

## Lecture 1 – Basic Definitions and Results

We are interested in the flow of radiation; what is often called “radiative transfer”. We follow the intensity,  $I_\nu$  (units of joule  $\text{m}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{ster}^{-1}$ ) and describe the sources and sinks of photons. The first sink term is absorption. In this process, a photon is destroyed and an atom (or molecule) goes from a lower energy state to a higher energy state.



where  $X$  denotes the atom in the lower energy level and  $X^*$  denotes the atom in the upper energy level. If  $ds$  denotes an increment in path length, then the absorption coefficient,  $\kappa_\nu$ , (units of  $\text{m}^{-1}$ ) is defined such that:

$$dI_\nu = -\kappa_\nu I_\nu ds \quad (2)$$

Therefore:

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu \quad (3)$$

If the atom has a cross section at frequency  $\nu$  of  $\sigma_\nu$  (units of  $\text{m}^2$ ) and the density of atoms is  $n$  ( $\text{m}^{-3}$ ), then we note that

$$\kappa_\nu = n \sigma_\nu \quad (4)$$

Often, we also use the opacity,  $\chi_\nu$  defined so that:

$$\chi_\nu = \frac{\kappa_\nu}{\rho} \quad (5)$$

where  $\rho$  is the mass density of the material. Since  $\rho = \mu n$  where  $\mu$  is the mean molecular weight, then

$$\chi_\nu = \frac{\sigma_\nu}{\mu} \quad (6)$$

If we define the mean free path for a photon of frequency  $\nu$  as  $l_\nu$ , then

$$l_\nu = \kappa_\nu^{-1} = \frac{1}{n\sigma_\nu} \quad (7)$$

If light with intensity,  $I_\nu^0$  is incident upon a medium of uniform  $\kappa_\nu$  and if we define  $s = 0$  to be the boundary of the medium, then the intensity as a function of  $s$  is given as the solution to the differential equation (3):

$$I_\nu = I_\nu^0 e^{-\kappa_\nu s} \quad (8)$$

Another important parameter to introduce is the dimensionless quantity, the optical depth,  $\tau_\nu$ . We define

$$d\tau_\nu = \kappa_\nu ds \quad (9)$$

For a homogeneous medium, then

$$\tau_\nu = \kappa_\nu s \quad (10)$$

The attenuation of the light is modest if  $\tau_\nu < 1$ ; such a situation is described as being “optically thin”. The attenuation of the light is large if  $\tau_\nu > 1$ ; such a situation is described as being optically thick.

In addition to absorption, light can be produced by emission. Schematically, this occurs when an atom (or molecule) undergoes a transition from an upper energy level ( $X^*$ ) to a lower energy level ( $X$ ). Thus:



We define the emissivity,  $\epsilon_\nu$  as the rate at which energy is emitted per unit solid angle. Then, if we only include the source term:

$$dI_\nu = \epsilon_\nu ds \quad (12)$$

or

$$\frac{dI_\nu}{ds} = \epsilon_\nu \quad (13)$$

If we include both the source term and the sink term, then we may write that

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \epsilon_\nu \quad (14)$$

This expression is called the equation of transfer and is the fundamental equation used to describe the flow of radiation energy.

The equation of transfer is often re-written in the following fashion. Divide the equation by  $\kappa_\nu$ , and use the definition of the optical depth to get:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{\epsilon_\nu}{\kappa_\nu} \quad (15)$$

We introduce a new quantity called the source function,  $S_\nu$  (units of joule  $\text{m}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{ster}^{-1}$ ) such that

$$S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \quad (16)$$

Thus the equation of transfer becomes:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \quad (17)$$

If we measure through a uniform medium ( $S_\nu$  is constant) in terms of optical depth instead of physical distance, then the solution to the equation of transfer for the emergent intensity from a uniform medium of optical depth,  $\tau$  is

$$I_\nu^0 = S_\nu (1 - e^{-\tau_\nu}) \quad (18)$$

Note that if the medium is optically thin ( $\tau_\nu < 1$ ), then

$$I_\nu^0 \approx S_\nu \tau_\nu \quad (19)$$

while if the medium is optically thick ( $\tau_\nu > 1$ ), then

$$I_\nu^0 \approx S_\nu \quad (20)$$

An opaque medium (that is, an optically thick medium) is a “black body”. Therefore, for black body radiation, we may set,  $S_\nu$  equal to the Planck function or

$$S_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad (21)$$

For an opaque object, the emergent intensity is independent of its composition and depends only upon its temperature.

