

## Radiative Transfer I: Fundamentals

## 1.1 INTRODUCTION

Radiative transfer describes how radiation changes as it travels through a medium. A wide variety of absorption and emission processes from atoms, molecules, and dust particles contribute to the radiation that emerges at the top of the planetary atmosphere. Although we can observe only the radiation emerging from the planet's atmosphere, much information is contained in the emergent spectrum, including details of the atmospheric temperature, pressure, and composition.

The radiative transfer equation describes the change in a beam of radiation as it travels some distance  $s$  through a volume of gas. The changes are due to losses from the beam  $\kappa(\mathbf{x}, \nu)I(\mathbf{x}, \hat{\mathbf{n}}, \nu)$ , where  $\kappa$  is the extinction coefficient and  $I$  is the intensity, and additions to the beam  $\varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu)$ , where  $\varepsilon$  is the emission coefficient. Adding up the losses and gains, for an atmosphere that is not changing with time (i.e., static), the radiative transfer equation is

$$\frac{dI(\mathbf{x}, \hat{\mathbf{n}}, \nu)}{ds} = -\kappa(\mathbf{x}, \nu)I(\mathbf{x}, \hat{\mathbf{n}}, \nu) + \varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu). \quad (5.1)$$

In this chapter we will omit the  $t$ -dependence of the intensity, flux, and other terms. Nevertheless, we must still keep in mind that radiation involves energy flow: energy per unit time.

The goal of this chapter is to lay the foundation for the equation of radiative transfer. We first introduce several fundamental quantities and concepts needed to characterize the radiation field. Next we describe the full equation of radiative transfer and derive the plane-parallel approximation. We finish with an outline of the formal solution to the radiative transfer equation. We leave details of the solutions of the equation of radiative transfer until Chapter 6.

## 1.2 OPACITY

## 1.2.1 Definition of Opacity

The most important physical quantity in the atmosphere that affects the transfer of radiation is the opacity. The opacity describes how opaque a substance is: how hard it is for radiation to pass through that substance. Opacity depends on the number density of particles in the atmosphere, and the particles' absorbing, scattering, and emitting properties, which may in turn depend on temperature, pressure, and frequency (i.e., wavelength). In order to describe radiative transfer we therefore must

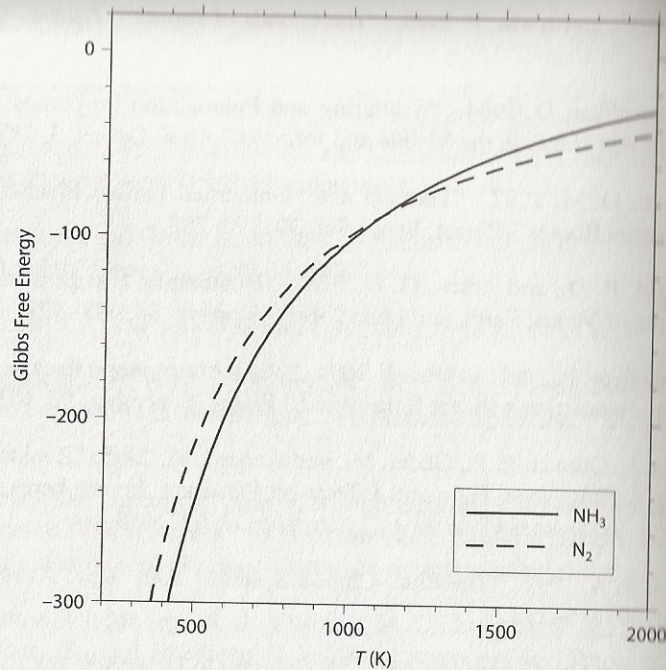


Figure 4.17 Gibbs free energies of  $\text{NH}_3$  and  $\text{N}_2$ .

## EXERCISES

1. Derive the Gibbs free energy equation (equation [4.11]) from the first law of thermodynamics (equation [4.3]) by filling in any missing steps.
2. Figure 4.17 shows the Gibbs free energy of the nitrogen compounds  $\text{NH}_3$  and  $\text{N}_2$ . Can you use Figure 4.17 to determine which nitrogen molecule will dominate at a given temperature and pressure? Why or why not?
3. Compute the escape velocity of the Earth, Venus, and Mars, in m/s and in eV. Based on their exobase temperatures compute the escape energies of H and O, as well as the Jeans escape parameter  $\lambda_c$ . Should H and O thermally escape on Earth, Venus, or Mars?
4. Make a plot of relative stellar heating (in units of the solar luminosity at Earth,  $L_\oplus = L_\odot/4\pi a_\oplus^2$ ) ( $y$ -axis) vs. escape velocity ( $x$ -axis) for all solar system planets, Pluto, Europa, Titan, Triton, and the Moon. On the same graph, plot all transiting exoplanets (take data from the Extrasolar Planet Encyclopaedia at <http://exoplanet.eu/catalog.php>, but consider only exoplanets with measured masses and radii). Because star luminosity data are not available, assume that  $L/L_\odot = (M/M_\odot)^{3/5}$ . Use a log-log plot. Comment on your findings.



define coefficients for the absorption and emission of radiation. The word opacity refers to absorption and scattering (i.e., extinction) coefficients of atmospheric particles. We emphasize that the underlying physical processes of photon absorption and emission, will, for now, be hidden in these macroscopic coefficients. The microscopic details of the absorption and emission coefficients will be covered in Chapter 8.

