UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2022

Problem Set 3
Due: Wednesday, Sep 21, 2022, before the start of class (by 1000), by email.
This problem set is worth **54 points**.

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, type the Pset, and submit a printed copy.
Recommended tools for typesetting your problem set are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **Equation of Radiative Transfer [21 pts]**.

   (a) Write the general form of the Equation of Radiative Transfer (not the solution to the EoRT; also, ignore terms of directionality such as μ or θ, which we haven't discussed in lecture yet). Thoroughly explain the meaning of all terms, and the meaning of the overall equation. [5 pts]

   Solution: The equation of radiative transfer is

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu}. \tag{1}$$

   i. The first (differential) term indicates that this is telling us how radiation (as measured by specific intensity) varies with optical depth.

   ii. The second term (S_ν) is the so-called "Source Function" which indicates the contribution (emission) of additional radiation into the beam. In many cases (when conditions are optically thick, and when we are in LTE=local thermodynamic equilibrium), $S_\nu \approx B_\nu(T)$, the Planck blackbody function.

   iii. The final term ($-I_\nu$) is negative and so indicates the removal of radiation from the beam. Since optical depth ($d\tau$) refers to a fractional absorption, this term is just "I_ν" (rather than being modulated by something like opacity or the absorption coefficient).

   (b) Show that if emission is negligible, that the solution to the equation of radiative transfer is

$$I(\tau) = I_0 e^{-\tau}. \tag{2}$$

   Explain what this expression means, in words. [3 pts]

   Solution: If emission is negligible, then S_ν in Eq. 1 is zero and we have a relatively simple differential equation. In this case, we can rearrange to get

$$\frac{dI_\nu}{I_\nu} = -d\tau. \tag{3}$$

   If we integrate this from where $\tau = 0$ (and where $I_\nu = I_{\nu,0}$) to some arbitrary optical depth τ (with intensity $I_\nu(\tau)$), we'll have

$$\int \frac{dI_\nu}{I_\nu} = -\int d\tau \tag{4}$$

$$\ln I_\nu \Big|_{I_0}^{I} = -\tau' \Big|_0^{\tau} \tag{5}$$

$$\ln\left(\frac{I(\tau)}{I_0}\right) = -\tau \tag{6}$$

   And so

$$\boxed{I(\tau) = I_0 e^{-\tau}}. \tag{7}$$

In words, this means that the initially incident intensity I_0 is attenuated by absorption (indicated by nonzero τ). The more optically thick is the path traversed by the radiations, the more of the light will be absorbed and the less total light will get through.

(c) Show that if the source function is constant and absorption is negligible, the solution is instead

$$I(\tau) = I_0 + \tau S. \tag{8}$$

Explain what this expression means, in words. [3 pts]

Solution: If absorption is negligible, then I_ν in Eq. 1 is zero and we have an even simpler differential equation. In this case, we can rearrange to get

$$dI_\nu = S_\nu d\tau. \tag{9}$$

If we integrate this from where $\tau = 0$ (and where $I_\nu = I_{\nu,0}$) to some arbitrary optical depth τ (with intensity $I_\nu(\tau)$), we'll have

$$\int dI_\nu = S_\nu \int d\tau \tag{10}$$

$$I_\nu \Big|_{I_0}^{I} = S_\nu \tau' \Big|_0^\tau \tag{11}$$

$$I_\nu(\tau_\nu) - I_{\nu,0} = S_\nu \tau_\nu \tag{12}$$

And so

$$\boxed{I(\tau) = I_0 + \tau S} \tag{13}$$

In words, this means that the initially incident intensity I_0 is never attenuated by any absorption (which we've set to zero) and that, furthermore, additional radiation is added to the beam via the source function S_ν. And the longer (i.e., the more optically thick) is the path traversed by the radiation, the more distance there is for additional light to be emitted and the more light will finally be measured.

Note that this solution is however nonphysical, since if there is no absorption (i.e. $\alpha = 0$) then actually $d\tau = \alpha ds = 0$ and so $\tau = 0$ always. In practice when $\tau > 1$ we still only get "S_ν" worth of emission added to the beam, though from the expression above we would have expected to get "τS_ν". Hence we need a more physically self-consistent solution; hence the next part of the problem.

