Chapter Nine

Vertical Thermal Structure of a Planetary Atmosphere

9.1 INTRODUCTION

The vertical temperature structure of the planet atmosphere is fundamental to understanding exoplanets for several different reasons. The most significant reason is that the temperature of the planetary surface tells us whether or not a planet is habitable. If the surface is too hot for covalent bonds to form complex molecules, then life cannot exist. More conventionally, people believe that liquid water is necessary at the surface for life to exist. The surface temperature and pressure define whether or not liquid water is stable. Beyond habitability, the vertical temperature pressure structure is the starting point for computing equilibrium and nonequilibrium chemistry to understand the composition of the planetary atmosphere.

Each of the solar system planet atmospheres has a qualitatively similar vertical temperature profile (Figure 9.1). This motivates us to begin with an understanding of the vertical temperature profile. Atmospheric temperatures can and do vary in the horizontal as well as in the vertical direction. This is especially true for hot exoplanets tidally locked to have permanent day and night sides. We defer a discussion of horizontal heat transport until the next chapter.

In this chapter we describe the 1D vertical pressure and temperature structure of a planet atmosphere. It is common to use pressure as a proxy for altitude (they are connected by hydrostatic equilibrium). We will use the abbrievation T-P profile. There is no simple equation to describe the vertical T-P structure, but rather many physical ingredients are at play.

9.2 EARTH'S VERTICAL ATMOSPHERIC STRUCTURE

We will begin with an overview of Earth's vertical temperature, pressure, and density structure, simply because Earth's atmosphere is the most accessible planetary atmosphere to us. Figure 9.2 shows the Earth's vertical structure typical of midlatitudes.

Earth has several different regions of the atmosphere, the lower four corresponding to altitudes of temperature reversals. These are regions of differing temperature profiles, caused by absorption of and subsequent heating by solar radiation. Solar radiation is absorbed at different altitudes, depending on the wavelength of light

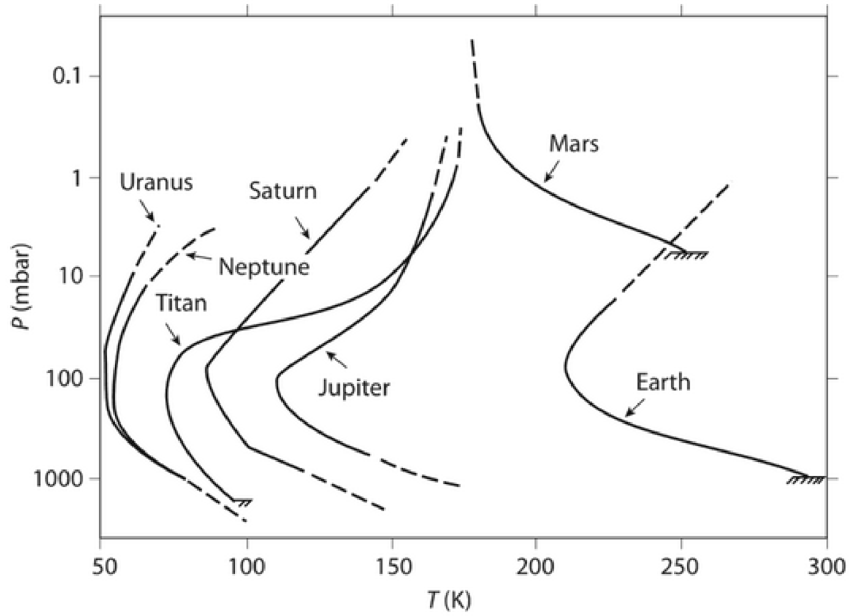

Figure 9.1 The vertical structure of solar system planet atmosphere temperatures. Notice the overall similarity in the temperature profiles. The thermal inversion, or stratosphere, on most planets comes from CH_4-induced hazes for the giant planets and from O_3 for Earth. Adapted from [1].

and the atmospheric composition. In Chapter 6 we described how radiation penetrates to different altitudes depending on the wavelength. For Earth, see Figure 9.3.

The lowest layer of Earth's atmosphere is the "troposphere." This is the region where we live and where most of what we call the weather occurs. For scale, note that Mt. Everest, at 8,856 m, reaches less than halfway through the troposphere. Most of the mass of Earth's atmosphere is in the troposphere—around 85%—because the atmospheric density decreases exponentially with increasing altitude (see the left-hand y-axis of Figure 9.2). The exponentially decreasing density is caused by hydrostatic equilibrium (Section 9.3.1) and is also familiar to Mt. Everest climbers, who require bottled oxygen to combat the decreasing amount of oxygen as they ascend. Because the troposphere contains the highest density, the troposphere is where most of the strong spectral features at visible and infrared wavelengths originate.

In the troposphere the temperature is hottest at the ground and decreases with increasing altitude. The ground is heated by solar radiation that makes it through the atmosphere unimpeded. We see from Figure 9.3 that most of the visible-wavelength solar radiation penetrates Earth's atmosphere to the surface. The ground reradiates the absorbed solar energy to heat the atmosphere. Some solar radiation is absorbed in the troposphere (by water vapor, carbon dioxide, and other gases at infrared wavelengths). The greenhouse effect also plays a role.

The stratosphere is the layer above the troposphere. There is a temperature inversion in the stratosphere. In other words, the temperature increases with increasing altitude, opposite to the troposphere. The stratopause marks the beginning of the temperature inversion. Why is there a temperature inversion in the strato-

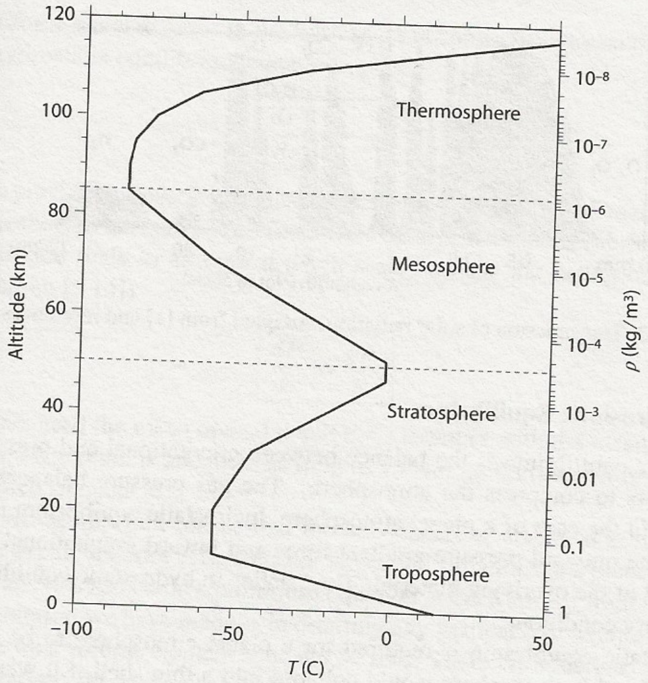

Figure 9.2 The vertical structure of Earth's atmosphere typical of midlatitudes. From the 1976 U.S. Standard Atmosphere [2]. Notice how the highest densities near the surface imply that most of Earth's mass is in the troposphere.

sphere? The stratosphere is heated from above by absorption of UV solar radiation by ozone. This UV radiation is detrimental to most biological cells, making the stratosphere a protective layer to life on Earth.

The stratosphere and the troposphere are the layers of the atmosphere that are most relevant for us. They are the regions where the spectral features occur and in turn where the surface lies and weather happens. The upper layers do have some effect on the lower levels, via photochemistry and atmospheric escape (Chapter 4).

9.3 HYDROSTATIC EQUILIBRIUM AND THE PRESSURE SCALE HEIGHT

Eighty-five percent of the mass of Earth's atmosphere is in the lower 10 km, out of an atmosphere that extends to about 100 km. Mountain climbers know this; for example, most climbers require bottled oxygen when hiking up Mt. Everest. Planetary atmospheres have an exponentially decreasing density and pressure with increasing altitude. In this section we derive hydrostatic equilibrium and the pressure scale height to show why.

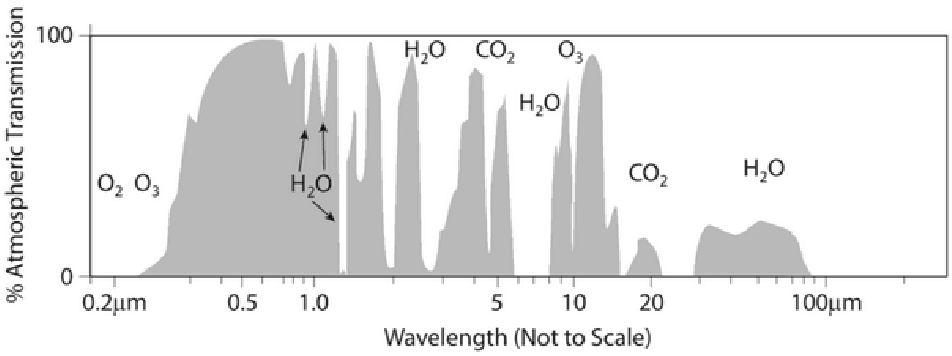

Figure 9.3 Transmission of solar radiation. Adapted from [1] and references therein.

9.3.1 Hydrostatic Equilibrium

Hydrostatic equilibrium is the balance between gravitational and pressure forces. Gravity acts to compress the atmosphere. The gas pressure balances this compression. In the case of a planet atmosphere, hydrostatic equilibrium is a balance between the outward pressure gradient force and inward gravitational force from the weight of the overlying material. The "static" in hydrostatic equilibrium refers to stationary conditions.

Hydrostatic equilibrium is required for a planet atmosphere to be stable. For example, Earth's atmosphere would collapse into a thin shell if it weren't for the outward pressure force. Without the force of gravity, the gas in a planetary atmosphere would diffuse into space, leaving an atmosphereless planet.

