Chapter Three

Temperature, Albedos, and Flux Ratios

3.1 INTRODUCTION

When observing a distant exoplanet, the only quantity we can measure is the radiation coming from the planet, in a form we have called flux. The temperature of exoplanet atmospheres, in contrast, is the quantity that is actually relevant for physics and chemistry of the exoplanet atmosphere. While only a single flux may be measured, real planets rarely have a single temperature throughout. Which temperature should represent the planet? Which temperature should represent the flux? In this chapter we explore the relationships among the planet flux, temperature, and albedo. We also derive planet-to-star flux ratios which are key for assessing what kind and size of telescope and instrumentation are required to detect an exoplanet—and whether or not an exoplanet is detectable by any means.

3.2 ENERGY BALANCE

The planet atmosphere temperature and albedo are related by the fundamental principle of conservation of energy. For planetary atmospheres the conservation of energy is described as "energy balance." The underlying concept is that no energy is created or destroyed in a planetary atmosphere. All of the energy in the planet atmosphere comes either from the parent star—in the form of absorbed incident radiation—or from the planetary interior.

We can describe the energy balance by

$$E_{\rm out}(t) = (1 - A_{\rm B})E_{\rm inc}(t) + E_{\rm int}(t). \tag{3.1}$$

Here $E_{\rm out}(t)$ is the energy per unit time leaving the planet, while $E_{\rm inc}(t)$ is the stellar energy per unit time incident on the planet and $E_{\rm int}(t)$ is the energy per unit time transferred to the atmosphere from the planetary interior. The factor $(1 - A_{\rm B})$ is the fraction of incident stellar energy absorbed to heat the atmosphere or surface (and subsequently reemitted as radiation at longer wavelengths). The fraction of incident stellar energy scattered back into space is the Bond albedo $A_{\rm B}$.

The planet exists in an equilibrium with the incident stellar radiation where the heating with time is constant. (Planets in eccentric orbits will not have constant heating with time.) For the rest of this chapter we adopt this energy balance by dropping the time dependence of the planet flux and related quantities. We note that energy per unit time is technically called power.

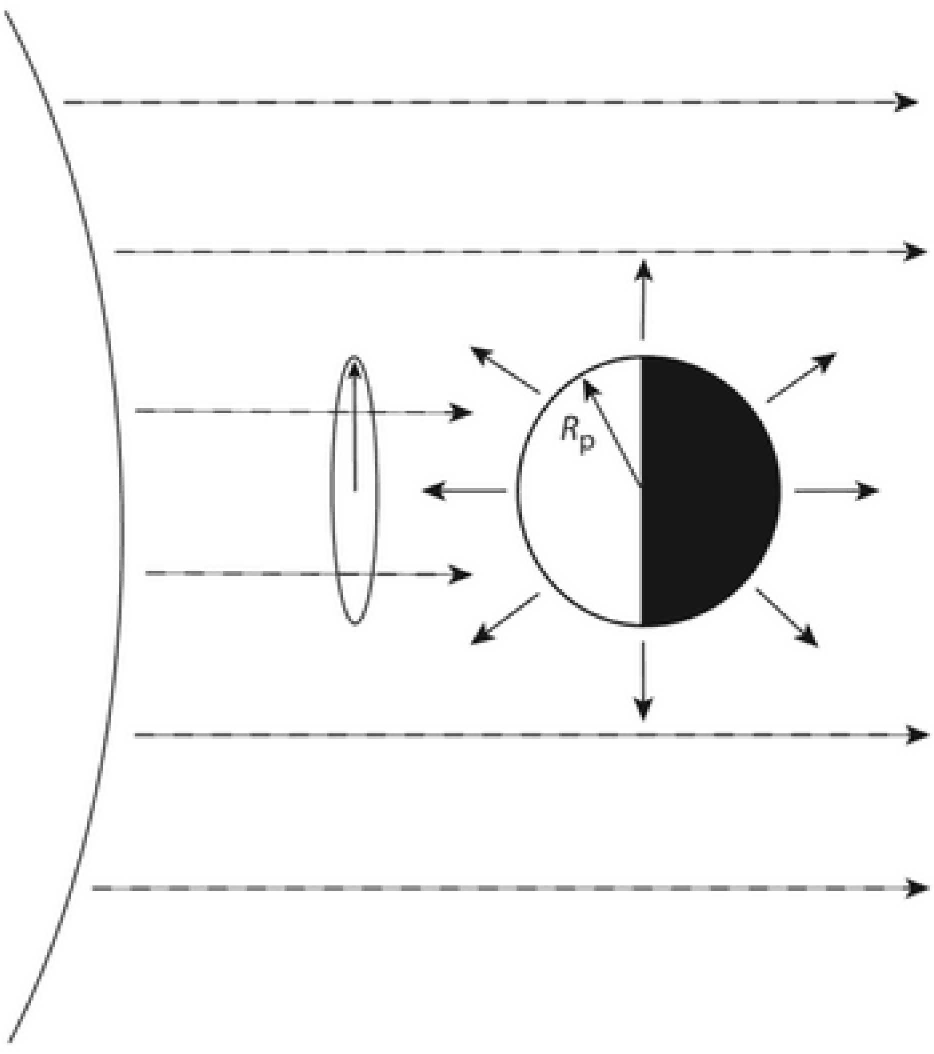


Figure 3.1 Schematic illustration for planetary energy balance. Stellar radiation falls onto the planet in an amount related to the planet's cross-sectional area of $\pi R_{\rm p}^2$. A fraction $(1-A_{\rm B})$ of this radiation is absorbed by the planet. In this illustration the absorbed stellar radiation is advected around the planet and uniformly reemitted into 4π.