(d) Show that if the source function is constant and neither absorption nor emission can be ignored, the general solution is

$$I(\tau) = I_0 e^{-\tau} + S\left(1 - e^{-\tau}\right). \tag{14}$$

Explain what each term in this expression means, in words. [5 pts]

Solution: If S_ν is constant but both emission and absorption are occuring, then things are a bit more complicated. We solve this by first multiplying both sides of the R.T. Equation by e^τ and rearranging:

$$\left(\frac{dI}{d\tau} + I\right) e^\tau = S e^\tau. \tag{15}$$

We can simplify the left-hand side by noting that

$$\frac{d}{d\tau}\left(I e^\tau\right) = \left(\frac{dI}{d\tau} + I\right) e^\tau \tag{16}$$

and so

$$d\left(I e^\tau\right) = S e^\tau d\tau. \tag{17}$$

If we integrate both sides of this from where $\tau = 0$ (and where $I_\nu = I_{\nu,0}$) to some arbitrary optical depth τ (with intensity $I_\nu(\tau)$), we'll have

$$\int d\left(I e^\tau\right) = S \int e^\tau d\tau \tag{18}$$

$$I_0 e^0 - I(\tau) e^\tau = S\left(e^0 - e^\tau\right) \tag{19}$$

$$I_0 - I(\tau) e^\tau = S\left(1 - e^\tau\right). \tag{20}$$

Multiplying each side by $e^{-\tau}$ and rearranging then yields the desired solution,

$$\boxed{I(\tau) = I_0 e^{-\tau} + S\left(1 - e^{-\tau}\right)}. \tag{21}$$

In words, this means that the specific intensity that would be measured at some optical depth τ depends on both the absorption of radiation by the medium, and on the emission of new radiation by the medium. The first term on the right is the exponential decay of the incident intensity I_0, as in the absorption-only case above. The second term on the right involves the source function S and so involves emission: it shows that extra radiation is emitted, but that as the optical depth of the path increases that emitted radiation will start to be re-absorbed by the medium before the radiation ever reaches its "destination" at optical depth τ. So for a perfectly crystal-clear path of $\tau = 0$ there will be no extra emission, but even the most optically thick path imagineable ($\tau \to \infty$) will still only have emission "S" .

(e) In Eq. 14, what is the emergent intensity $I(\tau)$: (i) if $\tau = 0$? (ii) if $\tau \gg 1$? (iii) if $0 < \tau \ll 1$? [5 pts]

Solution: For each of these cases, we just plug the given value of τ in (or take the corresponding limit).

i. If $\tau = 0$, then we simply have $\boxed{I(\tau = 0) = I_0}$. Outgoing radiation equals incident radiation; surface brightness is constant through empty space.

ii. For very large τ (i.e., as $\tau \to \infty$) our exponential terms will collapse down (since $e^{-\infty} \to 0$). We will just be left with $\boxed{I_\lambda = S_\lambda}$. An optically thick medium radiates as its source function, and if its in LTE then the medium is just radiating like a Planck blackbody (since in this case $S_\lambda = B_\lambda(T)$).

iii. In the optically thin (but not wholly clear) case we can Taylor-expand the exponentials, since for small τ we'll have $e^{-\tau} \approx 1 - \tau$. So the result will be: $\boxed{I(\tau) = I_0\left(1 - \tau\right) + \tau S}$.

2. **Light-Absorbing Clouds, Part I [14 pts]** In this and the next problem, you will consider how a few types of clouds overhead affect the amount of starlight that reaches a ground-based astronomical observatory. In all cases, assume that the cloud does not emit (or reflect, or scatter) any light. It only absorbs.

First, consider a homogeneous cloud deck: it extends all the way to the ground ($z = 0$) and up to an altitude of $z = H$; above this the cloud is gone and above it the sky is perfectly clear. The cloud has a constant extinction coefficient of α_λ throughout.

