UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2020

Problem Set 2
Due: Friday, Sep 11, 2020, before the start of class (by 0900)
This problem set is worth **59 points**.

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, type the Pset, and submit as a PDF or OpenDocument file that includes your name. Recommended software tools are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **How Do You Say "Absorption" in Astrophysics? [12 pts]**

   (a) Define α_λ, extinction coefficient (in words, not just an equation!), and give its typical units. [3 pts]
   Solution: The extinction coefficient α_λ has SI units of m^{-1} (inverse meters). It describes the fraction of light that would be absorbed in a one-meter path length of the given material. It is usually a wavelength- (or frequency-)dependent quantity, hence the λ subscript.

   (b) Define σ_λ, cross-section (in words, not just an equation!), and give its typical units. [3 pts]
   Solution: The cross-section σ_λ has SI units of m^2 (area). It describes the effective, physically absorbing area that the particles of some material present to incoming radiation. E.g. if our particle of interest is "big red barns," then σ would be roughly the cross-sectional area of a barn, something on the order of $\sim$100 m^2 for visible light (which can't penetrate a barn). At radio wavelengths, a parn is mostly transparent; if only 1% of radio intensity is blocked by barn material, then the barn's effective cross-section at radio wavelengths would be just $\sim$1 m^2.

   (c) Define κ_λ, opacity (in words, not just an equation!), and give its typical units. [3 pts]
   Solution: The opacity κ_λ has SI units of $\mathrm{m}^2\ \mathrm{kg}^{-1}$. This quantity describes the effective total cross-section (i.e. absorbing/blocking area) of a material per unit mass. E.g. if the barn described above weighs 100 tons ($\approx 10^5$ kg) then at visible wavelengths the opacity of big red barns would be roughly $10^{-3}\ \mathrm{m}^2\ \mathrm{kg}^{-1}$, while at radio wavelengths it would be a lower $10^{-5}\ \mathrm{m}^2\ \mathrm{kg}^{-1}$.

   (d) Define τ_λ, the optical depth (in words, not just an equation!), and give its typical units. [3 pts]
   Solution: Optical depth is dimensionless, i.e. it has no units. It relates to how opaque a material is over some defined length or path. The optical depth in a given situation depends both on the material doing the absorbing (e.g. fog particles in air, space dust, dense plasma in a star's interior) as well as the path length (just a nanometer? a mile? ten thousand km?).

2. **A Hazy Morning [15 pts]** On a given day the weather report mentions that visibility is about one km. Take that as the distance at which $\tau_\lambda \approx 1$:

   (a) Estimate the atmosphere's extinction coefficient α_λ that morning. [3 pts]
   Solution: We know that $\tau = \alpha \Delta x$, where Δx is the distance traveled. So the morning's extinction coefficient is
   $$\alpha_\lambda = \frac{\tau}{\Delta x} = \frac{1}{1000\ \mathrm{m}} = \boxed{10^{-3}\ \mathrm{m}^{-1}}. \tag{1}$$

   (b) If the visibility is mainly limited by smoke from nearby fires, estimate the effective cross-section σ_λ of the smoke particles and the number density n of these particles in the atmosphere. [4 pts]
   Solution: By searching online, I find that typical smoke particles have a size of approximately one micron. Assuming each little particle is opaque to visible light, their cross-section is
   $$\boxed{\sigma_\lambda = \pi r^2 \approx 3 \times 10^{-12}\ \mathrm{m}^2}. \tag{2}$$

(This assumption would probably start to break down once the particles were substantially smaller than the wavelength of light we care about, but it's good enough for now).

Since number density n is related to σ and α by $\alpha_\lambda = n\sigma_\lambda$, we then have

$$n_{\rm smoke} = \frac{\alpha}{\sigma} \approx \frac{10^{-3}\ {\rm m}^{-1}}{3\times 10^{-12}\ {\rm m}^2} \approx \boxed{3\times 10^8\ {\rm m}^{-3}}. \quad (3)$$

Almost a billion smoke particles per cubic meter! That's a lot – no wonder smoking (or coal-mining) affects the lungs so severely.

