UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2022

Problem Set 6
Due: Friday, October 21, 2022, at the start of class (1000 Kansas Time).
This problem set is worth **41 points**.

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, type the Pset, and submit a printed copy.
Recommended tools for typesetting your problem set are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **Stellar Structure – of the Earth! I. [17 pts]**. Planets like the Earth are governed by some of the same structure equations that we introduced for stellar interiors. Limit your answers below to two significant figures; the crude model we'll use below often isn't even that accurate.

   (a) Let's make the simplest possible model – assume that the Earth has a constant density, ρ_0. Given the known mass and radius of the Earth, calculate ρ_0. [2 pts]

   Solution: From Wikipedia, $M_\oplus \approx 6.0 \times 10^{24}$ kg/m^3 and $R_\oplus \approx 6.4 \times 10^6$ m. If we just want a constant density, we should just take $\rho = M/V$ and $V = \frac{4}{3}\pi R_\oplus^3$. So we get

$$V_\oplus = \frac{4}{3}\pi R_\oplus^3 \approx 1.1 \times 10^{21}\ \mathrm{m}^3. \tag{1}$$

   and so

$$\boxed{\rho_0 = \frac{M_\oplus}{V_\oplus} \approx 5500\ \mathrm{kg\ m^{-3}}}. \tag{2}$$

   (b) Assuming a constant density, calculate and plot the Earth's enclosed mass, $M_{\mathrm{enc}}(r)$, from $0 < r < R_\oplus$. What do you calculate for $M_{\mathrm{enc}}(r = R_\oplus)$, i.e. for the total mass of the Earth ($M_\oplus$)? Explain how well your answer compares to the true, measured mass of the Earth. [5 pts]

   Solution: We defined

$$M_{\mathrm{enc}}(r) = \int_0^r 4\pi r^2 \rho(r) dr. \tag{3}$$

   Since we have $\rho(r) = \rho_0$ = constant, we will just have

$$\boxed{M_{\mathrm{enc}}(r) = \frac{4}{3}\pi\rho_0 r^3}. \tag{4}$$

   If I plug in the approximate values assumed and calculated above (all rounded to just two significant figures), then I get

$$\boxed{M_{\mathrm{enc}}(r = R_\oplus) \approx 6.0 \times 10^{24}\ \mathrm{kg}}. \tag{5}$$

   This is within 10% of the measured value of $M_\oplus$, which is probably reasonable since I've been rounding all these values off.

   Note however that the mass calculated above is **not** discrepant with the Earth's measured mass because of our assumption of a constant density! If we had carried more significant figures through, we should recover the Earth's mass just as precisely as whatever initial values we input.

   The plot is shown in Fig. 1.

(c) Assuming a constant density, calculate and plot the Earth's internal gravity field, $g(r)$, for $0 < r < R_\oplus$. Explain how well your answer for $g(r = R_\oplus)$ compares to the true, measured surface gravity at the surface of the Earth. [5 pts]

Solution: We know that the local gravity field is

$$g(r) = \frac{GM_{\rm enc}(r)}{r^2} \tag{6}$$

Since Eq. 4 already gives us the enclosed mass, we can just plug this in:

$$g(r) = \frac{G}{r^2}\frac{4}{3}\pi\rho_0 r^3 = \boxed{\frac{4}{3}G\pi\rho_0 r}. \tag{7}$$

So the internal gravity field increases linearly with increasing radius. Even though as get get further from the Earth's center the gravity will decrease as $1/r^2$, the enclosed mass is increasing more quickly (as r^3) and so $g(r)$ gets stronger with increasing radius.

We therefore calculate

$$g(r = R_\oplus) = \frac{4}{3}G\pi\rho_0 R_\oplus = \boxed{9.8 \text{ m s}^{-2}}. \tag{8}$$

Happily, this is still quite consistent with the Earth's known surface gravity. A nice coincidence, considering how much rounding we've been doing!

The plot is shown in Fig. 1.

(d) Assuming a constant density, calculate and plot the Earth's internal pressure, $P(r)$, for $0 < r < R_\oplus$. Calculate the pressure at the very center of the Earth, $P_c = P(r = 0)$. [5 pts]

Solution: We know that the equation of hydrostatic equilibrium is $dP/dr = -\rho(r)g(r)$, and so to calculate $P(r)$ we need to take

$$\begin{aligned}
P(r) = &\int_{P=0}^{P(r)} dP = \int_{r=R_\oplus}^{r} -\rho(r)g(r)dr && (9)\\
= &-\rho_0 \int_{r=R_\oplus}^{r} \tfrac{4}{3}G\pi\rho_0 r dr && (10)\\
= &-\tfrac{4}{3}G\pi\rho_0^2 \int_{r=R_\oplus}^{r} r dr && (11)\\
= &\boxed{\frac{2}{3}G\pi\rho_0^2\left(R_\oplus^2 - r^2\right)}. && (12)
\end{aligned}$$

And so our simple model estimates the pressure at the center of the Earth to be roughly $\boxed{1.7 \times 10^{11} \text{ Pa}}$ or over 10^6 times atmospheric pressure.