A distant star's light, with intensity $I_{\lambda,0}$ at the top of the atmosphere, is coming down toward your observatory.

(a) How much optical depth has the starlight passed through when it first reaches the top of the cloud, at $z = H$? What fraction of the incident starlight ($I_{\lambda,0}$) reaches altitude $z = H$? [2 pts]

Solution: When the light reaches the top of the atmosphere it hasn't been absorbed by anything at all. So $\boxed{\tau = 0}$ and all of the light reaches $z = H$, i.e. $\boxed{I(z = H) = I_0}$.

(b) How much optical depth has the starlight passed through when it reaches your observatory, at $z = 0$? What fraction of the incident starlight reaches altitude $z = 0$? [4 pts]

Solution: Generally speaking $d\tau = \alpha ds$ for radiation traveling along a little increment of path length ds. Since our radiation is coming down from space, we have $ds = -dz$, α is constant, and $\tau(z = H) = 0$. So we just have

$$\tau(z = 0) = \int \alpha ds = \int_{z=H}^{z=0} -\alpha dz = -\alpha(0 - H) \tag{22}$$

or

$$\boxed{\tau(z = 0) = \alpha H}. \tag{23}$$

And so similarly,

$$\boxed{\frac{I}{I_0} = e^{-\alpha H}}. \tag{24}$$

(c) Write a general expression for $\tau_\lambda(z)$, for any altitude $z \leq H$. (Make sure that you get the expected behavior for small or large τ!). [4 pts]

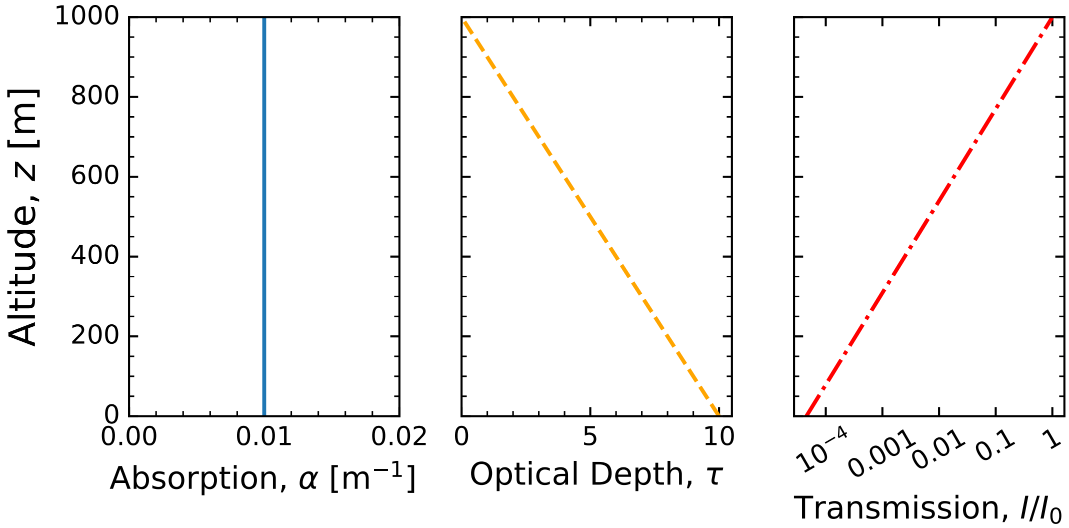

Figure 1: Plots for the Uniform Absorbing Cloud question. From left to right: absorption coefficient α, optical depth τ, and total transmission of the starlight down to the ground (I/I_0) for H=1 km.

Solution: Following the same approach as above, we have

$$\tau(z) = \int \alpha ds = \int_{z'=H}^{z'=0} -\alpha dz' = -\alpha(z - H) = \boxed{\alpha\,(H - z)}. \tag{25}$$

We should check that this has the desired behavior: at the top of the atmosphere ($z = H$) we have $\tau = 0$, and at ground level ($z = 0$) we have $\tau = \alpha H$. Looks right!