Before we describe the opacity in more detail, we should keep in mind that many different symbols for the opacity coefficients are used in other textbooks and various papers in the literature.

### 5.2.2 The Extinction Coefficient: Absorption and Scattering

The monochromatic extinction coefficient  $\kappa(\mathbf{x}, \nu)$  describes the amount of energy removed from a beam of radiation from a volume  $dV$  and a solid angle  $d\Omega$ , per unit time per unit frequency,

$$dE = \kappa(\mathbf{x}, \nu) I(\mathbf{x}, \hat{\mathbf{n}}, \nu) dV d\Omega d\nu dt. \quad (5.3)$$

See Chapter 2 and Figure 2.1 for a description of the intensity  $I$ .

We use  $\kappa$  (with units of  $\text{m}^{-1}$ ) to include all processes that remove energy from a beam of radiation. True absorption describes processes that destroy the photon. This can happen, for example, when a photon is absorbed by a particle, and then is converted to kinetic energy of the gas by a subsequent collision. We will denote the true absorption coefficient by  $\alpha(\mathbf{x}, \nu)$ . Scattering describes photons that are removed from the beam by a change of direction (and possibly a change of energy). We denote the scattering coefficient by  $\sigma_s(\mathbf{x}, \nu)$ , where we use the subscript  $s$  to distinguish the scattering coefficient  $\sigma_s$  in  $\text{m}^{-1}$  from the cross section  $\sigma$  in  $\text{m}^2$ . The extinction coefficient is therefore comprised of

$$\kappa(\mathbf{x}, \nu) = \alpha(\mathbf{x}, \nu) + \sigma_s(\mathbf{x}, \nu). \quad (5.4)$$

Extinction is isotropic in a static medium because the absorbing or scattering particle does not care which direction the photon is coming from. Hence  $\kappa$  has no  $\hat{\mathbf{n}}$ -dependence.

### 5.2.3 The Emission Coefficient: Thermal Emission and Scattering

The monochromatic emission coefficient  $\varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu)$  describes the amount of energy emitted into a volume  $dV$  within a solid angle  $d\Omega$  per unit time, per unit frequency,

$$dE = \varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu) dV d\Omega dt d\nu, \quad (5.5)$$

with dimensions  $\text{J m}^{-3} \text{sr}^{-1} \text{s}^{-1} \text{Hz}^{-1}$ . Planetary atmosphere emission includes both thermal emission and scattering, because both processes can add to the beam of radiation.

Thermal emission can be described by the Kirchhoff-Planck relation also known as Kirchhoff's Law of thermal radiation, or Kirchhoff's Law for short,

$$\varepsilon_{\text{therm}}(\mathbf{x}, \nu) = \alpha(\mathbf{x}, \nu) B(\mathbf{x}, \nu), \quad (5.6)$$

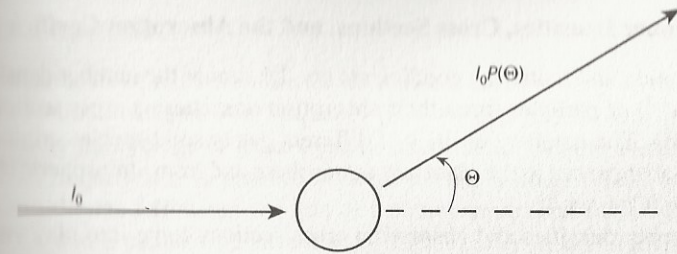


Figure 5.1 Definition of scattering angle used in  $P(\Theta)$ .

where we recall that  $B(\mathbf{x}, \nu)$  is black body radiation, and  $\mathbf{x}$  depends on temperature. This equation for thermal radiation is valid in local thermodynamic equilibrium, and is further described in Section 5.4. The thermal emission is isotropic in a static medium because atoms and molecules have no preferred direction of radiation.

The emission coefficient for pure scattering is

$$\varepsilon_{\text{scat}}(\mathbf{x}, \hat{\mathbf{n}}, \nu) = \sigma_s(\mathbf{x}, \nu) \frac{1}{4\pi} \int_{\Omega} P(\Theta) I(\mathbf{x}', \hat{\mathbf{n}}', \nu) d\Omega'. \quad (5.6)$$

Here,  $\Theta$  is the scattering angle,  $\Theta = \hat{\Omega} \cdot \hat{\Omega}'$ , where in general the terms with primes refer to the direction of incidence and the terms without primes refer to the direction after scattering (Figure 5.1 and see ahead to a related discussion in Section 8.5). Here  $\sigma_s(\mathbf{x}, \nu)$  is the scattering coefficient, with dimensions  $\text{m}^{-1}$ . In this book we will consider only coherent scattering, the case where the photon retains the same energy (i.e., frequency).

Scattering can be anisotropic and so we define  $P(\Theta)$  as the dimensionless single scattering phase function. The phase function denotes the redirection in the incident intensity to the outgoing intensity and describes the 3D directional scattering probability. The phase function is normalized to 1. ( $P$  should not be confused with the illumination phase function  $\Phi(\alpha)$  used in Section 3.4.3 to describe observations.) We will describe the origin of the single scattering phase function in Chapter 8. The phase function is normalized:

$$\frac{1}{4\pi} \int_{\Omega} P(\Theta) d\Omega = 1. \quad (5.7)$$

If the scattering is isotropic,  $P(\Theta) = 1$ , and the emission coefficient for scattering (equation [5.6]) reduces to

$$\varepsilon_{\text{scat}}(\mathbf{x}, \nu) = \sigma_s(\mathbf{x}, \nu) J(\mathbf{x}, \nu). \quad (5.8)$$

Recall that  $J(\mathbf{x}, \nu)$  is the mean intensity (equation [2.2]).