We now proceed with the derivation of the hydrostatic equilibrium equation. We consider a volume element of homogeneous gas where the pressure forces $F_P(z)$ in the volume element balance the gravitational forces $F_g(z)$. Here δV is the volume element, δA is the cross-sectional area, and δz is the distance along the column. We define the sign convention as positive downward toward the planetary surface.

The gravitational force on the volume element is described by

$$F_g(z) = mg = \rho g \delta V = \rho g \delta z \delta A, \tag{9.1}$$

where g is the gravitational acceleration. The pressure force

$$F_P(z) = dP \delta A \tag{9.2}$$

is related to the pressure difference dP between the pressure on the upper and lower surfaces of the volume element:

$$dP = P(z_1 + dz) - P(z_1) = P(z_1) + \frac{dP}{dz}\delta z - P(z_1) = \frac{dP}{dz}\delta z, \tag{9.3}$$

where the first term in the second equality is expanded to first order using Taylor's expansion. Equating the gravitational and pressure forces from the above three equations, we obtain

$$\frac{dP}{dz}\delta z \delta A = -g\rho \delta z \delta A. \tag{9.4}$$

Here the negative sign arises because the gravitational force is acting downward on the planetary volume element and the pressure force is acting upward, and we have set the net force equal to zero. In the limit as dz becomes infinitesimally small, we have the hydrostatic equilibrium equation,

$$\boxed{\frac{dP}{dz} = -g\rho.} \tag{9.5}$$

We have previously described a planetary atmosphere in terms of an optical depth scale rather than an altitude scale. To convert the hydrostatic equilibrium equation from an altitude scale to an optical depth scale, we use the definition of optical depth (equation [5.14])

$$\boxed{\frac{dP}{d\tau} = \frac{g\rho}{\overline{\kappa}}.} \tag{9.6}$$

Here we have used the mean optical depth scale together with the mean opacity $\overline{\kappa}$, but any reference scale is adequate as long as τ_ν and κ_ν correspond to each other at the specified frequency.

Hydrostatic equilibrium means that the atmosphere is stable; it is neither collapsing, nor expanding, nor escaping. For almost all situations in a planetary atmosphere, the above hydrostatic equilibrium equation is the form we want to use.

One assumption we have made is in omitting any vertical acceleration, whereby the net force on a volume element need not be zero. There are situations where vertical accelerations do occur, for example, hydrodynamic escape in the very upper atmosphere of an evolving planetary atmosphere. Another example is on very small scales—much smaller than we are considering in this book. Vigorous small-scale systems, such as tornadoes, thunderstorms, and convection, have nonzero vertical acceleration.

9.3.2 The Equation of State

The equation of state relates the temperature, pressure, and density of a given material. For planetary atmospheres it is adequate to use the ideal gas law

$$P = nkT = \frac{\rho kT}{\mu_{\rm m} m_{\rm H}}. \tag{9.7}$$

Here n is the number density, k is Boltzmann's constant and $\mu_{\rm m}$ is the mean molecular weight,

$$\mu_{\rm m} \equiv \frac{\overline{m}}{m_{\rm H}}, \tag{9.8}$$

where $\overline{m}$ is the number-weighted average mass of the molecules and atoms in the gas and $m_{\rm H}$ is the mass of the hydrogen atom.

9.3.3 The Pressure Scale Height

The pressure scale height H is a characteristic length scale of the planetary atmosphere. H is important because we can use it to estimate the total height and volume of the atmosphere.

As we shall see, the pressure scale height is the e-folding distance for pressure for an isothermal atmosphere with a constant mean molecular weight. The pressure scale height, in other words, is the altitude above which the pressure drops by a factor of e (e = 2.71). The pressure scale height is a characteristic length which is a good measure for the radial extent of a planetary atmosphere—the spectra we can observe come from a region in the atmosphere limited to several scale heights.

We derive H from the hydrostatic equilibrium equation [9.5]

$$\frac{dP}{dz} = -g\rho,$$

and the ideal gas law (equation [9.7]

$$P = nkT = \frac{\rho kT}{\mu_{\rm m} m_{\rm H}}.$$

Combining these equations gives

$$\frac{dP}{P} = -\frac{\mu_{\rm m} m_{\rm H} g}{kT} dz \tag{9.9}$$

with a solution

$$P = P_0 {\rm e}^{-z/H}, \tag{9.10}$$

where we have defined the pressure scale height to be

$$\boxed{H \equiv \frac{kT}{\mu_{\rm m} m_{\rm H} g}.} \tag{9.11}$$

Here again $m_{\rm H}$ is the mass of the hydrogen atom and $\mu_{\rm m}$ is the mean molecular weight. We emphasize that the pressure scale height we have derived is for constant T and constant $\mu_{\rm m}$, and we have neglected the variation of g with altitude.

The scale height H for solar system planets is on the order of 5 to 20 km, and is given in Table 3.1. For hot Jupiter exoplanets, the scale height can be several hundred kilometers. We may estimate the total atmosphere relevant for spectral lines as approximately 5 scale heights.

With the pressure scale height we can also get a handle on how the pressure and density vary as a function of altitude. Even though we have assumed an isothermal atmosphere, equation [9.11] shows us that the pressure (and density via the ideal gas law) varies exponentially with altitude. The temperature for a typical planetary atmosphere varies by only a factor of a few (see Figure 9.1).

9.4 SURFACE TEMPERATURE FOR A SIMPLIFIED ATMOSPHERE

We begin our investigation into the vertical thermal structure of a planetary atmosphere by estimating the difference between the atmosphere and surface temperatures in a very simple model. We will use a previous derivation of the equilibrium temperature $T_{\rm eq}$ in Chapter 3. In Chapter 3 and Figure 3.1 we used the concept of energy balance to equate the total flux from a star incident on a planet (equation [3.3]),

$$(1 - A_{\rm B})\mathcal{F}_{{\rm S},*} \left(\frac{R_*}{a}\right)^2 \pi R_{\rm p}^2, \tag{9.12}$$

with the total flux emerging from a planet,

$$4\pi R_{\mathrm{p}}^2 \mathcal{F}_{\mathrm{S,p}}, \tag{9.13}$$

to find

$$\sigma_{\mathrm{R}} T_{\mathrm{eq}}^4 = \frac{1}{4} \sigma_{\mathrm{R}} T_{\mathrm{eff},*}^4 \left(\frac{R_*}{a}\right)^2 (1 - A_{\mathrm{B}}) \tag{9.14}$$

and the equilibrium temperature (equation [3.3])

$$T_{\mathrm{eq}} = T_{\mathrm{eff},*} \left(\frac{R_*}{2a}\right)^{1/2} (1 - A_{\mathrm{B}})^{1/4}. \tag{9.15}$$

Here we have assumed that the absorbed stellar radiation is circulated evenly around the planet. We have also used the Stefan-Boltzmann Law (equation [3.6]), equating flux with temperature $\mathcal{F} = \sigma_{\mathrm{R}} T^4$ where σ_{R} is the radiation constant and T is temperature.

Recall that T_{eq} is a theoretical number that is the temperature attained by an isothermal planet after it has reached complete equilibrium with the radiation from its parent star. In the context of an idealized planetary atmosphere, the equilibrium temperature is essentially the temperature at the layer where most of the radiation is emitted. For this subsection we will refer to the equilibrium temperature as the emission temperature, $T_{\mathrm{eq}} = T_{\mathrm{e}}$. We also denote the stellar effective temperature $T_{\mathrm{eff},*} = T_*$.

We now move to describe a simple greenhouse atmosphere (Figure 9.4), still considering uniform redistribution of absorbed stellar radiation, and following [3]. In this atmosphere there are two layers. The bottom layer is the surface. We assume that all of the radiation from the star (in the form of stellar flux) reaches the surface, that is, there is no scattering. This is a reasonable approximation for stars with most of their energy output at visible wavelengths and for a planet atmosphere composed of molecules that primarly absorb and emit at infrared wavelengths. We assume the radiation that reaches the ground is absorbed, heats the surface, and is reemitted at longer wavelengths characteristic of the atmospheric temperature (see the discussion of black body radiation in Section 2.8 and Figure 2.7).

The reprocessed flux (in the amount of the absorbed incident stellar flux) then travels upward and is absorbed by the second layer. We will call this second layer the atmosphere layer. In this picture, the atmospheric layer absorbs at infrared wavelengths; we have already assumed the planetary atmosphere is composed of molecules that primarily emit at IR wavelengths.

We assign a temperature T_{a} to the atmosphere layer and T_{s} to the surface layer. Our goal is to estimate the surface temperature in terms of the planet's equilibrium temperature (T_{e}) and in terms of the atmosphere layer temperature. Let us focus first on the atmospheric layer. From energy balance, the radiation emerging from the top of the atmosphere layer must be equivalent to the absorbed stellar radiation,

$$\sigma_{\mathrm{R}} T_{\mathrm{a}}^4 = \sigma_{\mathrm{R}} T_*^4 \left(\frac{R_*}{2a}\right)^2 (1 - A_{\mathrm{B}}). \tag{9.16}$$

But the definition of equilibrium temperature T_{e} comes from the same energy balance requirement (equation [9.15]). We therefore equate the atmospheric and emission temperatures,

$$\sigma_{\mathrm{R}} T_{\mathrm{a}}^4 = \sigma_{\mathrm{R}} T_{\mathrm{e}}^4. \tag{9.17}$$

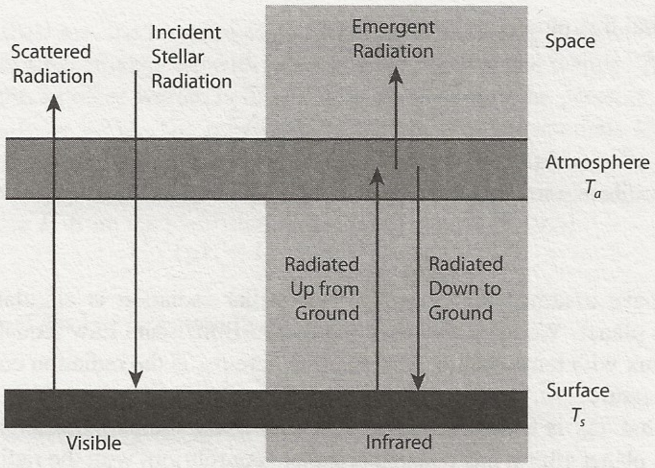

Figure 9.4 The simple greenhouse. Adapted from [3].