The plot is shown in Fig. 1.

2. **Stellar Structure – of the Earth! II. [24 pts]**. For this problem, download the "Preliminary Reference Earth Model" file from `https://crossfield.ku.edu/files/preliminary_reference_earth_model.csv`. For all parts of this problem that involve calculations, be sure to explain your methods and/or show your work. Note that you will need to do some simple numerical integration for this problem: this is best done using your favorite programming language, but spreadsheet programs such as LibreOffice Calc can be made to do this work in a pinch. (Limit numerical answers to three significant figures; I scraped the data file from a figure in a paper, so the data values aren't likely to be more precise than that.)

(a) Plot the Earth's density profile $\rho(r)$ (from the data file). Explain why you think the density profile has the general shape that it does (don't worry about explaining every little wiggle, but focus on the larger overall features). [4 pts]

Solution: The plot is shown in Fig. 1.

The density profile looks so disjointed because of the many, differentiated layers of the Earth. We say that the various materials in the Earth's interior have different "equations of state," e.g. the mantle responds differently to compression than does, say, the core or the crust.

(b) (i) Calculate and plot the Earth's enclosed mass, $M_{\rm enc}(r)$, from $0 < r < R_\oplus$. (ii) Describe and try to explain the general, overall features shown in your plot. (iii) What do you calculate for $M_{\rm enc}(r = R_\oplus)$, i.e. for the total mass of the Earth ($M_\oplus$)? Explain how well your answer compares to the true, measured mass of the Earth. [6 pts]

Solution: We calculate $M_{\rm enc}(r)$ by solving the same equation as in Eq. 3, but now we have to do it via direct numerical integration (rather than the analytic approach we used last time).

The plot is shown in Fig. 1.

I calculate $\boxed{M_\oplus = 6.05 \times 10^{24} \text{ kg}}$ which is surprisingly close to the simple, rounded estimate above in Eq. 5, and so it doesn't seem to provide a more accurate mass than that estimate did. The difference is that there we calculated the mass from a density that we got from the mass (a bit circular!) whereas here we calculated the mass solely from the provided density profile.

(c) Calculate and plot the Earth's internal gravity field, $g(r)$, for $0 < r < R_\oplus$. The plot you get may surprise you – it did me! Describe and try to explain the general, overall features shown in your plot. Explain how well your answer for $g(r = R_\oplus)$ compares to the true, measured surface gravity at the surface of the Earth. [6 pts]

Solution: We calculate $g(r)$ by solving the same equation as in Eq. 6, but now we have to do it via direct numerical integration (rather than the analytic approach we used last time).

The plot is shown in Fig. 1.

The plot of $g(r)$ surprised me so much because the gravity profile rises from the core as one goes outward, reaches a maximum at a point well inside the Earth, and then the gravity actually decreases again out toward the surface. So if one were to journey well inside the Earth, one would eventually start to experience a stronger gravitational acceleration!

This occurs because the density profile of the Earth is so unusual. And in particular, while going outward from the center, the gravity intensity "wants" to keep decreasing as $1/r^2$, and in the outermost layers the enclosed mass begins to increase more slowly than r^2 – since the gravity is given by the ratio of these, it starts decreasing with increasing r.

Out at the surface, I calculate $\boxed{g(r = R_\oplus) = 9.79 \text{ m s}^{-2}}$, which again is pleasantly close to the value that I'm familiar with.

(d) Calculate and plot the Earth's internal pressure, $P(r)$, for $0 < r < R_\oplus$. Give the central pressure, P_c, for this model and describe how it compares to the central pressure you calculated in the previous problem. [6 pts]

Solution: We calculate $P(r)$ by solving the same equation as in Eq. 9, but now we have to do it via direct numerical integration (rather than the analytic approach we used last time).

The plot is shown in Fig. 1.

In the Earth's center, I calculate a central pressure of

$$\boxed{P_c = 3.55 \times 10^{11} \text{ Pa} \approx 3.55 \times 10^{6} \text{ atmospheres}}. \tag{13}$$

This is about twice as high as the value calculated previously using the much simpler model. So even one of the simplest models we could construct got us within a factor of $\sim$2 of the correct interior values! By astronomical standards, that's pretty good.

Solution: Below, I give the full set of Python code that I used to calculate and plot the Earth's internal properties as a function of radius. Note, again, that you could use a spreadsheet to do all this as well – but it would probably be slower & less elegant. The resulting plots are all shown in Fig. 1.

```
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd

tab = pd.read_csv('preliminary_reference_earth_model.csv')
```

