UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2022

Problem Set 7
Due: Wednesday, Nov 02, 2022, before the start of class (by 1000).
This problem set is worth **37 points**.

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, type the Pset, and submit a printed copy.
Recommended tools for typesetting your problem set are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **The Gray Plane-Parallel Atmosphere [24 + 20 pts]**. In class we discussed at length the gray (i.e. wavelength-independent), plane-parallel (i.e. ignoring spherical geometry) stellar atmosphere. In this problem you will work through some of the calculations that we skimmed over in class.

   (a) (3 pts) Explain why the equation of radiative transfer in a plane-parallel atmosphere,

   $$\frac{dI}{d\tau}\cos\theta = I - S, \tag{1}$$

   is a bit different from the general form we saw earlier in class, namely

   $$\frac{dI}{d\tau} = S - I. \tag{2}$$

   (b) Show that when taking the first moment of the equation of radiative transfer, $\int \cos\theta \frac{dI}{d\tau} d\Omega = 0$ [3 pts].

   (c) Using the above result, show why (under these approximations) the average intensity will be equal to the source function, i.e. why $\langle I \rangle = S$ [3 pts].

   (d) Show that when taking the second moment of the equation of radiative transfer, $\int \cos^2\theta \frac{dI}{d\tau} d\Omega = \frac{4\pi}{3}\frac{d\langle I \rangle}{d\tau}$ [4 pts].

   (e) Show that when taking the second moment of the equation of radiative transfer, $\int \cos\theta \,(I - S)\, d\Omega = F$ (i.e., flux) [4 pts].

   (f) Using the above results, derive the expression for the source function S as a function of flux F and optical depth τ (with no other arbitrary constants) [3 pts].

   (g) (4 pts) Using the above results, show why (under these approximations) there *must* be a temperature gradient of the form

   $$T^4 = T_{\text{eff}}^4 \left(\frac{3}{4}\tau + \frac{1}{2}\right). \tag{3}$$

   (h) (BONUS 20 pts): Derive the linear limb-darkening law

   $$I_0 = a + b\cos\theta \tag{4}$$

   under the approximation of a gray, plane-parallel, two-stream atmosphere where $S(\tau) = a + b\tau$.

2. **Line Locations vs. Line Widths. [13 pts]** A number of students at times have expressed confusion that although for, e.g., spectral line wavelengths and frequencies $\lambda = c/\nu$ (clearly true!) it is nonetheless *not* the case that the *widths* of these lines are related by $\Delta\lambda = c/\Delta\nu$.

   (a) Pick your favorite spectral absorption line in the UV, optical, or infrared. Write its wavelength λ (in nm) and frequency ν (in Hz). [1 pt]

(b) Assume that when observed in a particular astronomical object, this absorption line (shown in Fig. 1) has a fractional full-width of 0.2% – i.e. $\gamma = 0.002$ (this "line width" γ has nothing to do with the speed of light).

Calculate the full line width, $\Delta\nu$, along with the maximum and minimum frequencies of the absorption line profile (marked by the vertical dashed lines in Fig. 1). Don't even think about wavelengths yet! [3 pts].

(c) Use those the maximum and minimum frequencies to calculate the minimum and maximum wavelength, respectively, of the line profile, as well as the line width in wavelength, $\Delta\lambda$. [3 pts]

(d) Compare your calculated $\Delta\lambda$ to the result you get by naively calculating $c/\Delta\nu$. Explain why they are not the same, and why the second value would be incorrect. [6 pts]

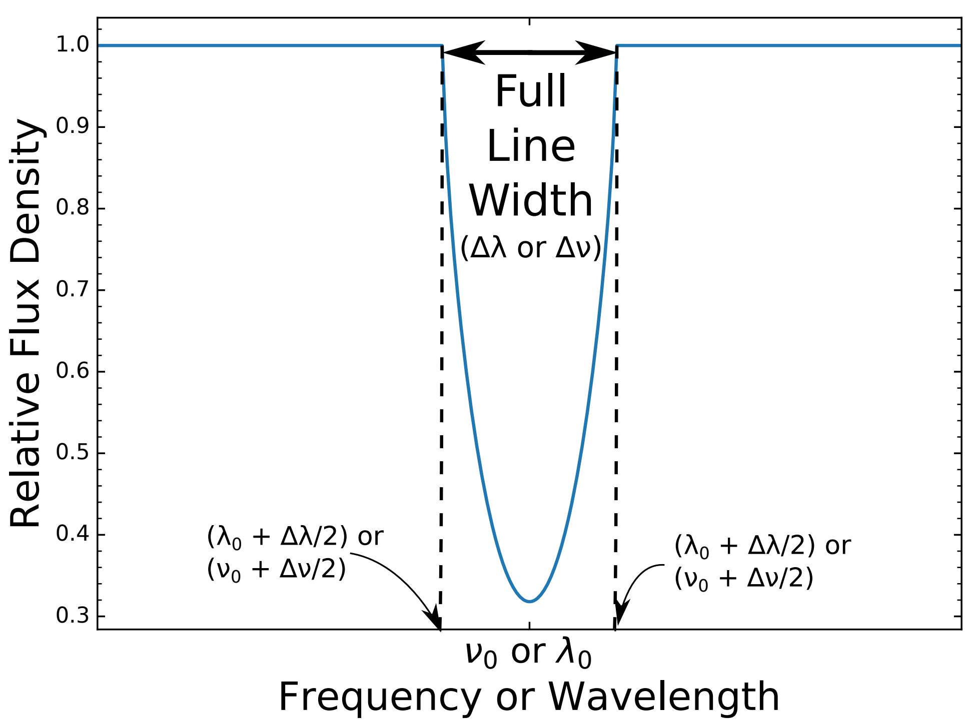

Figure 1: Arbitrary absorption line profile with central frequency and wavelength ν_0 and λ_0 and width $\Delta\nu$ or $\Delta\lambda$.