The atmosphere layer is the layer emitting to space, so it makes sense that the atmosphere and emission temperatures are equivalent.

At the surface, energy balance again means that all of the incoming stellar radiation absorbed must be subsequently reradiated. There are two contributions to the absorbed radiation at the surface: the radiation incoming from the star and the downward radiation from the atmosphere layer,

$$\sigma_R T_s^4 = \sigma_R T_*^4 \left(\frac{R_*}{2a}\right)^2 (1 - A_B) + \sigma_R T_a^4. \quad (9.18)$$

We have previously shown above that the first term on the right-hand side is equivalent to $\sigma_R T_e^4$ and the second term on the right-hand side is also equivalent to $\sigma_R T_e^4$, giving

$$\boxed{T_s = 2^{1/4} T_e.} \quad (9.19)$$

We see that the surface temperature is about 1.19 times the atmospheric temperature. In this simplified greenhouse model, the atmosphere layer contributes to heating the planet surface. This simple greenhouse model overestimates Earth's surface temperature but significantly underestimates Venus's surface temperature. For Earth the model gives $T_s \sim 303$ K, based on $T_e \sim 255$ K, compared to the average surface temperature $T_s \sim 280$ K. For Venus, the model surface temperature is $T_s \sim 274$ K based on $T_e = 230$ K, compared to the measured $T_s = 730$ K.

In the simple greenhouse model we have assumed that the absorbed stellar radiation is trapped by the atmospheric layer. In a slightly more realistic model we consider a "leaky" greenhouse [3]. In the leaky greenhouse (Figure 9.5) the single atmosphere layer is optically thin. Again, we make the simplifying assumption that all of the radiation from the star not scattered back to space reaches the planet surface. We will write down two equations to solve for T_s in terms of T_e.

By energy balance the radiation leaving the planet must be equal to the total amount of radiation absorbed by the planet. There are two contributions to the

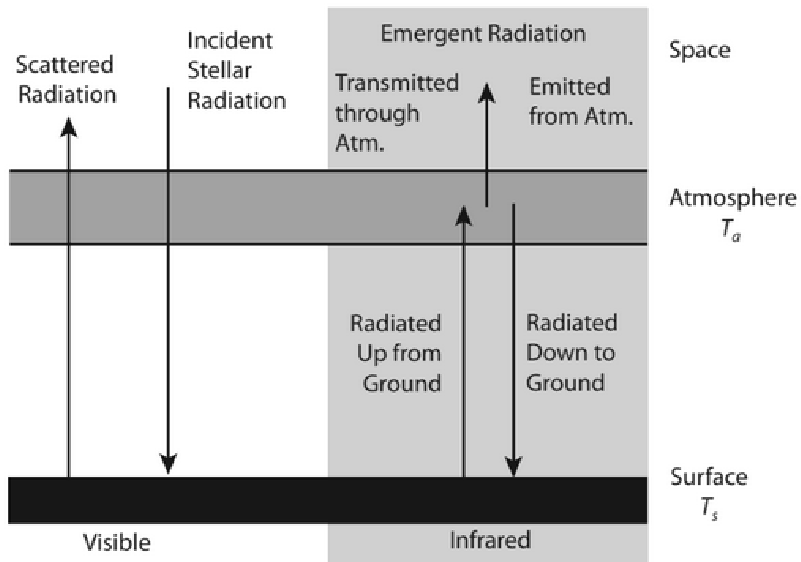

Figure 9.5 The leaky greenhouse. Adapted from [3].

radiation leaving the planet. The first contribution is from the atmosphere layer and the second contribution is the surface radiation that leaks through the atmosphere. We therefore have

$$\sigma_{\mathrm{R}} T_{\mathrm{e}}^4 = \sigma_{\mathrm{R}} T_{\mathrm{a}}^4 + \sigma_{\mathrm{R}} T_{\mathrm{s}}^4 (1 - \alpha), \tag{9.20}$$

where $\alpha \in (0, 1)$ is the fraction of radiation that is absorbed by the atmosphere layer. At the planet surface, we again have

$$\sigma_{\mathrm{R}} T_{\mathrm{s}}^4 = \sigma_{\mathrm{R}} T_{\mathrm{a}}^4 + \sigma_{\mathrm{R}} T_{\mathrm{e}}^4. \tag{9.21}$$

Solution of the above two equations yields

$$\boxed{T_{\mathrm{s}} = \left(\frac{2}{2 - \alpha} \right)^{1/4} T_{\mathrm{e}}.} \tag{9.22}$$

The greenhouse warming effect is reduced by the paritally transparent atmosphere layer. We can see that if $\alpha = 1$ all radiation is trapped by the atmosphere layer, in agreement with the simple greenhouse model (also showing that the simple greenhouse model is a special case of the more general leaky atmosphere model). In the limit $\alpha \to 0$, we recover $T_{\mathrm{s}} = T_{\mathrm{e}}$, the case where the atmosphere layer is transparent to radiation (or there is no atmosphere layer). The surface layer is always warmer than the atmosphere layer in the leaky greenhouse model. We could further show, using Kirchhoff's Law, that $T_{\mathrm{s}} \geq T_{\mathrm{e}} \geq T_{\mathrm{a}}$.

In the above simple greenhouse and leaky greenhouse models we have effectively explored a radiative equilibrium model for a one-layer atmosphere in the case of no scattering. If we were to extend the simple models to many layers, with the appropriate value for α in each layer, considering all wavelengths, molecules, and clouds, we would approach a multilayer radiative equilibrium model.

9.5 CONVECTION VERSUS RADIATION

In a planetary atmosphere energy can be transported by radiation, convection, or thermal conduction. The competing heat transport methods are radiation and convection. In the relatively dense lower atmospheres that we are considering, thermal conduction is not important. What determines whether or not convection is taking place in a planetary atmosphere? Energy will be transported by the most efficient method—that is, via the "path of least resistance." If the opacity (the absorption or scattering of photons) is low, then the energy is most likely to be transported by photons. If the opacity is high then the resistance to the flow of photons is high, and energy is more likely to be carried by convection.

To quantify which energy transport mechanism dominates, we turn to a discussion of temperature gradients (dT/dz). We begin with a qualitative discussion on how the temperature gradient determines whether or not convection will occur. If convection will occur we call the atmosphere unstable against convection. We call a situation stable if, after a disturbance, the system will return to its original state.

Consider an air parcel that is slightly unstable, rises a slight distance, and expands adiabatically to the ambient pressure. Adiabatic expansion means that there is no heat exchange with its surroundings; the air parcel will have the same pressure as the surrounding atmosphere, but its own (possibly different) temperature and density. If the air parcel is colder and therefore more dense than its surroundings it will sink, and convection will not occur. The atmosphere is said to be stable against convection. In contrast, the atmosphere is said to be unstable against convection if, after rising a slight distance and expanding, the air parcel continues to rise. This happens if the air parcel is hotter and therefore less dense than its surroundings, so that the buoyancy force will cause the air parcel to continue to rise.

The criterion for convection, then, is related to two temperature gradients: (1) the adiabatic temperature gradient (the temperature gradient followed by the rising air parcel) and (2) the surrounding atmospheric temperature gradient. If the adiabatic temperature gradient is shallower than the atmospheric temperature gradient, the atmosphere is unstable and convection occurs. This criterion for convection is then a criterion for buoyancy,

$$\left(\frac{dT}{dz}\right)_{\rm ad} > \left(\frac{dT}{dz}\right)_{\rm atm}. \tag{9.23}$$

Convective stability and instability in terms of the atmospheric and adiabatic temperature gradients are described in Figure 9.6.

In order to understand under what conditions a temperature gradient is small or large we now turn to a quantitative discussion of the adiabatic and radiative temperature gradients. The adiabatic temperature gradient is (derived in Section 9.7.2)

$$\frac{dT}{dz} = -\frac{g}{c_p}, \tag{9.24}$$

where g is the surface gravity and c_p is the specific heat capacity at constant pressure in units of J kg^{-1} s^{-1}.

Convection will set in when the adiabatic gradient becomes small, or when the radiative temperature gradient becomes large. The adiabatic temperature gradient

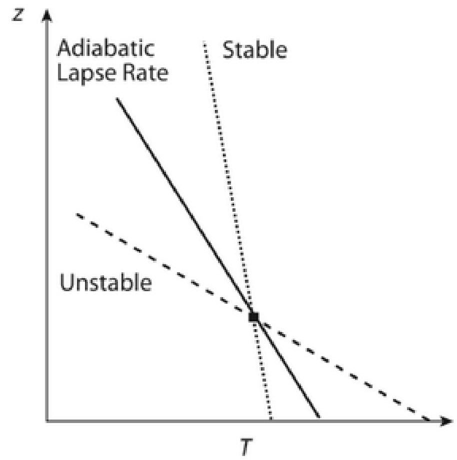

Figure 9.6 Illustration of temperature profiles stable or unstable against convection.

(equation [9.24]) becomes small when the heat capacity becomes large. The specific heat capacity is the amount of energy that must be transferred to the gas per unit mass at constant pressure to increase the temperature by 1 K. In a situation where a large amount of energy is needed to heat the gas by 1 K, the adiabatic temperature gradient will become very small. A typical example in stars is the layer where an abundant element such as hydrogen starts to ionize. The temperature increase goes into energy to remove the electrons from the atoms, rather than to increase the kinetic energy in the gas; the heat capacity becomes large because a lot of energy is needed to heat the gas. A similar situation also occurs in the hot interiors of giant planets like Jupiter, where at $T > 4000$ K hydrogen starts to ionize.