(c) Given the typical density ρ of air at sea level, estimate the opacity κ_λ of the smoky air (in SI units). [4 pts]

Solution: The course notes remind us that opacity is related to the other quantities discussed above via

$$\alpha_\lambda = n\sigma_\lambda = \rho\kappa_\lambda. \quad (4)$$

A quick internet search reveals that the density of air at sea level is roughly $\rho \approx 1$ kg m^{-3}, so we then have for the opacity

$$\kappa_\lambda = \frac{\alpha_\lambda}{\rho} \approx \frac{10^{-3}\ {\rm m}^{-1}}{1\ {\rm kg\ m}^{-3}} \approx \boxed{10^{-3}\ {\rm m}^2\ {\rm kg}^{-1}}. \quad (5)$$

(d) Now assume that this same smoke layer is confined to a thin layer near the ground with a thickness of just 100 m (due to some unknown atmospheric phenomenon; the reason isn't important here). What is the optical depth at visible wavelengths when looking straight overhead? and when looking up at a 45^o angle (halfway between straight up and parallel to the ground)? [4 pts]

Solution: We need to calculate an optical depth, and we've already used the expression we need here: $\tau = \alpha\Delta x$ (for a homogeneous, uniform medium). We already estimated α above, so we just need a path length Δx. If we're looking straight up and the smoke layer is only 100 m thick, then $\Delta x = 100$ m and so

$$\boxed{\tau_{\rm vertical} = \alpha\Delta x = \left(10^{-3}\ {\rm m}^{-1}\right)(100\ {\rm m}) = 0.1}. \quad (6)$$

If we're looking up at a 45^o angle, then our line of sight cuts diagonally through the smoke layer and so we have a longer total path length, specifically $\Delta x_2 = \sqrt{2}\times 100$ m≈ 141 m. So we have the same expression as above, but a slightly greater optical depth:

$$\boxed{\tau_{\rm vertical} = \left(10^{-3}\ {\rm m}^{-1}\right)(141\ {\rm m}) = 0.14}. \quad (7)$$

As our line-of-sight goes lower and lower, we look through more and more smoke and the optical depth steadily increases; as our gaze drifts closer to the perfectly horizontal (parallel to the ground) then we'll eventually approach $\tau \approx 1$ and not be able to see through the haze.

3. **The random walk of a photon [22 pts; Heng, 2.8.2]**. Photons created at the center of the Sun must travel a distance $R_\odot$ to escape from it, where $R_\odot \approx 700,000$ km is the solar radius. Each photon travels a distance of about the mean free path ($\ell_{\rm mfp}$) before being absorbed or re-emitted (or scattered).

(a) If the Sun were completely transparent, estimate the time it would take a photon to escape. [3 pts]

Solution: If photons could stream freely through the Sun at the speed of light c (i.e., if the Sun's opacity/extinction coefficient/optical depth were all zero), then their travel time would just be

$$t = \frac{R_\odot}{c} \approx \frac{7\times 10^8\ {\rm m}}{3\times 10^8\ {\rm m\ s}^{-1}} \approx \boxed{2.3\ \text{seconds}}. \quad (8)$$

Pretty quick! And this is actually just about the case for the neutrinos produced in the Sun's core by nuclear fusion. Neutrinos interact only weakly with regular matter, so most of them stream directly out of the Sun at almost the speed of light.

(b) Imagine if the Sun were not *too* opaque such that the center-to-surface optical depth (τ) were between 1 and 10. Convince yourself that the number of absorption/scattering events would be about $R_\odot/\ell_{\rm mfp}$. Show that the number of events would be $N \sim \tau$. [5 pts]

Solution: There are several ways to think about this one. An analytic approach is to recall that (i) the mean free path of photons of frequency ν through a material is the inverse of the material's extinction coefficient α_ν, i.e. $\ell_{\rm mfp} = 1\alpha$, and (ii) these quantities are related to optical depth (in a uniform medium) by $\tau_\nu = \alpha_\nu \Delta x$.

If only a few scattering events occur then the total distance traveled $\Delta x \approx R_\odot$, and so we have $\alpha = \tau/R_\odot$ and thus $\ell_{\rm mfp} = R_\odot/\tau$. If the mean free path is equal to or greater than $R_\odot$ then the photons typically stream out of the Sun without any interactions at all; if $\ell_{\rm mfp} < R_\odot$ then the number of legs of the photon's journey, and so the number of interactions it experiences, will be roughly

$$\boxed{N \approx \frac{R_\odot}{\ell_{\rm mfp}} \approx \frac{R_\odot}{R_\odot/\tau} = \tau}. \tag{9}$$

(c) Consider the case of the opaque Sun, [in which case $N \sim \tau^2$; see below]. Taking the number density to be $n \sim 10^{24}$ cm^{-3} (pure hydrogen) and the cross section to be $\sigma \sim 10^{-24}$ cm^2 (Thomson [free electron] scattering), estimate the value of $\ell_{\rm mfp}$. Estimate the time taken for a photon to leak out of the Sun from its center. [7 pts]

Solution: We want to calculate (i) the mean free path and (ii) the time taken to escape the Sun.