(d) Write a general expression for $I_\lambda(z)/I_{\lambda,0}$, the fraction of starlight reaching any altitude $z \leq H$. [4 pts]

Solution: We already know the form of the solution to the radiative transfer equation when the source function is constant – it's given by Eq. 14 as $I(\tau)$ – and that it reduces further to Eq. 2 when emission is negligible (as in this case). We now have an expression for $\tau(z)$, so we can just plug Eq. 25 into Eq. 2. The result will be

$$\boxed{\frac{I(z)}{I_0} = e^{-\tau(z)} = \exp\left[\alpha\,(H - z)\right]} \tag{26}$$

I didn't ask you to plot anything for this problem, but a plot is often a useful way to see how a situation is behaving. Using the same numerical values as in the variable-cloud case below, α, τ, and I/I_0 are plotted in Fig. 1.

3. **Ozone [19 pts]**. The Hartley band of the ozone molecule (O_3) is a broad absorption band in the UV, and its cross-section[1] has an approximately Gaussian functional form of

$$\sigma_\lambda = \left(10^{-17}\ \mathrm{cm}^2\right) \exp\left[-\frac{(\lambda - 255\ \mathrm{nm})^2}{2\,(17\ \mathrm{nm})^2}\right]. \tag{27}$$

(a) Plot the ozone cross-section across the full width of this absorption band. (For this and the other plots you will make below, use a wavelength range no narrower than 200–310 nm). [3 pts]

Solution:

As usual, I'll do this with Python (though you could use a spreadsheet program almost as easily). All plots for this problem are shown in Fig. 2.

[1] See e.g., `http://vpl.astro.washington.edu/spectra/o3uvimages.htm`.

```python
# Import necessary modules
import numpy as np
from pylab import *

# Define wavelength grid, then calculate sigma:
lam = np.linspace(.2, .32, 100)   # microns; I think in the infrared!
sigma = 1e-17 * exp(-(lam-.255)**2/(2*.017**2)) # cm^2

# Plot the figure:
plt.figure(1, figsize=[10, 10])
ax1=subplot(2,2,1)
plt.plot(lam, sigma / 1e-17, '--c', linewidth=3)
plt.xlabel('Wavelength [microns]', fontsize=16)
plt.minorticks_on()
plt.ylabel('$\sigma_\lambda$ -- Cross Section [$10^{-17}$ m$^2$]',
```