Before ending our description of scattering we emphasize two points. First, scattering is both a source and a sink for the beam of radiation in the planet atmosphere. As a source, scattering is counted as emission. As a sink, scattering is counted as extinction. Second, scattering is angle dependent, in contrast to the isotropic thermal emission. Hence the emission coefficient dependency on  $\hat{\mathbf{n}}$ .



### 5.2.4 Number Densities, Cross Sections, and the Absorption Coefficient

The absorption and scattering coefficients are the sum of the number density (in units of  $\text{m}^{-3}$ ) of particles times their absorption or scattering cross sections (in units of  $\text{m}^2$ ). The number densities of different gas or solid species originate from elemental abundances in the planetary atmosphere and from atmospheric chemistry and escape (Chapter 4).

The number densities and absorption cross sections come into play via the absorption coefficient

$$\alpha(T, P, \nu) = \sum_j \alpha_j(T, P, \nu) = \sum_j \sum_i n_{ji}(T, P) \sigma_{ji}(T, P, \nu). \quad (5.9)$$

Here  $j$  refers to different atomic and molecular species and  $i$  refers to different atomic or molecular states (i.e., energy level populations). We note that, when we have previously used the location  $\mathbf{x}$  to describe the absorption coefficient in an atmosphere (by  $\alpha(\mathbf{x}, \nu)$ ), for a general description we prefer to use the  $T, P$  dependencies. Recall that the temperature  $T$  and pressure  $P$  vary with location  $\mathbf{x}$  in an atmosphere; therefore for the extinction coefficient we can consider  $\mathbf{x}$  and  $T, P$  interchangeable. We could write an expression similar to equation [5.9] for the scattering coefficient  $\sigma_s$ , using number densities and scattering cross sections for solid particles or Rayleigh scattering.

Considerable inputs go into the computation of the absorption and emission coefficients. For example, to compute  $\alpha(T, P, \nu)$  for a broad range of frequencies for water vapor, up to hundreds of millions of molecular lines must be considered due to the numerous transitions possible among the many electronic, rotational, and vibrational states of the water vapor molecules.

### 5.2.5 Mass-Independent Opacity

Until now we have defined the opacity as the extinction coefficient in units of  $\text{m}^{-1}$ . Many books and articles instead use the mass-independent opacity, defined as

$$\kappa_m(T, P, \nu) = \sum_j \frac{\kappa(T, P, \nu)_j}{\rho_j}, \quad (5.10)$$

in units of  $\text{m}^2/\text{kg}$ . Here  $j$  refers to a gas species  $j$ .  $\rho$  is the gas density and for planetary atmospheres can be described by the ideal gas law.

The mass-independent opacity is useful because it is independent of the number density of molecules in the atmosphere. To further explore the utility of this opacity, let us revisit the opacity definition equation [5.9]. The opacity is equal to the number density times the absorption (or scattering) cross section. The number density itself depends heavily on pressure and temperature through the ideal gas law. The simplicity of the mass-independent opacity, therefore, is that it varies much less with pressure and temperature than does the mass-dependent opacity.

### 5.3 Mean Opacities

In many treatments of radiative transfer we desire a mean opacity so that we may consider equations averaged over frequency (i.e., wavelength). At first thought, we might consider the average or mean opacity taken literally as the average opacity over all frequencies. In practice such an opacity average would not be very useful. Consider the following: the flux of a planet approximated as a black body (Figure 3.8), and a planet with exceedingly high opacity at UV frequencies, but zero opacity at all other frequencies. In this case an opacity average is not useful—the high opacity is at UV frequencies where there is virtually no planetary flux. In this artificial example, a better representation would be simply zero opacity.

The point is that in radiative transfer we are interested in how much flux is blocked by (or allowed to pass through) the atmosphere at different frequencies by the absorbing or emitting particles. Useful mean opacities are therefore those that are weighted by a function of the intensity. Because the intensity of a planet's atmosphere is unknown—and indeed the quantity we are trying to solve for—the black body intensity is used.

The Planck mean opacity is defined as

$$\kappa_P(T, P, \nu) = \frac{\int_0^\infty \kappa(T, P, \nu) B(T, \nu) d\nu}{\int_0^\infty B(T, \nu) d\nu}. \quad (5.11)$$

The Planck mean opacity is the opacity weighted by the black body intensity at a given temperature. Where the opacity is high, the contribution to the mean opacity is also high. The Planck mean opacity is valid in “optically thin” regimes (see Section 5.3 for a definition of optically thin).