For (i), again recall that $\ell_{\rm mfp} = 1/\alpha$, and that $\alpha_\nu = n\sigma_\nu$; both of which quantities we're given. So

$$\alpha \approx 10^{24} \times 10^{-24}\ \mathrm{cm}^{-1} \approx 1\ \mathrm{cm}^{-1} = 100\ \mathrm{m}^{-1} \tag{10}$$

and thus

$$\boxed{\ell_{\rm mfp} = \frac{1}{\alpha} = \frac{1}{100\ \mathrm{c}^{-1}} = 0.01\ \mathrm{m}}. \tag{11}$$

For (ii), photons travel at the speed of light c and their journey consists of N legs each of length $\ell_{\rm mfp}$. So the total distance traveling by this bouncing photon is $d = N\ell_{\rm mfp}$ and the time taken will be

$$t = \frac{d}{c} = \frac{N\ell_{\rm mfp}}{c} = \frac{\tau^2 \ell_{\rm mfp}}{c}. \tag{12}$$

We lack only an expression for τ, but we again recall that $\tau \equiv \alpha\Delta x$, or in this case (from the center to the surface)

$$\tau = \alpha R_\odot = \left(100\ \mathrm{m}^{-1}\right)\left(7 \times 10^8\ \mathrm{m}\right) \approx 7 \times 10^{10}. \tag{13}$$

Now *that* is optically thick! So our estimate for the total travel time for photons to escape the Sun is therefore

$$t = \frac{\tau^2 \ell_{\rm mfp}}{c} = \frac{\left(7 \times 10^{10}\right)^2 (0.01\ \mathrm{m})}{3 \times 10^8\ \mathrm{m\ s}^{-1}} \approx \boxed{1.6 \times 10^{11}\ \mathrm{sec} \approx 5200\ \mathrm{yr}}. \tag{14}$$

In that sense, some of the photons we see every day were "born" before most of the pharoahs had ever ruled in ancient Egypt. (In reality each photon is absorbed and a new photon is re-emitted, but it's more fun to think of the photons as being as older than King Tut).

(d) What is a key weakness of this analysis [described in the above parts]? (Hint: it is *not* that the numbers are imprecise.) [5 pts]

Solution: In my mind, the biggest weakness in these estimates has been the assumption that the Sun is homogeneous throughout its interior – in fact, this is very far from the case. The Sun is extremely dense (and hot) in its core where nuclear fusion occurs, and in its outer layers it is much more rarified (and much cooler, though still hot by terrestrial standards).

We could make a *slightly* more accurate model by splitting the Sun into a denser inner half (with $\alpha' = 2\langle\alpha\rangle$ and $\tau'_{\rm inner} = (2\alpha)(R_\odot/2) = \tau$) and a more rarified outer half ($\alpha' = \langle\alpha\rangle/2$, and thus $\tau'_{\rm outer} = (\alpha/2)(R_\odot/2) = \tau/4$).

Since we saw above that $t \propto \tau^2 \propto (\alpha R_\odot)^2$, the time to escape just this inner half is the same as our previous estimate for the escape time from the *entire* homogeneous Sun (since $\tau'_{\rm inner} = \tau$) – and then it still has to escape from the outer half of the Sun! We could keep subdividing the Sun and get increasingly larger escape times, so our assumption of a uniform and homogeneous Sun is likely an underestimate (though still useful as a lower limit).

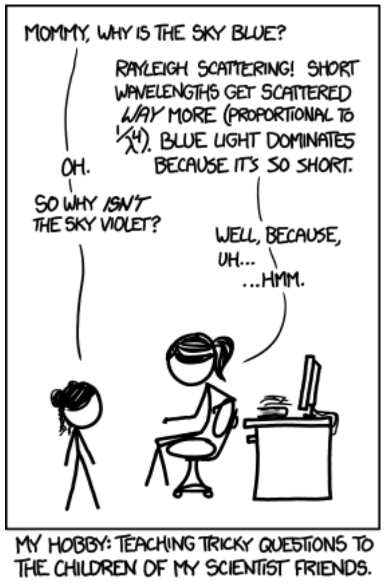

Figure 1: A cruel trick involving Rayleigh scattering, from `https://xkcd.com/1145/`. Bonus points if you answer this child's question.

4. **A Very Simple Scattering Model [10 pts]** As we talked about in class, very large objects (like basketballs) absorb visible light based on their geometric cross-section, $\sigma_\lambda = \rm constant$. But for small objects (or very long wavelengths), when $\lambda \gg r$, we enter the *Rayleigh Scattering* regime where $\sigma_\lambda \propto \lambda^{-4}$.