The radiative equilibrium temperature gradient in the limit of the diffusion approximation is (Section 6.4.4)

$$\left(\frac{dT}{dz}\right)_{\rm rad} = -\frac{3}{16} F_{\rm int} \frac{\overline{\kappa}}{\sigma_{\rm R} T^3}. \tag{9.25}$$

While the above equation is valid only deep in a planetary atmosphere where the diffusion approximation holds, the equation serves to illustrate criteria where the radiative temperature gradient becomes large. The radiative temperature gradient becomes large when either the radiative flux F or the mean opacity κ becomes large. In planet interiors, the fluxes may be large for giant planets, driving convective interiors. In the atmospheres, the fluxes may not be large enough to force convection.

The opacity, on the other hand, is a driving factor for convection. The opacity becomes large where the number density of absorbing particles becomes large. Hence, deep in a planetary atmosphere where high opacities and optically thick conditions prevail, convection almost always sets in. In the upper part of the planetary atmosphere, transport of energy by radiation usually dominates over convection. In addition, the opacity can change from large to small (or vice versa) due to changes in temperature and pressure versus altitude in a planetary atmosphere—these change the atomic or molecular makeup of the atmosphere. Jupiter, for example, follows the above description of a radiative zone in the upper atmosphere and a convection zone in the lower atmosphere. But, beneath the convection zone, Jupiter may have another radiative zone, because of a minimum in the mean opacities at high temperatures (1300–1800 K [4]). Hot Jupiter exoplanets are different examples of the interplay between radiative and convective regions. In contrast to Jupiter, the radiative zones in hot Jupiter atmospheres are expected to continue deep into the atmosphere on their permanent day sides because the strong heating from the parent star causes the temperature gradient to be close to isothermal or even inverted.

Planets like Earth with solid surfaces beneath relatively thin atmospheres are always expected to have convection zones just above the solid surface. The reason is that most of the visible-wavelength incident stellar energy is absorbed at the planet surface, while only some is absorbed in the planet atmosphere. This uneven energy absorption by the atmosphere and surface makes the surface hotter than the overlying atmosphere layers heated only by absorption of radiation. A significant temperature discontinuity will arise between the atmosphere and the planetary surface. This temperature discontinuity will drive convection. On Earth convection is occurring in most of the troposphere, but may be limited to a thinner layer on exoplanets with different atmospheric conditions. According to the criterion for convection (equation [9.23]) and the radiative equilibrium temperature gradient (equation [9.25]), we can think of a very large radiative flux being emitted from the ground layer. This causes the radiative temperature gradient to be large enough that convection sets in.

Atmosphere layers with temperature inversions are very stable against convection. Almost all of the solar system planets have temperature inversions in the upper atmospheres, where the temperature rises with increasing altitude from the planet surface. Recall that these temperature inversions are typically caused by absorption of UV radiation. The temperature inversion layers are stable against convection because the restoring force to a perturbed, lifted air parcel is very strong. Recall that as an air parcel rises, it expands and cools. Being cooler—and hence denser—than the surrouding temperature-inverted atmosphere (which increases with rising height), the air parcel will sink again.

In the next two sections we will describe the T-P profile in the atmosphere from radiative equilibrium and from convective equilibrium.

9.6 THE RADIATIVE EQUILIBRIUM TEMPERATURE PROFILE

We embark on a description of the 1D vertical temperature profile in the case where energy is transported only by radiation. We will show that, if we know the total amount of energy passing through the planetary atmosphere, we can derive the temperature profile. We begin by outlining the general case where there is no analytic solution.

9.6.1 Radiative Equilibrium

In a planetary atmosphere, no energy is created or destroyed. Therefore, the amount of radiation passing into a given volume of the atmosphere must equal the amount of radiation leaving that volume. This is called radiative equilibrium. We can also express radiative equilibrium as a flux constancy,

$$\frac{dF(\tau)}{d\tau} = 0. \tag{9.26}$$

Here $F(\tau)$ is the total radiative flux (equation [5.37]) integrated over all frequencies, and τ is the optical depth scale (Section 5.3). The radiative equilibrium temperature profile is the temperature profile that satisfies the radiative equilibrium constraint.

To continue with an expression for radiative equilibrium we consider the total amount of radiation absorbed in a given atmospheric volume:

$$\int_0^\infty \int_\Omega \kappa(\tau_\nu, \nu) I(\tau_\nu, \mu, \nu) d\Omega d\nu. \tag{9.27}$$

Here "total" means integrated over all angles and frequencies. Because absorption is isotropic we may integrate over the solid angle Ω to find

$$\int_0^\infty \kappa(\tau_\nu, \nu) J(\tau_\nu, \nu) d\nu. \tag{9.28}$$

The total amount of radiation emitted in a given atmospheric volume is

$$\int_0^\infty \int_\Omega \varepsilon(\tau_\nu, \mu, \nu) d\Omega d\nu. \tag{9.29}$$

If we make the assumption of isotropic emission, and consider the definition of the source function (Section 5.5 and equation [5.24]), we have for the total emission

$$\int_0^\infty \kappa(\tau_\nu, \nu) S(\tau_\nu, \nu) d\nu. \tag{9.30}$$

We may now equate the total amount of radiation absorbed (equation [9.28]) and emitted (equation [9.30]) in a given volume to find

$$\boxed{\frac{dF_{\mathrm{R}}(\tau)}{d\tau} = \int_0^\infty \kappa(\tau_\nu, \nu) \left[J(\tau_\nu, \nu) - S(\tau_\nu, \nu)\right] d\nu = 0.} \tag{9.31}$$

We now have a mathematical expression for radiative equilibrium: radiative absorption is balanced by radiative emission, in a given layer. It is important to realize that radiative emission and absorption in a given layer do not have to balance *at a given frequency*.

What is the constant flux being driven through the atmosphere? By radiative energy balance, the flux in each layer is the same as emitted at the top of the atmosphere, namely,

$$F(\tau) = \sigma_{\mathrm{R}} T_{\mathrm{int}}^4 + \mu_0 I_0. \tag{9.32}$$

The first term $\sigma_{\mathrm{R}} T_{\mathrm{int}}^4$ is the flux coming from the planetary interior that passes through the planet atmosphere. Giant planets have a source of interior energy from the gradual loss of residual gravitational potential energy from the planet's formation. Indeed, Jupiter has an internal luminosity over twice as high as its luminosity from reradiated absorbed stellar energy. Earth has an interior energy source partly from residual gravitational potential energy but mostly from decay of radioactive isotopes (of uranium, thorium, and potassium). For many planets, including Earth and the hot exoplanets in short-period orbits, the second term, the flux from the star, overwhelms the interior flux. (For a derivation of the term $\mu_0 I_0$ see Section 6.4.3.1 and equation [6.37].)

The equation of radiative equilibrium is coupled to the radiation field, through the angle-averaged quantity $J(\tau_\nu, \nu)$. The radiative equilibrium equation and the radiative transfer equation [5.39] must therefore be solved simultaneously. We emphasize that this radiative equilibrium equation shows how the opacity as a function of frequency plays into determining the temperature profile. A temperature inversion could arise naturally by solving the radiative transfer and radiative equilibrium equations together.

There is no analytic solution of the radiative equilibrium temperature profile in the general case. We now proceed with a gray atmosphere, where analytic solutions to the temperature profile are possible.

9.6.2 Gray Atmosphere Heated from Below

We will derive a temperature profile using the condition of radiative equilibrium (equation [9.31]) under the assumption of LTE and in an atmosphere with no scattering. In order to find an analytical solution, we must make a further simplification: that of a gray atmosphere. A gray atmosphere is one where the radiation quantities (i.e., intensity, absorption, and emission coefficients) do not vary with frequency.

We first consider an atmosphere heated from below. By this, we mean an atmosphere for which the source of radiation is from the interior or ground only. One example is a giant planet atmosphere, dominated by interior flux and heated as the interior flux travels through the atmosphere out to space. Another example is a planet with a thin atmosphere such as Mars (or even, approximately, Earth) which is completely transparent at visible wavelengths. The bulk of the stellar energy is at visible wavelength and reaches the planet surface. There, the stellar radiation is absorbed by and heats the surface and is reemitted at longer wavelengths related to the characteristic temperature of the planet. Terrestrial planet atmospheres are typically not transparent at IR wavelengths where molecules can absorb and reemit radiation, heating the atmosphere in the process. The infrared radiation coming from below therefore heats the atmosphere as it travels out through the atmosphere to space. We call the interior temperature $T_{\rm int}$. For the giant planet example above, $T_{\rm int} = T_{\rm eff}$. For the thin atmosphere transparent to visible radiation, $T_{\rm int}$ is the surface temperature $T_{\rm s}$.

For a gray atmosphere we use the total mean radiation field $J(\tau) = \int_0^\infty J(\tau_\nu, \nu) d\nu$, $I(\tau) = \int_0^\infty I(\tau_\nu, \nu) d\nu$, and $S(\tau) = \int_0^\infty S(\tau_\nu, \nu) d\nu$. In the gray case, the radiative equilibrium equation [9.31] becomes

$$J(\tau) = S(\tau). \tag{9.33}$$

With the assumption of LTE and the frequency-integrated black body flux, we have

$$\boxed{J(\tau) = S(\tau) = B(T(\tau)) = \frac{\sigma_R T^4}{\pi}.} \tag{9.34}$$

We now have enough background information to derive a temperature profile. Our goal is to show that the temperature profile in a gray, LTE, radiative equilbrium atmosphere with no scattering is

$$T^4 \approx \frac{3}{4} T_{\text{int}}^4 \left[\tau + 2/3\right]. \tag{9.35}$$

Using the definition of optical depth (equation [5.13]) the temperature versus optical depth can subsequently be converted to a temperature versus altitude.