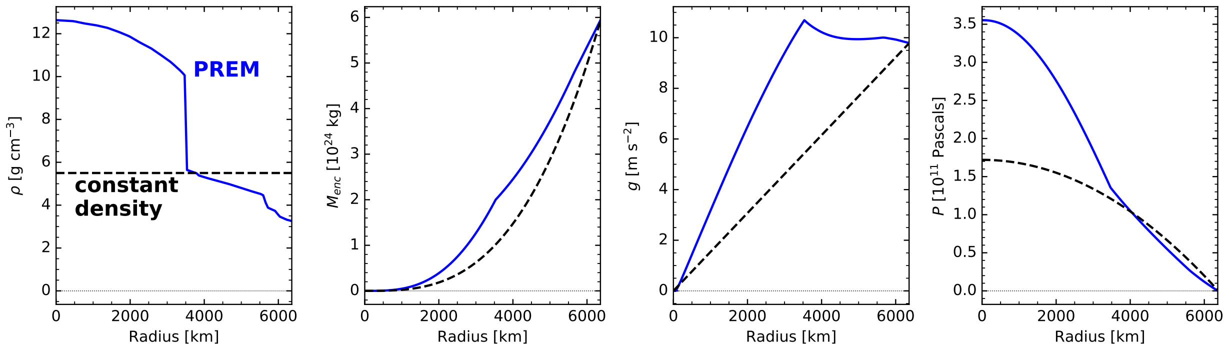

Figure 1: Interior structure of the Earth, assuming a constant density (black dashed lines) and using data from PREM (Preliminary Reference Earth Model, blue solid lines). From left: density $\rho(r)$, enclosed mass $M_{\rm enc}(r)$, gravity strength $g(r)$, and internal pressure $P(r)$. Somewhat surprisingly, even the very crude constant-density model is never off by more than a factor of $\sim$2 — not too bad!

```
r = tab.radius_meters.values # radius in meters
rho = tab.density_kg_per_m3.values # density in kg/m3
dr = np.diff(r).mean() # width of each row of the data table, in meters
G = 6.673e-11 # Newton's Grav. constant

def m_enc(rarray):
   """Give enclosed mass for all r values in radius array "rarray" """
   # Initialize:
   enclosed_mass = np.zeros(np.array(rarray).size, dtype=float)
   integrand = (4 * np.pi * r**2 * rho)
   for ii in range(rarray.size): # Loop over each radius value:
      this_radius = np.array(rarray).ravel()[ii]
      enclosed_mass[ii] = (integrand * dr)[r < this_radius].sum()
   enclosed_mass = enclosed_mass.reshape(rarray.shape)
   return enclosed_mass

def g_internal(rarray):
   """Give internal gravity for all r values in radius array "rarray" """
   enclosed_mass = m_enc(rarray) # use the function we defined
   gravity = G*enclosed_mass/rarray**2
   gravity[rarray==0] = 0. # fix divide-by-zero errors
   return gravity

def p_internal(rarray):
   """Give internal pressure for all r values in radius array "rarray" """
   # Initialize:
   pressure = np.zeros(np.array(rarray).size, dtype=float)
   gravity = g_internal(rarray)
   integrand = (rho * gravity)
   for ii in range(rarray.size): # Loop over each radius value:
      this_radius = np.array(rarray).ravel()[ii]
      pressure[ii] = (integrand * dr)[r > this_radius].sum()
   pressure = pressure.reshape(rarray.shape)
   pressure[rarray > r.max()] = 0. # pressure is zero outside the star
   return pressure
```

```
# Now get ready to plot:
r_km = r / 1000.
r_limits = [r_km.min(), r_km.max()]
rho_0 = 5500. # kg/m3, for constant-density model
rearth = 6378000. # Earth's radius, in meters

# Now: plot!
plt.figure()
ax1=plt.subplot(1,4,1)
ax1.text(3700, 10, 'PREM', color='blue', fontsize=14, weight='bold')
ax1.text(500, 4.0, 'constant\ndensity', color='black', fontsize=14,weight='bold')
ax1.plot(r_km, rho / 1000, '-b')
ax1.plot(r_limits, [5.500,5.500], '--k')
ax1.set_ylabel('$\\rho$ [g cm$^{-3}$]')
ax2=plt.subplot(1,4,2)
ax2.plot(r_km, m_enc(r)/1e24, '-b')
ax2.plot(r_km, (4./3.)*np.pi*r**3*rho_0/1e24, '--k')
ax2.set_ylabel('$M_{enc}$ [$10^{24}$ kg]')
ax3=plt.subplot(1,4,3)
ax3.plot(r_km, g_internal(r), '-b')
ax3.plot(r_km, (4./3.)*np.pi*r*rho_0*G, '--k')
ax3.set_ylabel('$g$ [m s$^{-2}$]')
ax4=plt.subplot(1,4,4)
ax4.plot(r_km, p_internal(r)/1e11, '-b')
ax4.plot(r_km, (2./3.)*np.pi*rho_0**2*G * (rearth**2 - r**2) / 1e11, '--k')
ax4.set_ylabel('$P$ [$10^{11}$ Pascals]')
[ax.set_xlabel('Radius [km]') for ax in [ax1, ax2, ax3, ax4]]
[ax.plot(r_limits, [0,0], ':k', linewidth=0.5) for ax in [ax1, ax2, ax3, ax4]]
[ax.set_xlim(r_limits) for ax in [ax1, ax2, ax3, ax4]]

plt.tight_layout()
```