The Rosseland mean opacity is defined as

$$\frac{1}{\kappa_R(T, P, \nu)} = \frac{\int_0^\infty \frac{1}{\kappa(T, P, \nu)} \frac{dB(T, \nu)}{dT} d\nu}{\int_0^\infty \frac{dB(T, \nu)}{dT} d\nu}. \quad (5.12)$$

The Rosseland mean opacity is a harmonic mean, weighted by the temperature derivative of the black body intensity. The Rosseland mean is useful because it gives a higher weight to frequencies with a small opacity than to frequencies with a large opacity—this captures the physical situation where more radiation travels through the atmosphere at frequencies where opacity is smallest. This is in contrast to the Planck mean opacity. Typically the Planck mean opacity is valid in optically thin regions of the atmosphere and the Rosseland mean opacity is valid in optically thick regimes (see Section 5.3 for definitions of optically thin and thick). To further understand the definition of the Rosseland mean opacity, we must wait until a discussion of radiative diffusion in Section 6.4.4.

## 5.4 OPTICAL DEPTH

In radiative transfer we are aiming to understand the interactions of photons (or a beam of photons) as they travel through a planetary atmosphere. From our viewpoint, it is straightforward to think of a distance in meters or kilometers in a planetary atmosphere. A more natural distance scale for photon interactions is the optical



depth scale. As we shall see in this chapter and the next, the optical depth scale enables a simplification of the radiative transfer equation and its solutions.

The optical depth  $\tau$  is a measure of transparency and describes how opaque a part of the planetary atmosphere is to radiation traveling through it. A medium that is completely transparent has an optical depth of zero. A planetary atmosphere usually has some optical depth. In the optically thin case,  $\tau \ll 1$ ; a photon could travel the distance  $s$  without being absorbed or scattered. The opposite is the optically thick case,  $\tau > 1$ . We will later see that  $\tau$  is the dimensionless e-folding factor for absorption of hot radiation through a cooler gas layer. The optical depth is related to the absorbing particles in a planetary atmosphere. Because the extinction coefficient depends on frequency and on location in the planet atmosphere, so too does the optical depth. We therefore write the optical depth as  $\tau(x, \nu)$ .

The definition of the optical depth along a one-dimensional path  $s$  is

$$d\tau(s, \nu) = -\kappa(s, \nu) ds. \quad (5.18)$$

The optical depth is dimensionless.

The optical depth is often used as a distance scale in planetary atmospheres. In plane-parallel atmospheres (see Section 5.6.2) the optical depth scale often takes the convention of being measured backward along the ray of traveling photons. In other words, the plane-parallel atmosphere optical depth scale has  $\tau = 0$  defined at the top of the atmosphere, because the observer is looking down into the planetary atmosphere. This convention introduces a negative sign for the optical depth scale.

$$\tau(z, \nu) = -\int_{z=0}^{z_{\max}} \kappa(z', \nu) dz' \equiv \int_{z_{\max}}^{z=0} \kappa(z', \nu) dz'. \quad (5.19)$$

For notational simplicity we use  $\tau_\nu$  and take the  $z$ -dependence as implied.

We can relate the optical depth to the mean free path of a photon,  $l$ . (In this description we will drop the variable dependencies for clarity.) The mean free path is defined as the mean distance a photon can travel before it is stopped by interaction with a molecule or other atmospheric particle. Let us consider a slab with area  $A^2$ , volume  $A^2 dz$ , and number density  $n$ . The probability of an incident photon stopping in the slab can be calculated as the ratio of the net area of the molecules to the slab area,

$$P(z) = \frac{n\sigma A^2 dz}{A^2} = n\sigma dz, \quad (5.20)$$

where  $\sigma$  is the cross section. But we have already described the drop in intensity of a beam of radiation,

$$dI = -\kappa I dz = -n\sigma I dz. \quad (5.21)$$

We can now see a reasonable definition for the mean free path

$$l = \frac{1}{n\sigma} = \frac{1}{\kappa}, \quad (5.22)$$

and we can relate

$$d\tau = \frac{1}{l} dz. \quad (5.23)$$

We can now recover the mean free path more formally by starting with  $dI = -\kappa I dz$ ,

$$\frac{dI}{I} = -\frac{dz}{l}, \quad (5.19)$$

with a solution

$$I = I_0 e^{-z/l}. \quad (5.20)$$

Now, the probability that a photon is absorbed between  $z$  and  $z + dz$  is

$$dP(z) = \frac{I(z) - I(z + dz)}{I_0} = -\frac{1}{l} e^{-z/l} dz. \quad (5.21)$$

We can finally use the expectation value to show that the mean free path is  $l$ ,

$$\langle z \rangle = \int_0^\infty z dP(z) = \int_0^\infty \frac{z}{l} e^{-z/l} dz = l. \quad (5.22)$$

## 5.1 LOCAL THERMODYNAMIC EQUILIBRIUM

### 5.1.1 LTE Definition

We now arrive at one of the most important fundamental concepts in radiative transfer. This is the concept of local thermodynamic equilibrium (LTE). Complete thermodynamic equilibrium applies when the material is in thermal, chemical, and mechanical equilibrium. On the contrary, across a planet atmosphere, we expect huge differences in temperature and pressure, especially because of the open boundary at the top of the atmosphere. LTE is valid in a local area of the atmosphere where the temperature, pressure, or chemical gradients are small compared to the photon mean free path. LTE is therefore a local version of complete thermodynamic equilibrium. More specifically, in LTE, we assume that all of the conditions in thermodynamic equilibrium hold, except we let the radiation field depart from that of a black body. This is because one of the fundamental properties of planetary atmospheres is a radiation field that is very different from the black body radiation. We can use the radiative transfer equation to compute the intensity.