We follow [5] and start with the radiative transfer equation [5.39]

$$\mu \frac{dI(\tau, \mu, \nu)}{d\tau} = I(\tau, \mu, \nu) - S(\tau, \nu).$$

Integrating over all frequencies, we have

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - S(\tau). \tag{9.36}$$

Given $J(\tau) = S(\tau)$ for a gray radiative equilibrium atmosphere (equation [9.34]), the gray radiative transfer equation is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - J(\tau). \tag{9.37}$$

We now proceed to multiply the radiative equilibrium version of the gray radiative transfer equation by successive orders of μ and integrate over solid angle (the so-called moments of the radiative transfer equation). For the zeroth moment we integrate the above equation [9.37] over the range $-1 < \mu < 1$ and $0 \leq \phi < 2\pi$ to find

$$\frac{1}{4\pi} \frac{dF(\tau)}{d\tau} = J(\tau) - J(\tau) = 0. \tag{9.38}$$

This result is the flux constancy that we had already assumed at the beginning of this subsection. We will now refer to $F(\tau)$ as a constant F. The first moment, again integrating over the range $-1 < \mu < 1$ and $0 < \phi < 2\pi$, is

$$\frac{dK(\tau)}{d\tau} = \frac{1}{4\pi} F \tag{9.39}$$

with a solution using $dF(\tau)/d\tau = 0$,

$$K(\tau) = \frac{1}{4\pi} F\tau + c, \tag{9.40}$$

where c is an integration constant we wish to find. We can relate $K(\tau)$ to $J(\tau)$ by the Eddington approximation $K(\tau) = \frac{1}{3} J(\tau)$ (equation [6.34]) to find

$$J(\tau) = \frac{3}{4\pi} F\tau + 3c. \tag{9.41}$$

We use the radiative equilibrium LTE relationship in equation [9.34] for $J(\tau) = \sigma_{\rm R}T^4/\pi$ and the constant flux that is passing through each layer of atmosphere $F = \sigma_{\rm R}T_{\rm int}^4$ to conclude that

$$\boxed{T^4 \approx \frac{3}{4}T_{\rm int}^4\,[\tau + 2/3]\,.} \tag{9.42}$$

We leave the evaluation of $3c = \sigma T_{\rm int}^4/2\pi$ as an exercise, with a general outline here. To find c we calculate the flux emerging at the top of the atmosphere using a frequency-integrated version of the emergent flux (equation [5.46]),

$$F(0) = 2\pi \int_0^1 \int_0^\infty S(\tau,\mu)\mathrm{e}^{-\tau/\mu} d\tau d\mu. \tag{9.43}$$

Using the condition for radiative equilibrium $S(\tau) = J(\tau)$, our solution for $J(\tau)$ in equation [9.41], and the upper boundary condition that the emergent flux at the top of the atmosphere is $\sigma_{\rm R}T_{\rm int}^4$, we have to solve the equation

$$F(0) = 2\pi \int_0^1 \int_0^\infty \left[\frac{3}{4\pi}F\tau + 3c\right] \mathrm{e}^{-\tau/\mu} d\tau d\mu \equiv \sigma_{\rm R}T_{\rm int}^4, \tag{9.44}$$

in order to derive c. We reiterate that $F(\tau)$ is a constant denoted here by F and we use $F(0)$ for the same constant flux that is emerging at the top of the planet atmosphere.

What can the gray atmosphere radiative equilibrium temperature profile tell us about planetary atmospheres? First, the derived temperature profile gives us an approximate prescription for a temperature profile in a planetary atmosphere, given that we have the appropriate atmospheric composition and opacities and their dependence on temperature and pressure (Figure 9.7).

Second, the gray atmosphere radiative equilibrium temperature at the top of the atmosphere (equation [9.42] at $\tau = 0$) matches our estimate for a simple greenhouse atmosphere (equation [9.19]), $T_{\rm e} \sim (1/2)^{1/4}T_{\rm int}$, where here $T_{\rm int}$ is the surface temperature.

Third, taking the derivative of our temperature profile (equation [9.42]) with respect to τ, we find a temperature gradient

$$\frac{dT}{dz} \approx -\frac{3}{16}\kappa(z)\frac{T_{\rm int}^4}{T^3}, \tag{9.45}$$

recovering the diffusion approximation temperature gradient if we associate the optical depth scale and the $\kappa(z)$ with the Rosseland mean. This temperature gradient, again, shows that either a high net radiative flux (in the form of $F = \sigma_{\rm R}T_{\rm int}^4$) or a high κ generates a large temperature gradient.

9.6.3 Gray Atmosphere Heated from Above and Below

We have above just computed a temperature profile for a gray, LTE, radiative equilibrium atmosphere heated from below. Such an atmosphere could represent a giant planet with an interior energy source heating the atmosphere. Alternatively, the heating from below scenario could be a rocky planet with a surface, where all of

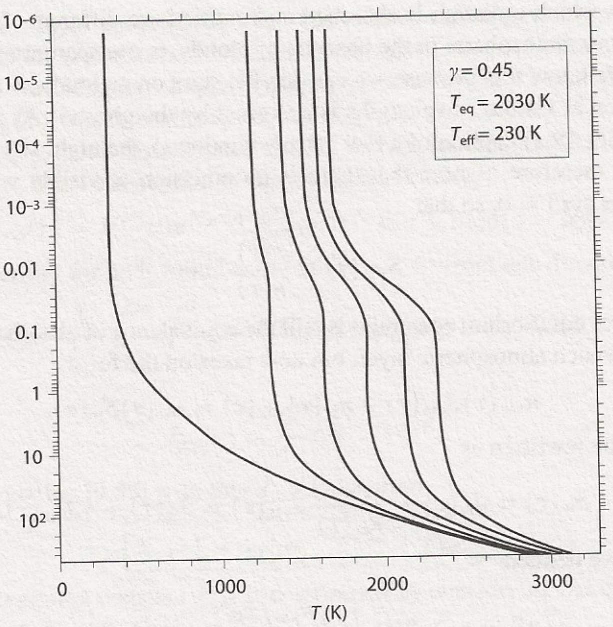

Figure 9.7 Temperature-pressure profiles computed using the gray atmosphere approximation in equation [9.62] with a scaling factor to convert from τ to P. The curves are for different incoming μ_0; from left to right for $\mu_0 = \cos\theta_0 = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$. Adapted from [5].

the incident radiation travels down through a transparent atmosphere, is absorbed by the ground, and is reradiated upward.

We now turn to the case of an atmosphere heated from above and below, where the stellar radiation plays a role throughout the atmosphere, following [5]. In order to arrive at an analytic equation for a temperature profile, we still work under the same simplifying assumptions, that of a gray atmosphere in LTE and in radiative equilibrium with no scattering. Recall that the gray atmosphere assumption means we use frequency-integrated terms of radiation quantities.

To consider heating from stellar radiation, we must consider the basic point common to all planetary atmospheres: the star is always hotter than the planet by definition. This implies that stellar radiation is absorbed at visible wavelengths and reradiated at infrared wavelengths. Molecular absorption typically dominates at IR wavelengths.

The approach to finding a temperature profile in an atmosphere heated both from above and below therefore involves two different intensities (and other quantities of radiation). One intensity is at visible wavelengths, I_{vis}, where we assume radiation is only absorbed and not emitted. The second intensity is at infrared wavelengths I_{ir}, where we assume that all of the absorbed energy is reradiated. The total intensity is $I_{\mathrm{vis}} + I_{\mathrm{ir}}$, and similar addition rules apply for other radiation quantities.

The motivation of the visible versus infrared separation of radiation is that the altitudes at which radiation is absorbed and emitted are different. Earth is a good example: our atmosphere, in the absence of clouds, is transparent at visible wavelengths. We know this because we can see the stars on a cloudless night. Most of the radiation at visible wavelengths is absorbed by the ground. At most IR wavelengths (with the exception of a few narrow windows), the night sky is opaque.

We will therefore assume that there is no emission at visible wavelengths, in other words $\varepsilon(\tau) = 0$, so that

$$S_{\text{vis}}(\tau) \equiv \frac{\varepsilon(\tau)}{\kappa(\tau)} = 0. \tag{9.46}$$

Our radiative equilibrium constraint is still the equivalency of absorbed and emitted radiation in each atmospheric layer, but now takes on the form

$$\kappa_{\text{vis}}(\tau)J_{\text{vis}}(\tau) + \kappa_{\text{ir}}(\tau)J_{\text{ir}}(\tau) = \kappa_{\text{ir}}(\tau)S_{\text{ir}}(\tau), \tag{9.47}$$

which can be rewritten as

$$S_{\text{ir}}(\tau) = J_{\text{ir}}(\tau) + \frac{\kappa_{\text{vis}}(\tau)}{\kappa_{\text{ir}}(\tau)}J_{\text{vis}}(\tau) = J_{\text{ir}}(\tau) + \gamma J_{\text{vis}}(\tau). \tag{9.48}$$

Here we have defined

$$\gamma \equiv \kappa_{\text{vis}}/\kappa_{\text{ir}} \tag{9.49}$$

as the ratio of the visible and infrared mean absorption coefficients.