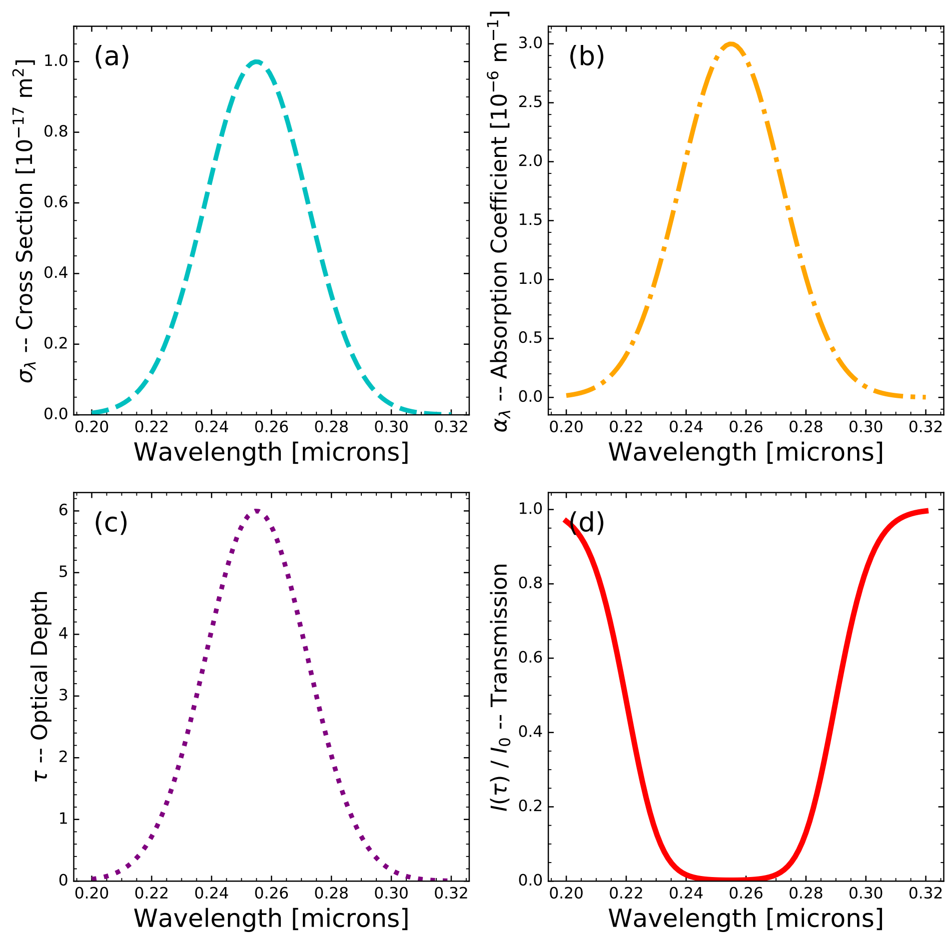

Figure 2: Plots for the Ozone question: (a) σ; (b) α; (c) τ; and (d) total transmission. [Note that the units in (a) and (b) should be cm^2 and cm^{-1}.]

```
    fontsize=14)
plt.ylim(0,1.1)
plt.text(.05, .9, '(a)', fontsize=17, transform=ax1.transAxes)
```

(b) Ozone is mainly found in the Earth's stratosphere, where a (very!) rough average for the number density of ozone particles is $n \sim 3 \times 10^{11}$ cm^{-3}. Calculate and plot the absorption coefficient α_λ for ozone across the full width of the Hartley band. [4 pts]

Solution: Assuming everything is just constant, we can calculate

$$\alpha_\lambda = n\sigma_\lambda \tag{28}$$

with $n = 3 \times 10^{11}$ cm^{-3}, as given in the problem statement. So continuing the Python code above, we have:

```
n = 3e11       # per cm^3
alpha = n * sigma # units are now 1/cm
ax2=subplot(2,2,2)
plt.plot(lam, alpha / 1e-6, '-.', color='orange', linewidth=3)
plt.xlabel('Wavelength [microns]', fontsize=16)
plt.minorticks_on()
plt.ylabel('$\\alpha_\lambda$ -- Absorption Coefficient [$10^{-6}$
    m$^{-1}$]', fontsize=14)
plt.text(.05, .9, '(b)', fontsize=17, transform=ax2.transAxes)
```

(c) Earth's ozone is mainly concentrated at altitudes from 15–35 km above the Earth's surface. Assume that n is constant with altitude in that range, and then calculate and plot the optical depth τ_λ of the Hartley band in Earth's ozone layer. [4 pts]

Solution: In our simple model, we have an ozone layer that is 20 km thick. Since generally $d\tau_\lambda = \alpha_\lambda dz$, if we still assume everything is just constant we can calculate

$$\tau_\lambda = \Delta z \alpha_\lambda \tag{29}$$

where $\Delta z = 20$ km $= 2 \times 10^6$ cm. Still continuing from the Python code above, we now have:

```
deltaZ = 20 * 1000 * 100 # convert from km to cm
tau = deltaZ * alpha # unitless

ax3=subplot(2,2,3)
plt.plot(lam, tau, ':', color='purple', linewidth=3)
plt.xlabel('Wavelength [microns]', fontsize=16)
plt.minorticks_on()
plt.ylabel('$\\tau$ -- Optical Depth', fontsize=14)
plt.text(.05, .9, '(c)', fontsize=17, transform=ax3.transAxes)
plt.ylim(0, ax3.get_ylim()[1])
```

(d) Considering only this absorption band (and no other atmospheric absorption features), (i) calculate and plot what fraction of incident Sunlight reaches the surface of the Earth across this ozone absorption band, i.e. as a function of wavelength. What fraction of light reaches the Earth's surface (ii) in the central 'core' of the Hartley band, around 255 nm, and (iii) far in the 'wings' of the band, say at 320 nm? [4 pts]