The reason LTE is one of the most important fundamental concepts in radiative transfer is that it provides a sweeping simplification of the radiative transfer problem. LTE will, in fact, enable us both to understand radiative transfer and spectral line formation conceptually, and to simplify the numerical solutions to the equation of radiative transfer.

### 5.1.1 LTE and Level Populations

Now you may be wondering: what is the magical simplification using LTE and how does it help to solve the radiative transfer equation? Let us go back to the absorption coefficient  $\alpha$  (equation [5.3]) and the emission coefficient  $\epsilon$  (equation [5.4]). Recall that these coefficients are made up of a number density times an



absorption (or scattering) cross section (see Section 5.2.4 and equation [5.9]). The total number density times cross sections are summed over all level populations. By level populations we mean the numbers of an atom or molecule in the different quantum mechanical energy states. In order to compute  $\kappa$  and  $\varepsilon$ , we need to know (1) the overall number densities of atoms or molecules and (2) the fraction of atoms or molecules in each quantum state. More formally, we would say that we need to know the state of the matter, the chemical partitioning of the matter, and the level populations.

The heart of the problem is that all of the above (in particular the level populations) are determined by the radiation field—yet the radiation field is the variable that we are trying to solve for in the radiative transfer equation. As an example of how the radiation field and the level populations are coupled, let us take photoexcitation followed by photodeexcitation. An electron in an atom or molecule may be excited to a higher level by absorption of a photon. Later, due to either spontaneous or stimulated emission, the electron will cascade downward. This electron transition will release photons, that is, release energy as radiation. We could say that this photoexcitation (and stimulated emission) is controlled by properties of the radiation field. Therefore, in this example, the level populations are controlled by the radiation field. This coupled nature of the radiation field and the level populations is a significant hurdle in solving the radiative transfer equation. In principle a full solution of the radiation field and the level populations is possible; in practice this is onerous.

LTE makes the solution of radiative transfer easier because we can assume we know how energy levels of different atoms and molecules are populated: they depend only on the local kinetic temperature and one other variable (such as  $\rho$  or  $P$ ). More formally we say that LTE assumes a decoupling of the state of the matter and the radiation field by explicitly assuming that all properties of the matter depend only on the local kinetic temperature and density (or pressure). In LTE, the atomic and molecular energy level populations can be computed using the equilibrium relations of statistical mechanics (see Chapter 8). For atomic and molecular energy levels, the relevant equation is the Boltzmann distribution, equation [8.30].

### 5.4.3 Conditions for LTE

Under what conditions is LTE valid? The concept of LTE implies a strict connection of the matter component to the local temperature. LTE is therefore valid when this matter-temperature connection is physically realized.

The conditions relevant for planetary atmospheres are at high enough densities that collisional processes dominate over competing radiative processes. Constant collisions enable matter and radiation to share the same temperature, and hence LTE to be valid. For example, an electron in an atom or molecule may be excited to a higher level by absorption of a photon. Later, due to a collision, the electron will cascade downward, releasing energy as the colliding particles' kinetic energy. This energy eventually ends up as a part of the thermal pool, after the particles undergo further elastic collisions with other atoms or molecules in the gas. In this case

of collisional deexcitation, the photon is said to be destroyed and converted into kinetic energy of the gas. In this way the collisional processes couple the photons (radiation field) to the matter temperature via the kinetic energy of the gas. (On a more subtle note, as long as the colliding particles have velocities in a Maxwellian distribution, the energy level populations will have their Boltzmann “equilibrium” values.)

The second situation in which LTE is valid is not really relevant for planetary atmospheres. This situation is where the mean radiation field is a black body. (Note that, from the above example, this is not a necessary condition of LTE). In the case that  $I(\mathbf{x}, \nu) = B(\mathbf{x}, \nu)$ , the radiative rate equations that control the level populations will give an equilibrium (i.e., Boltzmann) distribution for the level populations (see Section 8.2.4).

For a given situation one could try to determine the validity of LTE by computing and comparing radiative and collisional rates, but to accurately validate LTE a full LTE calculation itself may be necessary. LTE is actually valid in Earth's atmosphere from the ground up to about 60 km. This is largely because collisional rates between molecules dominate radiative rates, driving a Boltzmann distribution. We know from spectral measurements of Earth at the top of the atmosphere and also at different altitudes that the Earth atmosphere radiation field is not a black body.

Despite the fact that LTE is commonly used in exoplanet atmosphere calculations, there are some regions of the atmosphere where LTE is not valid. One example is in the very upper layers of the planet atmosphere where radiation freely escapes through the open boundary. Here, both the radiation field is very different from a black body and collisions do not dominate over radiative transition rates, causing a departure from LTE.

### 5.4.4 LTE and Kirchhoff's Law

We now turn to Kirchhoff's Law, one of the LTE expressions that simplifies the radiative transfer equation. Kirchhoff's Law of radiation states that at thermal equilibrium the emissivity of a body equals its absorptivity. Kirchhoff's Law is

$$\varepsilon_{\text{therm}}(\mathbf{x}, \nu) = \alpha(\mathbf{x}, \nu)B(\mathbf{x}, \nu). \quad (5.23)$$

This means the thermal emission is related to the absorptive properties of the gas. The thermal emission at a given location  $\mathbf{x}$  in the planetary atmosphere is black body radiation weighted by the absorptive properties of the gas. In other words, any time a photon is absorbed in a gas in LTE, the energy will be reemitted as radiation, with the amount and wavelength depending on the Planck function weighted by the absorption coefficient.