   Construct and write down an approximate, continuous expression for σ_λ, and then plot it for a range of λ extending from $\lambda < r$ to $\lambda > r$.

   Solution: We need to come up with an expression for the cross-section of these generic particles; the easiest way will just be a two-part piecewise function. At $r > \lambda$ we have $\sigma_\lambda = \pi r^2$, and at $r < \lambda$ we have $\sigma_\lambda = \sigma_0 \left(\lambda\right)^{-4}$. To keep everything continuous, these two expressions must be equal when $r = \lambda$.

$$\sigma_\lambda = \pi r^2 \quad (r \geq \lambda) \tag{15}$$

$$\sigma_\lambda = \pi r^2 \left(\frac{\lambda}{r}\right)^{-4} \quad (r < \lambda) \tag{16}$$

   You can quickly verify that the two expressions are equal when $r = \lambda$ and that it has the required λ^{-4} scaling.

   To plot it, we could do something like the following Python code. See Fig. 2.

```
# Import necessary modules
import pylab as py # for plotting
import numpy as np # for numerical stuff

nPoints = 1000 # <-- always nice to keep this as a free parameter

# Cast horizongal axis in terms of lambda/r:
lambdaOverR = np.linspace(.1, 2, nPoints) # generate X-coordinates

# Cast vertical axis, sigma, in terms of pi r^2:
sigmaGeometric = np.ones(nPoints, dtype=float)
sigmaRayleigh = lambdaOverR**(-4)
```

```
# Determine where the Geometric and Rayleigh regimes dominate:
geometricIndex = (lambdaOverR <= 1)
rayleighIndex = (lambdaOverR > 1)

# Initialize and Populate cross-section array:
sigma = np.zeros(nPoints, dtype=float)
sigma[geometricIndex] = sigmaGeometric[geometricIndex]
sigma[rayleighIndex] = sigmaRayleigh[rayleighIndex]

# Make figure:
py.figure()
py.plot(lambdaOverR, sigma, '--r', linewidth=3)
py.xlabel('Wavelength, in $\lambda / r$', fontsize=16)
py.ylabel('Cross Section, in $\sigma / (\pi r^2)$', fontsize=16)
py.minorticks_on() # <-- always! looks prettier.
py.text(.5, 0.7, 'Geometric\nRegime,\n$\sigma=\pi r^2$',
    horizontalalignment='center', fontsize=14)
py.text(1.6, 0.5, 'Rayleigh\nRegime,\n$\sigma \propto \lambda^{-4}$',
    horizontalalignment='center', fontsize=14)
py.title('APPROXIMATE Rayleigh-Scattering Model', fontsize=18)
plot([1,1], [0, 2], ':k')
py.xlim(0.1, 2)
py.ylim(0, 1.1)
```

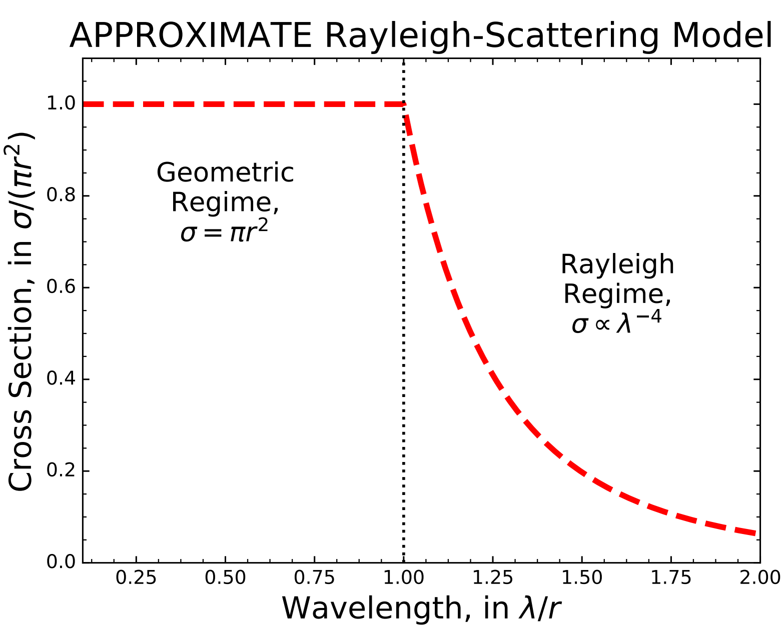

Figure 2: Crude Rayleigh-scattering model.