The two radiative transfer equations for the visible and infrared beams of inten-esity are

$$\mu\frac{dI_{\text{vis}}(\tau,\mu)}{d\tau_{\text{vis}}} = I_{\text{vis}}(\tau,\mu), \tag{9.50}$$

$$\mu\frac{dI_{\text{ir}}(\tau,\mu)}{d\tau_{\text{ir}}} = I_{\text{ir}}(\tau,\mu) - S_{\text{ir}}(\tau,\mu) = I_{\text{ir}}(\tau,\mu) - J_{\text{ir}}(\tau) - \gamma J_{\text{vis}}(\tau). \tag{9.51}$$

The zeroth-order moment equations are

$$\frac{1}{4\pi}\frac{dF_{\text{vis}}(\tau)}{d\tau_{\text{vis}}} = J_{\text{vis}}(\tau), \tag{9.52}$$

$$\frac{1}{4\pi}\frac{dF_{\text{ir}}(\tau)}{d\tau_{\text{ir}}} = -\gamma J_{\text{vis}}(\tau). \tag{9.53}$$

Adding the above two equations together, considering that $d\tau_{\text{vis}} = \gamma d\tau_{\text{ir}}$ we again see in the expression of radiative equilibrium that the total flux derivative is zero, meaning that a constant net flux passes through the atmosphere.

The first-order moment equations of the infrared radiative transfer equation are

$$\frac{dJ_{\text{ir}}(\tau)}{d\tau_{\text{ir}}} = \frac{3}{4\pi}F_{\text{ir}}(\tau), \tag{9.54}$$

where we have used the Eddington approximation $J = 3K$ (equation [6.34]).

As before, for the gray atmosphere heated from below, we find the temperature profile by starting with an expression for $J_{\text{ir}}(\tau)$. The complication compared to the case of the atmosphere heated from below is that the infrared equations also depend on the visible wavelength radiation quantities. Conceptually, this is because although in each layer the total absorbed and emitted radiation is constant, the radiation is absorbed at visible and infrared wavelenghts, and in this framework only emitted in IR wavelengths. Explicitly, the equation for $J_{\rm ir}(\tau)$ (equation [9.54]) depends on $F_{\rm ir}(\tau)$, which itself depends on $J_{\rm vis}(\tau)$,

To find $J_{\rm vis}(\tau)$, recall from Section 6.4.3.1 that the incoming intensity from one direction μ_0, ϕ_0 is the attenuation of the incident intensity,

$$I_{\rm vis}(\tau) = I(0,\mu)\mathrm{e}^{\gamma\tau_{\rm ir}/\mu} = I_0\mathrm{e}^{\gamma\tau_{\rm ir}/\mu}\delta(\mu+\mu_0)\delta(\phi-\phi_0). \qquad (9.55)$$

The mean intensity for an inward beam ($-1 < \mu < 0$) from one direction μ_0 is

$$J_{\rm vis}(\tau) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{0} I_0\mathrm{e}^{\gamma\tau_{\rm ir}/\mu}\delta(\mu+\mu_0)\delta(\phi-\phi_0)d\mu d\phi \qquad (9.56)$$

or

$$J_{\rm vis}(\tau) = \frac{1}{4\pi}I_0\mathrm{e}^{-\gamma\tau_{\rm ir}/\mu_0}. \qquad (9.57)$$

Integrating equation [9.53] with this $J_{\rm vis}(\tau)$, we find

$$\boxed{F_{\rm ir}(\tau) = \mu_0 I_0\mathrm{e}^{-\gamma\tau_{\rm ir}/\mu_0} + F_{\rm int},} \qquad (9.58)$$

where the integration constant $F_{\rm int}$ is determined by considering the upper boundary condition at $\tau_{\rm ir} = 0$, $F_{\rm ir}(0) = \mu_0 I_0 + F_{\rm int}$. Here $F_{\rm int} = \sigma_{\rm R}T_{\rm int}^4$.

With expressions for $F_{\rm ir}(\tau)$ and $J_{\rm vis}(\tau)$, we can now find an expression for $J_{\rm ir}(\tau)$ by integrating equation [9.54],

$$J_{\rm ir}(\tau) = -\frac{3}{4\pi}\left[\frac{\mu_0^2}{\gamma}I_0\mathrm{e}^{-\gamma\tau_{\rm ir}/\mu_0}\right] + \frac{3}{4\pi}F_{\rm int}\tau_{\rm ir} + c'. \qquad (9.59)$$

To reach a temperature profile we use $J_{\rm ir}(\tau) = \sigma_{\rm R}T^4/\pi$ and $F_{\rm int} = \sigma_{\rm R}T_{\rm int}^4$ to find

$$T^4(\tau) = \frac{3}{4}T_{\rm int}^4\tau_{\rm ir} - \frac{3}{4}\left[\frac{1}{\sigma_{\rm R}}\frac{\mu_0^2}{\gamma}I_0\mathrm{e}^{-\gamma\tau_{\rm ir}/\mu_0}\right] + \frac{c'\pi}{\sigma_{\rm R}}. \qquad (9.60)$$

We may find the constant c' as before for the atmosphere heated from below, by writing down the integral for the emergent flux and solving for the constant. We also use the condition for radiative equilibrium (equation [9.48]), the equations for $J_{\rm ir}(\tau)$ and $J_{\rm vis}(\tau)$, and the upper boundary condition that the emergent flux is equal to $\mu_0 I_0 + \sigma_{\rm R}T_{\rm int}^4$. To find c' we then have to solve

$$F_{\rm ir}(0) = 2\pi\int_0^1\int_0^\infty\left[S_{\rm ir}(\tau_{\rm ir})\mathrm{e}^{-\tau_{\rm ir}/\mu}\right]d\tau_{\rm ir}d\mu \equiv \sigma_{\rm R}T_{\rm int}^4 + \mu_0 I_0. \qquad (9.61)$$

Evaluating c' leads to

$$\boxed{\begin{aligned} T(\tau)^4 &= \frac{3}{4}T_{\rm int}^4\left[\tau_{\rm ir} + 2/3\right] + \\ \mu_0 T_0^4 &\left[-\frac{3}{4}\frac{\mu_0}{\gamma}\mathrm{e}^{-\gamma\tau_{\rm ir}/\mu_0} + \frac{3}{2}\left(\frac{2}{3} + \left(\frac{\mu_0}{\gamma}\right)^2 - \left(\frac{\mu_0}{\gamma}\right)^3\ln\left(1+\frac{\gamma}{\mu_0}\right)\right)\right].\end{aligned}} \qquad (9.62)$$

Here we have associated a temperature with the incident intensity via $I_0 = \sigma_{\rm R}T_0^4$.

The temperature profile we have just derived can be used for exoplanet atmospheres. Figure 9.7 shows temperature profiles for different values of μ_0.

9.7 THE ADIABATIC TEMPERATURE PROFILE

9.7.1 Convection Basics

In a planetary atmosphere there is a net transport of energy outward. So far we have discussed energy transport by radiation (i.e., the microscopic interactions of photons with atoms and molecules). In this section we will discuss energy transport by convection.

Convection is the transport of energy by bulk motions of matter in the atmosphere in the vertical direction. During convection, heat flows from hotter to cooler regions by the macroscopic movement of matter. Convection in a planetary atmosphere is driven by the temperature gradient and is enabled because of the gravity field.

To describe an overview of convection we consider a local air parcel that is heated from below. The heated air parcel becomes less dense than its surroundings and buoyancy forces will cause the air parcel to rise. As the air parcel rises, it experiences a lower ambient pressure, and expands and releases heat. Now cooler than the surrounding atmosphere, the air parcel sinks to a level with higher pressure and higher temperature. The parcel is again heated, again rises to release heat, sinks again, and repeats this in the convective energy cycle.

Everyday examples of convection occur on many different scales and include boiling water, lava lamps, and "shimmering" air above hot pavement. Convection in the Sun's atmosphere is evident from images that show a large granulation pattern. The bright spots are centers of convective cells—the tops of columns of rising hot gas. The dark areas are the areas of cooled gas beginning to descend. The solar convective cells are large—typically 1000 km in diameter.

9.7.2 Derivation of the Adiabatic Temperature Profile

We want to derive a temperature change with altitude (a temperature profile) that occurs in a gas as it adiabatically expands or is compressed as it rises under hydrostatic equilibrium. We begin with basic thermodynamic principles; describing them in terms of temperature and pressure will lead us to a formulation of the adiabatic temperature profile.

The first law of thermodynamics is the conservation of energy applied to thermodynamic processes. Energy can be transferred from one system to another but it cannot be created or destroyed. Work done on or by a system must change the internal energy of that system. The change in internal energy dU is

$$dU = dQ - dW, \tag{9.63}$$

where dQ is the amount of energy added to the system by heating and dW is the the amount of energy lost by via work done by the system. We should consider this conservation of energy description as applied to an air parcel, and we will use this equation to derive the adiabatic temperature gradient. We will now take each term of equation [9.63] in turn and describe it in terms of T and P. From the definition of specific heat capacity and the relation between c_v and c_p we rewrite the change in internal energy dU as

$$dU = mc_v dT = m(c_p - R_s)dT, \tag{9.64}$$

where m is mass, c_v and c_p are the specific heat capacities at constant volume and constant pressure respectively, and R_s is the specific gas constant; all have units of J kg^{-1} K^{-1}. An adiabatic process has no heat exchanged with the surroundings so that $dQ = 0$. In an expanding volume work is defined as $dW = PdV$. Taking the derivative of the equation of state for an ideal gas, we obtain

$$PdV = mR_s dT - VdP = mR_s dT - \frac{m}{\rho}dP. \tag{9.65}$$

We can now rewrite the conservation of energy in equation [9.63] as

$$\frac{dT}{dP} = \frac{1}{\rho c_p}. \tag{9.66}$$

To convert this temperature-pressure relation to a temperature gradient we relate the pressure P to altitude z by using the hydrostatic equilibrium equation [9.5]

$$dP = -\rho g dz \tag{9.67}$$

to derive the adiabatic temperature gradient

$$\boxed{\Gamma \equiv -\frac{dT}{dz} = \frac{g}{c_p}.} \tag{9.68}$$

We may also write the adiabatic temperature gradient as a function of optical depth

$$\Gamma_\tau \equiv \frac{dT}{d\tau} = \frac{g}{\overline{\kappa}(\tau)c_p}, \tag{9.69}$$

where the negative sign on the right-hand side comes from the increase of the optical depth scale with decreasing altitude. The adiabatic temperature gradient is called the adiabatic lapse rate, Γ, for planets. Note the sign convention; Γ is positive for a decrease in temperature with altitude. The adiabatic lapse rate is the temperature change that occurs in the gas as it adiabatically expands or is compressed. Returning to the air parcel scenario, the adiabatic lapse rate means that an air parcel moving in a hydrostatic atmosphere has a fixed rate of change of both temperature and density with altitude. The air parcel will have its own temperature and density, as given by the adiabatic lapse rate, but will share the pressure of the surrounding atmosphere.