Kirchhoff's Law, is, technically, valid only under conditions of complete thermodynamic equilibrium. In planetary atmospheres we justify using Kirchhoff's Law for small, localized areas of the atmosphere where gradients in thermodynamic properties are small compared to a photon mean free path.



### 5.5 THE SOURCE FUNCTION

The source function (also sometimes called the contribution function) is defined as the ratio of the emission coefficient to the extinction coefficient:

$$S(\mathbf{x}, \hat{\mathbf{n}}, \nu) = \frac{\varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu)}{\kappa(\mathbf{x}, \nu)}. \quad (5.24)$$

The source function is used to simplify the radiative transfer equation and solution and to provide physical insight. The source function has the same dimensions as intensity.

We can expand the source function by breaking it up into a thermal emission component and a scattering component. We have previously used Kirchhoff's Law for thermal emission,  $\varepsilon_{\text{therm}} = \alpha B$ . For isotropic scattering,  $\varepsilon_{\text{scat}} = \sigma_s J$ . In conditions of LTE where Kirchhoff's Law is valid and where scattering is coherent and isotropic we may write the source function as

$$S(\mathbf{x}, \nu) = \frac{\alpha(\mathbf{x}, \nu)B(\mathbf{x}, \nu) + \sigma_s(\mathbf{x}, \nu)J(\mathbf{x}, \nu)}{\alpha(\mathbf{x}, \nu) + \sigma_s(\mathbf{x}, \nu)}. \quad (5.25)$$

This expression differs from the strict LTE value of the source function. In LTE,  $S(\mathbf{x}, \nu) = B(\mathbf{x}, \nu)$ , according to Kirchhoff's Law, equation [5.23]. The above source function includes scattering terms and therefore allows a slight deviation from LTE.

In LTE with no scattering we can use Kirchhoff's Law to write the source function as

$$S(\mathbf{x}, \nu) = B(\mathbf{x}, \nu). \quad (5.26)$$

In general, though, while  $I(\mathbf{x}, \hat{\mathbf{n}}, \nu) = B(\mathbf{x}, \nu)$  for black body radiation, LTE radiation is not necessarily black body radiation. Therefore in LTE  $I$  is not required to equal  $B$ .

### 5.6 THE EQUATION OF RADIATIVE TRANSFER

#### 5.6.1 The Time-Dependent Equation

For completeness we now return to a derivation of the *time-dependent* radiative transfer equation. Here we follow both the outline at the beginning of this chapter as well as the derivation in [1].

We first point out that, for moving material, both the extinction coefficient  $\kappa$  and the emission coefficient  $\varepsilon$  have a dependence on angle. This is because in moving material, changes in direction (and hence angle) result from the Doppler shift.

We choose an inertial coordinate system and want to understand the energy flow along a path  $s$  in a direction  $\hat{\mathbf{n}}$  into a differential solid angle  $d\Omega$ . The intensity beam is traveling through a fixed volume element of length  $ds$  and cross section  $dA$  normal to  $\hat{\mathbf{n}}$ , in a time interval  $dt$ . As before, we add up the losses from and

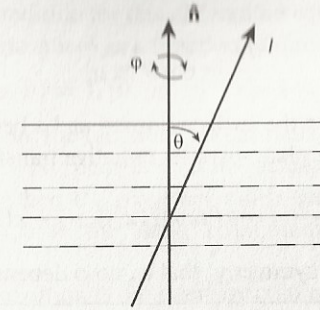


Figure 5.2 Schematic description of a 1D plane-parallel atmosphere.

gains into the beam of radiation along the pathlength of the beam, but now we also consider the losses and gains as functions of time,

$$[I(\mathbf{x} + \Delta\mathbf{x}, \hat{\mathbf{n}}, \nu, t + \Delta t) - I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)]dAd\Omega d\nu dt = [-\kappa(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t) + \varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)]dsdAd\Omega d\nu dt. \quad (5.27)$$

The pathlength  $\Delta s$  and the time interval  $\Delta t$  are related by  $\Delta t = \Delta s/c$ , and so

$$[I(\mathbf{x} + \Delta\mathbf{x}, \hat{\mathbf{n}}, \nu, t + \Delta t) - I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)] = \left[ \frac{1}{c} \left( \frac{\partial I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)}{\partial t} \right) + \left( \frac{\partial I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)}{\partial s} \right) \right] ds. \quad (5.28)$$

We now substitute equation [5.28] into equation [5.27] to get the time-dependent radiative transfer equation

$$\left[ \frac{1}{c} \left( \frac{\partial}{\partial t} \right) + \left( \frac{\partial}{\partial s} \right) \right] I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t) = -\kappa(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t) + \varepsilon(\mathbf{x}, \hat{\mathbf{n}}, \nu, t). \quad (5.29)$$

For the rest of this chapter (and for some future chapters) we will discuss the static case, independent of time. We will therefore drop the time-dependent terms of the above radiative transfer equation, including the  $\hat{\mathbf{n}}$ -dependence of the extinction coefficient  $\kappa$ . As previously in this chapter, we will omit the  $t$ -dependence of the intensity, flux, and other terms. Nevertheless, we must still keep in mind that radiation involves energy flow: energy per unit time.