Two related quantities to the intensity of radiation are the mean intensity and the flux or flow of energy. The mean intensity,  $J_\nu$ , is defined as the average over solid angle of the intensity. Therefore, at any particular location, we define

$$J_\nu = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi I_\nu(\theta, \phi) \sin \theta \, d\theta \, d\phi \quad (22)$$

Note that  $J_\nu$  and  $I_\nu$  have the same units.

If we define  $\theta$  relative to the  $Z$ -axis, then the flux of energy along the  $Z$  axis,  $F_\nu$ , is given by the expression

$$F_\nu = \int_0^{2\pi} \int_0^\pi I_\nu(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \quad (23)$$

The flux has units of joule  $\text{m}^{-2} \text{s}^{-1} \text{Hz}^{-1}$  or watts  $\text{m}^{-2} \text{Hz}^{-1}$ .

In an isotropic radiation field where the light moves equally in all directions, let  $I_\nu(\theta, \phi) = I_\nu^0$ . Then:

$$J_\nu = I_\nu^0 \quad (24)$$

and

$$F_\nu = 0 \quad (25)$$

In an isotropic radiation field, there is no flux because there is no *net* transport of energy.

Another important case is where a surface is radiating out into space. Assume a situation (such as the surface of a star or an aperture looking into a cavity.) In this case we assume that  $I_\nu$  is only a function of  $\theta$  and is independent of  $\phi$ . We write that

$$I_\nu(\theta) = I_\nu^0 \quad (26)$$

for  $0 \leq \theta \leq \pi/2$ . and

$$I_\nu(\theta) = 0 \quad (27)$$

Therefore

$$F_\nu = 2\pi \int_0^{\pi/2} I_\nu^0 \cos \theta \sin \theta d\theta \quad (28)$$

By setting  $x = \sin \theta$ , then it is easy to evaluate the integral and we find that

$$F_\nu = \pi I_\nu^0 \quad (29)$$

From above, we may therefore write that for a black body,

$$F_\nu = 2\pi \frac{h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad (30)$$

Often we are interested in the flow of all the energy and not just the energy at a particular frequency. We can write for the total flux,  $F$ , that

$$F = \int_0^\infty F_\nu d\nu \quad (31)$$

The integrated flux has units of watts  $\text{m}^{-2}$ . For radiation from a plane surface, then

$$F = \int_0^\infty 2\pi \frac{h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} d\nu \quad (32)$$

With the substitution

$$x = \frac{h\nu}{kT} \quad (33)$$

Then:

$$F = 2\pi \frac{k^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx \quad (34)$$

Consider an atom with an upper energy level  $U$  and a lower energy level,  $L$ . We now want to describe the emission and absorption by this atom. In order to do this exactly, we need to understand how the transition occurs by using quantum mechanics. However, even without a detailed understanding of the system, we can determine some general properties of a spectral line.

Assume that the atom can undergo a “spontaneous” transition from the upper to the lower level with a rate (units of  $\text{s}^{-1}$ ) of  $A_{UL}$  where this quantity  $A_{UL}$  is often called the “Einstein A”. The mean lifetime of the atom in the upper level is the inverse of the spontaneous decay rate or  $A_{UL}^{-1}$ . The units of this mean lifetime are  $s$ . The emissivity of the atom is given by the expression:

$$\epsilon_\nu = \frac{1}{4\pi} A_{UL} h\nu n_U \phi(\Delta\nu) \quad (35)$$

where  $\phi(\Delta\nu)$  (units of  $\text{Hz}^{-1}$ ) is called the line profile. The rate of production of line photons depends upon the number of atoms in the upper level, the energy per photon, the Einstein  $A$ . We also include the factor  $\frac{1}{4\pi}$  because we are interested in the production of photons into each solid angle as well as the total rate of production of photons. Finally, we include  $\phi(\Delta\nu)$ , the line profile, because we want to know the spectral energy distribution of the emission at different frequencies. The line photons are not all emitted at exactly the same frequency. Instead there is a spread of frequencies, and the function  $\phi(\Delta\nu)$  describes this spread.