The above equation for the adiabatic lapse rate is for a "dry" atmosphere. In Earth's atmosphere, the release of latent heat from condensation of water vapor must be considered in the lapse rate. Although it is not strictly adiabatic, we can compute the temperature profile by considering the heat deposited in the condensing layer, $dQ = -dm_s L$, where dm_s is the change in the mass of the condensing vapor per unit mass of noncondensible gas and L is the specific latent heat for the vapor (in units of J kg^{-1}). The negative sign arises because the air parcel absorbs heat—the latent heat deposited in the layer by the condensing vapor. Keeping dQ in the first law of thermodynamics we may follow the above derivation to find the moist adiabatic lapse rate

$$\boxed{\Gamma \equiv -\frac{dT}{dz} = \frac{g}{c_p}\left[\frac{1}{1 + (L/c_p)(dm_s/dT)}\right].} \tag{9.70}$$

The difference between the dry and moist adiabatic lapse rates can be significant. For Earth's troposphere, the dry adiabatic lapse rate is about 9.8 K/km. The moist adiabatic lapse rate is about 5 K/km. Typically, an average value of 6.5 K/km can be used, because convection in Earth's trophosphere is partially moist and partially dry. Remarkably, with the dry adiabatic lapse rate Earth's atmosphere is stable against convection, and with the moist adiabatic lapse rate, convection sets in.

Many exoplanets with different atmospheric temperatures from Earth's will have condensation of gas other than water vapor. We could use the same formulation for the moist adiabatic lapse rate, substituting the relevant change in masses of saturated gas and specific latent heat values.

9.8 THE ONE-DIMENSIONAL TEMPERATURE-PRESSURE PROFILE

9.8.1 Conceptual Overview

The approach to determining the temperature profile with altitude in a planetary atmosphere in radiative-convective equilibrium is to solve a set of three equations (the radiative transfer equation, the radiative and convective equilibrium equation, and the hydrostatic equilibrium equation) for three unknowns (the temperature, pressure, and radiation field). The approach to the solution is to adopt a starting solution via a temperature pressure profile and then to iterate until a T-P profile is computed that satisfies the three equations.

For example, we could start with a T-P profile from the gray atmosphere (Section 9.6.3) using the appropriate temperature- and pressure-dependent opacities. We would then solve the radiative transfer equation, the radiative equilibrium equation, and the hydrostatic equilibrium equation by any one of the number of numerical methods available. These numerical methods typically derive temperature corrections and iterate until a T-P profile that satisfies all three equations is obtained. With the new T-P profile, we would test at each altitude whether or not the atmosphere is stable against convection. If the atmosphere were stable against convection, our solution would be completed.

If in this example the atmosphere were to be unstable, convection would be occuring and we would use the adiabatic lapse rate as the temperature profile in the convective part of the atmosphere. We would iterate again to find a T-P profile in the radiative region, and one that smoothly connects to the adiabatic temperature profile.

9.8.2 A Radiative Equilibrium Atmosphere

For an atmosphere in radiative equilibrium, we wish to solve three equations: the equation of hydrostatic equilibrium (equation [9.6]),

$$\frac{dP}{d\tau} = \frac{g\rho}{\bar{\kappa}}; \tag{9.71}$$

the equation of radiative transfer (equation [5.39]),

$$\mu \frac{dI(\tau,\mu,\nu)}{d\tau} = I(\tau,\mu,\nu) - S(\tau,\mu,\nu); \tag{9.72}$$

and the equation of radiative equilibrium (equation [9.31]).

$$\frac{dF(\tau)}{d\tau} = \int_0^\infty \kappa(\tau,\nu)\,[J(\tau,\nu) - S(\tau,\nu)]\,d\nu = 0. \tag{9.73}$$

Outputs. In these equations there are three unknowns: the radiation field $I(\tau,\mu,\nu)$, the temperature $T(\tau)$, and the pressure $P(\tau)$. The three equations can be solved simultaneously for the three unknowns; thus the atmospheric T-P profile and the radiation field emerge simultaneously.

Inputs. The additional parameters in the three vertical structure equations are the interior temperature $T_{\rm int}$ and the surface gravity g and these are classifications of the model. g is known for transiting exoplanets, and $T_{\rm int}$ comes from evolution calculations or is a free parameter of the model. The opacities are another set of inputs. The opacities depend upon the composition of the atmosphere. The elemental abundance is an input variable and chemical equilibrium is assumed or a photochemical model must be used (Chapter 4).

Subtleties. The above system of equations is highly coupled and benefits from a simultaneous solution. For example, if the radiative transfer equation were solved on its own, and the solution did not satisfy radiative equilibrium, a new temperature profile $T(\tau)$ would have to be found that did satisfy radiative equilibrium. But, for a different temperature structure, the number densities of different chemical species would change. As a result the gas pressure would change, as well as the opacities and emissivities $\kappa(\tau,\nu)$ and $\varepsilon(\tau,\mu,\nu)$, leading to a change in the radiation field $I(\tau,\mu,\nu)$ at all depths.

The radiative transfer equation itself is highly nonlinear, as the scattering term means that photons decouple the radiation field from the local temperature. For the case of exoplanets irradiated by their parent stars, the radiation field at the top of the atmosphere is coupled to the deeper atmosphere by scattering—photons can travel long distances down into the atmosphere before heating the atmosphere.

To solve the equations, iterations are usually required, beginning with an estimated temperature pressure profile, typically from a gray atmosphere solution.

9.8.3 Radiative and Convective Equilibrium

To determine whether or not convection should be occurring in a planetary atmosphere, we take a radiative equilibrium temperature profile, and check if the atmosphere is stable against convection, with the previously described criterion for convection in equation [9.23] (but replacing "atm" with "rad"),

$$\left(\frac{dT}{dz}\right)_{\rm ad} > \left(\frac{dT}{dz}\right)_{\rm rad}. \tag{9.74}$$

In planetary atmosphere models one usually assumes that if convection is occurring, convection is so efficient that it is the dominant process for energy transport. This means that the temperature gradient comes directly from the adiabatic lapse rate (equation [9.69]). In the lapse rate equation, hydrostatic equilibrium has already been included. The temperature as a function of pressure can be computed from the adiabatic lapse rate and with the ideal gas law.

The adiabatic lapse rate by itself gives the slope of the temperature gradient, but describes any of a number of "parallel" lines in temperature versus altitude (or in temperature versus pressure depth). We need to derive a temperature profile consistent with the temperature profile in the radiative equilibrium part of the planetary atmosphere. To do this we use as a boundary condition the radiative equilibrium temperature at the upper end of the convection zone. Here we are assuming that convective equilibrium holds, that is,

$$F_{\rm conv}(\tau) = \sigma_{\rm R} T_{\rm int}^4 + \mu_0 I_0. \tag{9.75}$$

There may be a region in a real atmosphere where some of the energy is transported by radiation and the rest by convection. It is complicated to solve explicitly for the fraction of the radiation transported by convection. To proceed we would substitute radiative and convective equilibrium

$$F_{\rm rad}(\tau) + F_{\rm conv}(\tau) = \sigma_{\rm R} T_{\rm eff}^4 + \mu_0 I_0 \tag{9.76}$$

for the radiative equilibrium in equations [9.71]–[9.73] above. The major complication arises in developing an expression for the convective flux. Indeed, only an approximate expression is derivable, and here we will provide only an overview.

In order to determine how much energy is transported by convection compared to radiation, we will need to know the convective flux. Recall from Section 9.7.1 that during convection, heat flows from hotter to cooler regions by the macroscopic movement of matter. The excess energy deposited per unit volume when a mass element merges with the surrounding atmosphere is $\rho c_p \Delta T$. The heat flux is then

$$F_{\rm conv} = \bar{v} \rho c_p \Delta T, \tag{9.77}$$

where $\bar{v}$ is the mean velocity of the mass element. Here ΔT arises from the difference between the temperature gradient of the rising material and the temperature gradient of the surrounding atmosphere. Following [8], ΔT can be expressed in terms of the temperature gradients for mass elements moving over the distance Δz,

$$\Delta T = \left[\left(-\frac{dT}{dz} \right) - \left(-\frac{dT}{dz} \right)_{\rm c} \right] \Delta z, \tag{9.78}$$

where the first temperature gradient on the right hand side is that of the surrounding atmosphere in the sought after state of radiative and convective equilibrium. The second temperature gradient is that of the convective elements.

Let us take a look at the above two equations to see what we would need to derive the convective flux. The heat capacity should be known, the temperature is what we are trying to solve for, and the density comes from the ideal gas via the hydrostatic equilibrium equation. We are left with the convective velocity and the distance convective elements travel before releasing heat to the surroundings.

The derivation of an expression for the convective velocity, the temperature gradient, and the distance a convective element travels is complex with many caveats and takes several pages. We leave the details to the excellent references [6,7,8]. These references are to stellar atmospheres, where the theory of convection is a 1D approximation called the mixing length theory. This theory is a local theory because no definitive 3D convective theory is available. The main assumption in the mixing length theory is the scale length l, the distance over which a convective bubble rises and releases its heat, before sinking again. The choice of l is somewhat arbitrary, but it is approximately comparable to the pressure scale height.