#### 5.6.2 The Plane-Parallel Approximation

The plane-parallel atmosphere is a good framework in which to study the radiative transfer equation (Figure 5.2). For a 1D planar atmosphere the atmosphere is defined by a stratified plane with each layer having homogeneous properties such as  $T$ ,  $P$ , and  $\rho$ . This 1D plane represents a location on the surface of the planet. In what case is the plane-parallel approximation valid? If the radial depth of the atmosphere is much smaller than the planetary radius.

The plane-parallel definition assumes axial symmetry. Considering the axial symmetry and adopting  $z$  as our (1D) vertical coordinate, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \quad (5.30)$$



and

$$\frac{dz}{ds} = \cos \theta \equiv \mu, \quad (5.31)$$

where  $\theta$  is the angle between the surface normal and a beam of intensity. With the above we therefore have the plane-parallel radiative transfer equation

$$\mu \frac{dI(z, \hat{\mathbf{n}}, \nu)}{dz} = -\kappa(z, \nu)I(z, \hat{\mathbf{n}}, \nu) + \varepsilon(z, \hat{\mathbf{n}}, \nu). \quad (5.32)$$

If we further consider axial symmetry, that is, no  $\phi$  dependence,

$$\mu \frac{dI(z, \mu, \nu)}{dz} = -\kappa(z, \nu)I(z, \mu, \nu) + \varepsilon(z, \mu, \nu). \quad (5.33)$$

As written, the plane-parallel radiative transfer equation is an integro-differential equation: because  $\varepsilon = \varepsilon_{\text{scat}} + \varepsilon_{\text{therm}}$  and the scattering term contains angle integrals of  $I$ ,

$$\varepsilon_{\text{scat}}(z, \mu, \nu) = \sigma_s(z, \nu) \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 P(\mu, \phi; \mu', \phi') I(z, \mu', \nu) d\mu' d\phi'. \quad (5.34)$$

In a plane-parallel atmosphere the mean intensity (equation [2.2]) and flux (equation [2.5]) can also be rewritten considering azimuthal symmetry and the definition of  $\mu$ ,

$$J(z, \nu) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 I(z, \mu, \nu) d\mu d\phi, \quad (5.35)$$

or

$$J(z, \nu) = \frac{1}{2} \int_{-1}^1 I(z, \mu, \nu) d\mu, \quad (5.36)$$

$$F(z, \nu) = \int_0^{2\pi} \int_{-1}^1 \mu I(z, \mu, \nu) d\mu d\phi, \quad (5.37)$$

or

$$F(z, \nu) = 2\pi \int_{-1}^1 \mu I(z, \mu, \nu) d\mu. \quad (5.38)$$

(We note that in the stellar atmosphere literature the so-called astrophysical flux  $H = \frac{1}{4\pi} F$  is often used because it is similar to the form for  $J$  and  $K$ .)

In preparation for solving the radiative transfer equation we first rewrite the equation [5.33] using the optical depth distance scale (described in Section 5.3) and the definition of the source function (described in Section 5.5):

$$\mu \frac{dI(\tau_\nu, \mu, \nu)}{d\tau_\nu} = I(\tau_\nu, \mu, \nu) - S(\tau_\nu, \mu, \nu). \quad (5.39)$$

We emphasize the difference between incoming rays ( $\mu < 0$ ) and outgoing rays ( $\mu > 0$ ).

The upper boundary condition for this differential equation is the stellar radiation incident on the planet atmosphere as a function of direction  $\mu$ ,

$$I(0, \mu, \nu) = I_*(0, \mu_0, \nu) \quad (-1 \leq \mu \leq 0). \quad (5.40)$$

For plane-parallel radiation, the incoming stellar radiation is incident at one angle only (see Section 2.7.) For planets with no incident radiation, for calculations at infrared wavelengths when the planetary emission completely dominates any incoming stellar radiation, or for the “dark,” nonilluminated hemisphere of a planet,  $I_* = 0$ .

The lower boundary condition is the intensity coming from the planet interior

$$I(\tau_{\text{max}, \nu}, \mu, \nu) = I_{\text{int}}(\tau_{\text{max}, \nu}, \mu, \nu) \quad (0 \leq \mu \leq 1). \quad (5.41)$$

The intensity in this lower boundary condition is matched to the planet’s interior energy—the energy coming from the deep interior and incident on the lower boundary of the atmosphere.

### 5.3 The Formal Solution

A formal solution to the 1D plane-parallel radiative transfer equation (equation [5.39]) may be obtained with the integrating factor  $e^{-\tau_\nu/\mu}$ , whereby the radiative transfer equation is written

$$\frac{d(I(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu})}{d\tau_\nu} - I(\tau_\nu, \mu, \nu) \frac{1}{\mu} e^{-\tau_\nu/\mu} = -\frac{1}{\mu} S(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu}. \quad (5.42)$$

Integration from an initial optical depth  $\tau_{\nu, i}$  to a final optical depth  $\tau_{\nu, f}$  gives the solution

$$I(\tau_{\nu, f}, \mu, \nu) = I(\tau_{\nu, i}, \mu, \nu) e^{-(\tau_{\nu, f} - \tau_{\nu, i})/\mu} - \frac{1}{\mu} \int_{\tau_{\nu, i}}^{\tau_{\nu, f}} S(\tau'_\nu, \mu, \nu) e^{-(\tau'_\nu - \tau_{\nu, f})/\mu} d\tau'_\nu. \quad (5.43)$$

The solution has two terms on the right side of the equation. The first term describes the initial intensity diminished by exponential attenuation of absorption. The second term describes the emission from the atmosphere: an exponentially weighted average of the source function along the beam up to the location of interest.