Solution: (i): We're only interested in the fraction of incident light that reaches the surface through the ozone layer: so, all we need is to calculate the absorption-only solution to the equation of radiative transfer, namely

$$\frac{I_\lambda(\tau)}{I_{\lambda,0}} = e^{-\tau}. \tag{30}$$

```
transmission = np.exp(-tau)

ax4=subplot(2,2,4)
plt.plot(lam, transmission, '-r', linewidth=3.5)
plt.xlabel('Wavelength [microns]', fontsize=16)
plt.minorticks_on()
plt.ylabel('$I(\\tau)\ /\ I_0$ -- Transmission', fontsize=14)
plt.text(.05, .9, '(d)', fontsize=17, transform=ax4.transAxes)
plt.ylim(0, ax4.get_ylim()[1])
```

(ii)

The optical depth in the core of this absorption band is $\tau \approx 6$, so the UV light at this wavelength is roughly e^{-6}. A handy trick (impress your friends!) is that $e^3 \approx 20$, so only $\boxed{e^{-6} \approx 1/400 \approx 0.25\%}$ of the UV light gets through. Basically none!

(iii)

We can see from the plot that this particular ozone band hardly absorbs at all at 320 nm. I calculate $\tau \approx 0.004$, so hardly any of this light is blocked: approximately $\boxed{e^{-0.004} \approx 1 - 0.004 \approx 99.6\%}$ gets through.

(e) Fig. 3 shows the transmission of light through Earth's atmosphere across a much wider range of wavelengths. Explain how one might extend the analysis you just completed (i.e., one absorption band of one molecule over a narrow wavelength range) to generate a more complete transmission model such as this. [4 pts]

Solution: One would need detailed models of all of the components we used in our simple analysis. We would need to know (i) all important molecules, atoms, and other absorbing materials in the atmosphere; (ii) the cross-sections per particle for all of these absorbers; (iii) and a detailed model of how the number density $n(z)$ of all these particles changes with altitude throughout the Earth's atmosphere. Once we had this, we could calculate $\alpha_\lambda(z)$ for each type of particle, add them up for a total $\alpha_{\lambda,tot}(z)$, and integrate $d\tau = \alpha_\lambda dz$ to calculate the total optical depth at each wavelength.

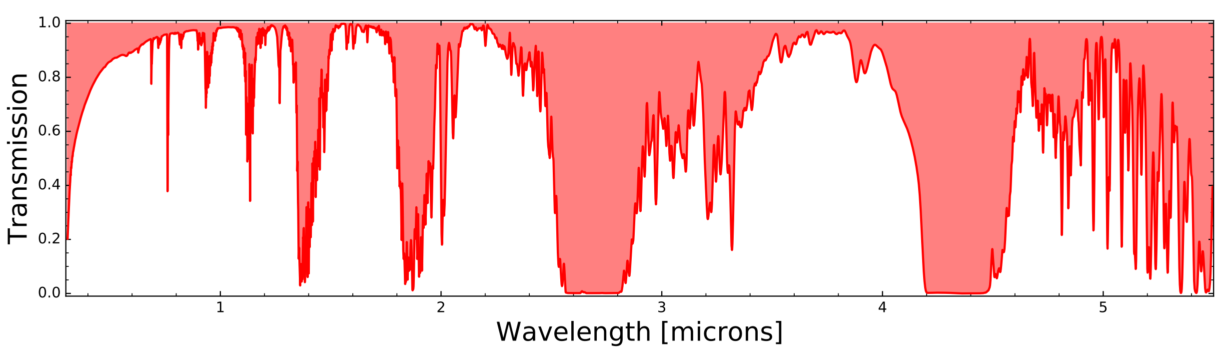

Figure 3: Transmission of the Earth's atmosphere vs. wavelength over optical and near-infrared wavelengths. Shaded wavelengths are more opaque, while white wavelengths are more transparent.