frac{M_*}{R_*} && (21)\\ & && (22) \end{aligned}$$

   Taking $m_{\rm avg} = m_H \approx (1/6) \times 10^{-26}$ kg, $k_B \approx (1/7) \times 10^{22}$ J/K, and $G \approx (2/3) \times 10^{-11}$ (in SI units), we then find

$$\boxed{T_c \approx 2.3 \times 10^7 \text{ K}} \tag{23}$$

   for both the Sun and for the M dwarf.

6. BONUS [7 pts]: Use the equation of hydrostatic equilibrium to calculate the full pressure profile, $P(r)$, throughout the star described in parts 1–3 above. Plot $P(r)$.