Our goal in solving the radiative transfer equation is to derive a planet spectrum: the emergent flux at the top of the atmosphere. The emergent flux is the measurable quantity for exoplanetary atmospheres. For a semi-infinite atmosphere, integration from deep in the planetary atmosphere  $\tau_{\nu, i} = \infty$  to the top of the planet atmosphere at  $\tau_{\nu, f} = 0$ , the emergent intensity is

$$I(0, \mu, \nu) = \frac{1}{\mu} \int_0^\infty S(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu} d\tau_\nu, \quad (5.44)$$

where we have used

$$\lim_{\tau_{\nu, i} \rightarrow \infty} I(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu} = 0. \quad (5.45)$$

The emergent intensity is the amount of intensity at each altitude that reaches the surface along a path with angle  $\theta$  to the line of sight.



The emergent surface flux can be derived from equation [2.10],

$$F(0, \nu, t) = 2\pi \int_0^1 \int_0^\infty S(\tau_\nu, \mu, \nu) e^{-\tau_\nu/\mu} d\tau_\nu d\mu. \quad (5.46)$$

If the source function  $S(\tau, \nu, \mu)$  is known, the emergent intensity and the emergent flux (i.e., the planet's spectrum) can be computed directly from the above two equations. In many cases, however, this straightforward solution is not possible. The main complication is that the source function itself depends on  $I(\tau, \mu, \nu)$ , the quantity we are trying to solve for.

As a specific example let us consider that the source function depends on the intensity through any scattering terms that may also be  $\mu$ -dependent. Recall that the source function may be described as

$$S(\tau, \nu, \mu) = \frac{\varepsilon(\tau, \nu, \mu)}{\kappa(\tau, \nu)}, \quad (5.47)$$

where the emission coefficient  $\varepsilon$  has an angle-independent thermal emission component  $\varepsilon_{\text{therm}}(\tau_\nu, \nu) = \alpha B(\tau_\nu, \nu)$  and an angle dependent scattering component of  $\varepsilon$  (equation [5.34]),

$$\varepsilon_{\text{scat}}(\tau, \mu, \nu) = \sigma(\tau, \nu) \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 P(\mu, \phi; \mu', \phi') I(\tau, \mu', \nu) d\mu' d\phi'. \quad (5.48)$$

Physically the scattering term (whether angle dependent or isotropic) means that the intensity is decoupled from local conditions: the photons may scatter through large distances in the atmosphere without interacting with the thermal pool of the gas via absorption and thermal reemission. The scattering term requires a numerical solution of an integro-differential equation.

To further investigate the hidden complication in solving the 1D plane-parallel radiative transfer equation [5.39] we return to a discussion of the boundary conditions, equations [5.40] and [5.41]. To solve for intensity from this first-order ordinary differential equation we require two full boundary conditions, that is, an upper and a lower boundary condition on the full range ( $-1 \leq \mu \leq 1$ ). Yet, the information we have is the stellar radiation incident on the planet and traveling downward ( $-1 \leq \mu < 0$ ), and an estimate of the interior energy that we may convert to an outward-going intensity ( $0 < \mu \leq 1$ ) at the lower boundary. Because the boundary conditions are not fully specified, iterative techniques or a different formulation of the radiative transfer equation are needed to solve for  $I$ .

## 5.7 SUMMARY

We have presented fundamental concepts leading up to the foundational equation of radiative transfer. We started with opacity, the macroscopic description that captures the interaction of radiation with gases or solids in the atmosphere. Opacity is a major component of the radiative transfer equation. The concept of optical depth is related to opacity; an opaque atmosphere is optically thick and a transparent atmosphere is optically thin. The quantitative optical depth scale is a useful

substitution for the distance scale in solving the radiative transfer equation. We defined the source function, a convenient ratio of emission to extinction for solving the radiative transfer equation. The concept of local thermodynamic equilibrium (LTE) was described in some detail. The situation of LTE enables a decoupling of radiation from the local temperature, which greatly simplifies the radiative transfer problem and solution; the atomic and molecular energy population levels are specified by the local temperature and do not have to be determined by a simultaneous solution with the radiative transfer equation. We finally came to the radiative transfer equation itself, an equation that can be set out in 1D as the rate of change of intensity (a beam of traveling photons) with distance is equal to the loss from the beam and the additions to the beam. We are now ready to proceed to solutions of the radiative transfer equation.

## REFERENCES

For further reading

For the most thorough description of LTE in atmospheres:

- Mihalas D. 1978. *Stellar Atmospheres* (2nd ed.; San Francisco: W. H. Freeman).

For a concise outline of the radiative transfer fundamental concepts:

- Chapter 1 in Rybicki, G. B., and Lightman, A. P. 1986. *Radiative Processes in Astrophysics* (New York: J. Wiley and Sons).

See Chapter 6 for a more complete annotated list of radiative transfer textbooks.

## EXERCISE

1. Kirchhoff's Law of radiation. Explain why Kirchhoff's Law is valid in thermodynamic equilibrium. Use a conceptual explanation, based on absorption and emission in a black body enclosure. Also use the 1D plane-parallel equation of radiative transfer.