Because the line photons are emitted near the frequency  $\nu$ , we can define the frequency offset from line center,  $\Delta\nu$  as

$$\Delta\nu = \nu - \nu_0 \quad (36)$$

where  $\nu_0$  is the frequency at line center. Then we expect that

$$\int_{-\infty}^{+\infty} \phi(\Delta\nu) d\Delta\nu = 1 \quad (37)$$

Alternatively, we may write this equation as:

$$\int_0^{+\infty} \phi(\Delta\nu) d\nu = 1 \quad (38)$$

since:

$$d\nu = d\Delta\nu \quad (39)$$

The difference between these two equations is the lower limit of the integral. Since the emission always occurs at frequencies relatively near  $\nu_0$  so it does not make a real difference whether we integrate  $\Delta\nu$  to  $-\infty$  (physically slightly unrealistic) or to  $-\nu_0$  (physically realistic but mathematically more complex).

An important example of line broadening is that produced by Doppler motions of the atoms in the system. If we observe emission from a gas, then along the line of sight, some atoms will be approaching us and others receding. We expect that:

$$\frac{\Delta\nu}{\nu} = -\frac{v_r}{c} \quad (40)$$

where  $v_r$  is the radial velocity of the gas atom and  $c$  is the speed of light. In a 1-dimensional Maxwell-Boltzmann distribution,

$$f(v_r) dv_r \propto \exp\left(-\frac{mv_r^2}{2kT}\right) dv_r \quad (41)$$

where  $m$  denotes the mass of the atom. The velocity distribution is symmetric around its mean value which here we take to be  $0 \text{ m s}^{-1}$ . We therefore expect that

$$f(\Delta\nu) d(\Delta\nu) \propto \exp\left(-\frac{mc^2\Delta\nu^2}{2kT\nu^2}\right) d(\Delta\nu) \quad (42)$$

We expect that

$$\phi(\Delta\nu) \propto f(\Delta\nu) \quad (43)$$

Therefore, with the normalization condition, we find for a line undergoing thermal broadening that:

$$\phi(\Delta\nu) = \sqrt{\frac{mc^2}{2\pi kT\nu^2}} \exp\left(-\frac{mc^2\Delta\nu^2}{2kT\nu^2}\right) \quad (44)$$

In addition to emission, the atom can absorb light. We define the ‘‘Einstein B’’ such that  $B_{LU}$

$$\kappa_\nu = n_L B_{LU} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \quad (45)$$

With this definition,  $B_{LU}$  has a similar appearance to  $A_{UL}$ . Warning: while everyone agrees about the definition of the Einstein  $A$ , different authors do or do not include the  $\frac{1}{4\pi}$  term in their definition of the Einstein ‘‘B’’. The units of  $B_{LU}$  (or  $B_{UL}$ ) are different from those of  $A_{UL}$ . In particular, the units of  $B_{LU}$  are equal to the units of  $A_{UL}$  divided by an intensity. Thus  $B_{LU}$  has units of  $\text{m}^2 \text{s}^{-1} \text{joule}^{-1}$ . You may also think of the cross section in the line and write,

$$\sigma_\nu = B_{LU} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \quad (46)$$

As pointed out by no lesser an authority than Einstein, we must also allow for the possibility of stimulated emission. That is, as with all harmonic oscillators, there can be forced oscillations. We therefore, assume that there may be ‘‘stimulated emission’’ which is the reverse of absorption. In stimulated emission, we expect that:

$$h\nu + X^* \rightarrow h\nu + h\nu + X \quad (47)$$

In this scheme, there is conservation of energy, and a photon is produced from an atom which is already excited. We denote the coefficient for stimulated emission as  $B_{UL}$ , and it acts like a ‘‘negative absorption’’ Thus, it contributes to the opacity as:

$$\kappa_\nu = -n_U B_{UL} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \quad (48)$$

We now estimate the relationship between the Einstein  $A$  and  $B$ 's. Consider the two level atom. In a steady state, the rate at which atoms leave level  $U$  equals the rate at which they enter level  $U$ . The rate at which they leave  $U$  depends both upon the rate of spontaneous emission and the rate of stimulated emission. The rate per unit volume of spontaneous emission is

$$\frac{1}{4\pi} A_{UL} n_U \phi(\Delta\nu) \quad (49)$$

The rate per unit volume of stimulated emission is the result of photons arriving from all directions and therefore depends upon the mean intensity,  $J_\nu$ . We can write that the total