9.9 TEMPERATURE RETRIEVAL

Up until now we have described the so-called forward problem, starting with basic physics to derive a planet's vertical temperature structure. The forward problem involves solution of a set of coupled equations, beginning with a starting solution and iterating to convergence. A direct solution for the temperature, or the "inverse problem," is possible if a highly detailed spectrum is available. Studies of solar system planetary atmospheres use temperature retrieval methods by first inferring a basic T-P profile and then perturbing that fiducial temperature profile until data is fit well. The fiducial temperature profile is perturbed over both temperature and molecular abundances (see, e.g., [1]).

For exoplanets—in contrast to solar system planets—the spectra are not of sufficient quality to infer a unique fiducial temperature-pressure profile (see Figures 6.6, 6.7, and 6.8). In other words, there is no starting point from the data to derive a fiducial model. A suitable approach is to computationally derive the *range* of temperature-pressure profiles and molecular and atomic abundances allowed by a given spectrum. The forward temperature-pressure structure determination described in this chapter, however, is too computationally intensive to use to run millions of models to find a good fit.

A new method for exoplanet temperature and abundance retrieval has therefore been proposed [9]. This new method uses a parametric T-P profile (Figure 9.8), running tens of thousands of them that fit the data, and requiring T-P profiles to satisfy hydrostatic equilibrium and global energy balance. The quantitatively allowed ranges of T-P profiles and molecular abundances can be described. In addition, constraints on the albedo and day-night energy redistribution and on the effective temperature can be determined.

The parametric T-P profile in this example is motivated by physical principles, solar system planet T-P profiles, and 1D exoplanet T-P profiles generated from model atmosphere calculations reported in the literature. The P-T parametric profile is a generalized exponential profile of the form

$$P = P_0 e^{\alpha (T - T_0)^{\beta}}, \tag{9.79}$$

where P is the pressure in bars, T is the temperature in K, and P_0, T_0, α, and β are free parameters. For layer 3, the model profile is given by $T = T_3$, where T_3 is a free parameter. Furthermore,

$$\begin{array}{lll} P_0 < P < P_1: & P = P_0 e^{\alpha_1 (T - T_0)^{\beta_1}} & \text{layer1} \\ P_1 < P < P_3: & P = P_2 e^{\alpha_2 (T - T_2)^{\beta_2}} & \text{layer2} \\ P > P_3: & T = T_3 & \text{layer3} \end{array} \tag{9.80}$$

and P is typically used as the independent coordinate. The above parametric profile is in fact overspecified. The β parameter turns out to be a redundant parameter and can be set $\beta_1 = \beta_2 = 0.5$. Then the model profile in (9.80) has nine parameters, namely, $P_0, T_0, \alpha_1, P_1, P_2, T_2, \alpha_2, P_3$, and T_3. Furthermore, two of the parameters can be eliminated based on the two constraints of continuity at the two layer boundaries, that is, layers 1–2 and layers 2–3. P_0 is generally set to $P_0 = 10^{-5}$ bar,

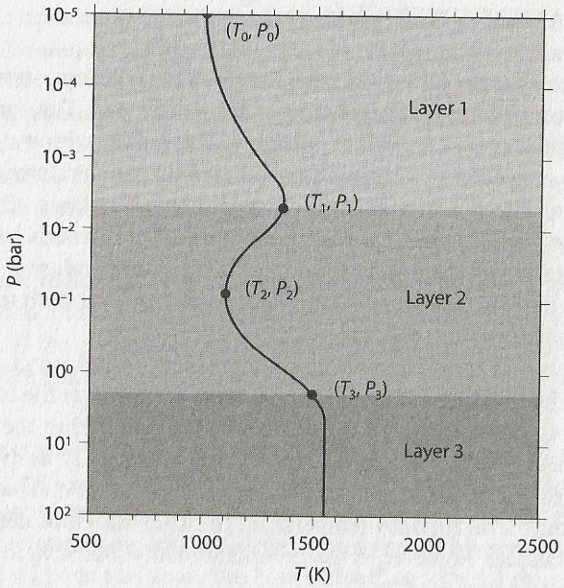

Figure 9.8 The parametric pressure-temperature profile. In the general form, the profile includes a thermal inversion layer (layer 2) and has six free parameters. An isothermal profile is assumed below the pressure P_3 (layer 3). Alternatively, for cooler atmospheres with no isothermal layer, layer 2 could extend to deeper layers and layer 3 could be absent. Adapted from [9].

that is, at the top of the model atmosphere. The parametric profile in its complete generality therefore has six free parameters.

The parametric T-P profile shown in Figure 9.8 is motivated by the physics that sets a planet atmosphere's vertical structure. In general the temperature structure at a given altitude depends on the opacity at that altitude, along with density and gravity.

The T-P profile shown in Figure 9.8 is divided into three representative layers. Above layer 1, at pressures below $P \sim 10^{-5}$ bar, the optical depth at all wavelengths becomes low enough so that the layers of the atmosphere are transparent to the incoming and outgoing radiation and not relevant for spectral features. The uppermost layer, layer 1, has no thermal inversions. Here the atmosphere is being heated by lower layers and cools with increasing altitude. The middle layer, layer 2, is where most spectral features are formed. In layer 2, the temperature structure is governed by radiative process and possibly by atmospheric dynamics. These optically thin layers are at altitudes where thermal inversions may be formed, depending on the level of irradiation from the parent star and the presence of strong absorbing gases or solid particles (see Figure 9.1). The bottom-most layer, layer 3, is the regime where a high optical depth leads to radiative diffusion and the related isothermal temperature structure. Essentially the strong irradiation heating from above does not reach the deep atmosphere layer, which is heated from the interior energy outflow. Below layer 3, in the deepest layers of the planet atmosphere, convection is the dominant energy transport mechanism. The high pressure (equivalently, high density) implies a high opacity, making energy transport by convection a more efficient energy transport mechanism than radiation.

The description of the T-P profile in Figure 9.8 is focused on hot Jupiters. For cooler planets layer 3 could be absent, with layer 2 extending to deeper layers. In addition, the radiative-convective boundary occurs at a higher altitude in the planet atmosphere than for hot Jupiters, meaning convection may play a role in layers 2 and 3, making the temperature profile an adiabat.

9.10 SUMMARY

The vertical temperature structure of a planetary atmosphere is intimately connected to energy conservation: energy is neither created nor destroyed in an atmospheric layer. We began with a description of Earth's vertical atmospheric structure. We continued with a derivation of the hydrostatic equilibrium equation, which describes how atmospheric pressure supports the atmosphere against gravity. The vertical thermal structure is also connected to energy transport. We described the two major mechanisms of energy transport, radiation and convection, and the criterion that determines which energy transport mechanism dominates. If energy transport by radiation dominates, then the vertical thermal structure comes from radiative equilibrium. If convection dominates, the vertical thermal structure is defined by the adiabatic lapse rate.

The planetary spectrum can be connected to the vertical thermal structure—that is, the temperature gradient—by energy conservation, hydrostatic equilibrium, radiative transfer, and opacities. The 1D temperature structure (by which we really mean the temperature-pressure structure) and the emergent planetary spectrum can be solved with three equations (radiative transfer, radiative and convective equilibrium, and hydrostatic equilibrium) for three unknowns as a function of altitude (the temperature, pressure, and frequency-dependent radiation field).

Ultimately the vertical temperature-pressure structure is tied to the opacities and radiative transfer: if energy transport is by radiation, the opacity governs the temperature structure. If energy transport by radiation is inhibited, then convection takes over.

REFERENCES

For further reading

For a description of convection, including mixing length theory:

- Bohm-Vitense, E. 1989. *Stellar Atmospheres, vol. 2 and 3* (Cambridge: Cambridge University Press).

- Hansen, J., Kawaler, S. D., and Trimble, V. 2004. *Stellar Interiors: Physical Principles, Structure, and Evolution* (2nd ed; New York: Springer).

For a basic description of convection as applied to Earth's atmosphere:

- Marshall, J., and Plumb, A. 2008. *Atmosphere, Ocean, and Climate Dynamics: An Introductory Text* (London: Elsevier Academic Press).

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1. Goody, R. M., and Yung, Y. L. 1989. *Atmospheric Radiation: Theoretical Basis* (2nd ed; Oxford: Oxford University Press).
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EXERCISES

1. Rewrite the ideal gas law in a form that uses the universal gas constant R.
2. What is the value of the scale height H for exoplanets? Consider a hot Jupiter, a hot super Earth with an H_2-dominated atmosphere, a hot super Earth dominated by a CO_2 atmosphere, and Earth itself. Use equation [9.11].
3. Show that for an n-layer leaky greenhouse model, $T_s = (n+1)^{1/4} T_e$, where T_e is the emission temperature in the highest atmosphere layer.

4. Complete the derivation of the gray, radiative equilibrium, LTE, no-scattering atmosphere.

   a. Derive $3c = \frac{\sigma_R T_{int}^4}{2\pi}$ by solution of the equation [9.44].

   b. Do the same for equation [9.62].

5. Assumptions in the gray radiative equilibrum temperature profile.

   a. In the derivation of the gray radiative equilibrum temperature profile, we have used the Eddington approximation $J = 3K$. What limitations does this assumption put on the temperature profile?

   b. We derived the same temperature gradient from the gray radiative equilibrum temperature profile (equation [9.45]) as was derived in the diffusion approximation (Section 6.4.4, equation [6.53]). What assumptions that went into the gray radiative equilibrium temperature profile cause this to be the same?

6. Estimate the surface pressure of an exoplanet, given an atmospheric mass and composition.
7. Derive the equation for the moist adiabatic lapse rate, equation [9.70].
8. The solar system planet atmosphere vertical temperature-pressure profiles shown in Figure 9.1 have remarkable similarity because temperature profiles of planetary atmospheres are governed by basic physics. Qualitatively describe the physics that causes the characteristic shape of the temperature-pressure profiles.