rate per unit volume of stimulated emission is

$$J_\nu n_U B_{UL} \frac{1}{4\pi} \phi(\Delta\nu) \quad (50)$$

The rate per unit volume at which atoms enter level  $U$  is given by the rate per unit volume of absorptions from level  $L$  and is given by the expression:

$$J_\nu n_L B_{LU} \frac{1}{4\pi} \phi(\Delta\nu) \quad (51)$$

Therefore, in a steady state,

$$\frac{1}{4\pi} A_{UL} n_U \phi(\Delta\nu) + J_\nu n_U B_{UL} \frac{1}{4\pi} \phi(\Delta\nu) = J_\nu n_L B_{LU} \frac{1}{4\pi} \phi(\Delta\nu) \quad (52)$$

This equation can be re-written as:

$$A_{UL} n_U + J_\nu n_U B_{UL} = J_\nu n_L B_{LU} \quad (53)$$

We can re-arrange the terms to find:

$$J_\nu = \frac{A_{UL} n_U}{n_L B_{LU} - n_U B_{UL}} \quad (54)$$

or

$$J_\nu = \frac{\frac{A_{UL}}{B_{UL}}}{\frac{n_L B_{LU}}{n_U B_{UL}} - 1} \quad (55)$$

This relationship is derived for a steady state at any temperature. We may therefore set  $J_\nu$  equal to the Planck function at any temperature. This gives the following:

$$\frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1} = J_\nu = \frac{\frac{A_{UL}}{B_{UL}}}{\frac{n_L B_{LU}}{n_U B_{UL}} - 1} \quad (56)$$

The solution to this expression is that

$$\frac{2h\nu^3}{c^2} = \frac{A_{UL}}{B_{UL}} \quad (57)$$

and

$$\exp\left(\frac{h\nu}{kT}\right) = \frac{n_L B_{LU}}{n_U B_{UL}} \quad (58)$$

which is the same as:

$$\exp\left(-\frac{h\nu}{kT}\right) = \frac{n_U B_{UL}}{n_L B_{LU}} \quad (59)$$

In atomic spectroscopy, it is often found that a level is “degenerate”. This means that there might be more than one “sublevel” at the same energy. If  $g$  is used to denote the number of sublevels in a level, then we have both  $g_L$  and  $g_U$  for the lower and upper energy levels, respectively. We can generalize the usual Boltzmann relationship so that

$$\frac{n_U}{n_L} = \frac{g_U}{g_L} \exp\left(-\frac{h\nu}{kT}\right) \quad (60)$$

Therefore:

$$g_L B_{LU} = g_U B_{UL} \quad (61)$$

At this point, we have now established that there is a simple, relationship between the Einstein  $A$  and the Einstein  $B$ 's. First, any atom which has a high value of  $A$ , that is any atom which can be a strong emitter, also must have a large value of  $B$  which means that it also is a strong absorber. Second, of necessity,  $B_{UL}$  is not zero and is positive. The process of stimulated emission must occur if thermodynamic equilibrium can be achieved. The Einstein  $A$  and  $B$ 's are properties of the atom and *not* of the gas temperature or pressure.

The total opacity of the gas depends upon the difference between true absorptions and stimulated emissions. We may write that

$$\kappa_\nu = n_L B_{LU} h\nu \frac{1}{4\pi} \phi(\Delta\nu) - n_U B_{UL} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \quad (62)$$

Collecting terms, this means that:

$$\kappa_\nu = n_L B_{LU} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \left(1 - \frac{n_U B_{UL}}{n_L B_{LU}}\right) \quad (63)$$

In thermodynamic equilibrium, we therefore have that:

$$\kappa_\nu = n_L B_{LU} h\nu \frac{1}{4\pi} \phi(\Delta\nu) \left(1 - \exp\left[-\frac{h\nu}{kT}\right]\right) \quad (64)$$

This is the “normal” condition of a gas.

A gas need not be in thermodynamic equilibrium. If, by some process the gas can be kept out of equilibrium and if, in fact, the population in the upper energy level can be kept high by some process, then,  $\kappa_\nu$  can become negative. That is, there can be more stimulated emissions than absorptions. Therefore, when we write that

$$I_\nu = I_\nu^0 e^{-\kappa_\nu s} \quad (65)$$

we find a net effect in the emergent intensity compared to the incident intensity,  $I_\nu^0$ , *not* a decrease. A device which can do this is called a laser = light amplification [by] stimulated emission [of] radiation.