We can describe the energy balance in equation [3.1] using the planetary luminosity $L_{\rm p} = E_{\rm out}$ for outgoing energy and describing the incident stellar energy in terms of stellar flux and planet semimajor axis (see equations [2.23] and [2.29]),

$$4\pi R_{\rm p}^2 F_{\rm S,p} = (1 - A_{\rm B}) F_{\rm S,*} \left(\frac{R_*}{a}\right)^2 \pi R_{\rm p}^2 + L_{\rm p,int}. \qquad (3.2)$$

On the right-hand side of equation [3.2] we see the two terms that describe the energy sources for the planetary atmosphere: the absorbed (and reradiated) stellar energy and the planet's own interior energy or luminosity $L_{\rm p,int}$. The first term on the right-hand side is the energy absorbed by the planet (see Figure 3.1). It consists of the incident energy on the planet from the star, derived in equation [2.29], multiplied by the fraction of energy absorbed. This factor is $(1 - A_{\rm B})$, where $A_{\rm B}$ is the Bond albedo. We assume that this absorbed energy is eventually reradiated to space as long-wavelength radiation. Regarding the planet's interior luminosity $L_{\rm p,int}$ (i.e., the interior energy), we recall the definition of luminosity: the flux passing through a surface encompassing the planet. The planet's internal luminosity has several different sources.

For giant planets such as Jupiter which are composed predominantly of hydrogen and helium, $L_{\rm p,int}$ is the gradual loss of residual gravitational potential energy from the planet's formation. Indeed, Jupiter has an internal luminosity over twice as high as its luminosity from reradiated absorbed stellar energy. This is an indication of how long it takes for energy to travel from a giant planet interior out to space.

Earth's $L_{\rm p,int}$ arises partly from residual gravitational potential energy but mostly from decay of radioactive isotopes (of uranium, thorium, and potassium). For many planets, including the hot Jupiter exoplanets in very short-period orbits (less than four days) and terrestrial planets with typically limited amounts of interior energy, the luminosity from external heating overwhelms the interior luminosity by many orders of magntiude. We therefore write the energy balance equation

$$\boxed{4\pi R_{\rm p}^2 F_{\rm S,p} = (1 - A_{\rm B}) F_{\rm S,*} \left(\frac{R_*}{a}\right)^2 \pi R_{\rm p}^2.} \tag{3.3}$$

3.3 PLANETARY TEMPERATURES

There is no single temperature to describe the planet atmosphere. The temperature of a planet varies with altitude, with horizontal location around the planet, and possibly in time from day to night or from season to season. Which temperature should we use to describe the planet? Temperature is an important planet parameter since it governs to first order the chemical equilibrium state of the planet and hence the emergent spectrum. Here we describe three commonly used temperatures: the effective temperature, the equilibrium temperature, and the brightness temperature, and how they relate to the planet flux.

3.3.1 The Effective Temperature $T_{\rm eff}$

The effective temperature $T_{\rm eff}$ is used as a proxy for the global temperature of a planet atmosphere. $T_{\rm eff}$ is defined as the temperature of a black body of the same shape and at the same distance as the planet and with the same total flux as the planet. (Recall that here total flux refers to the flux integrated over all wavelengths or frequencies.) $T_{\rm eff}$ is in principle a measured quantity, taking the total flux of a planet, converting it to surface flux (i.e., flux radiated at the planet), and finding the temperature of a black body with the same total flux. Based on this definition, we can derive an equation for $T_{\rm eff}$ by: using the definition of black body flux (equation [2.37]); integrating the black body flux over all frequencies; and assuming the surface flux to be uniform across the object's surface

$$F_{\rm S} \equiv \int_0^\infty F_{\rm S}(\nu) d\nu = \pi \int_0^\infty B(T, \nu) d\nu \equiv \sigma_{\rm R} T_{\rm eff}^4. \tag{3.4}$$

Here $\sigma_{\rm R}$ is known as the Stefan-Boltzmann constant,

$$\sigma_{\rm R} = \frac{2\pi^5}{15} \frac{h}{c^2} \left(\frac{k}{h}\right)^4, \tag{3.5}$$

which has a value 5.670×10^{-8} J K^{-4} m^{-2} s^{-1}. $F_{\rm S}$ is the planetary surface flux (i.e., radiated at the planet surface) in units of J m^{-2} s^{-1}, as derived in Chapter 2. This relation is the Stefan-Boltzmann Law:

$$\boxed{F_{\rm S} = \sigma_{\rm R} T_{\rm eff}^4.} \tag{3.6}$$

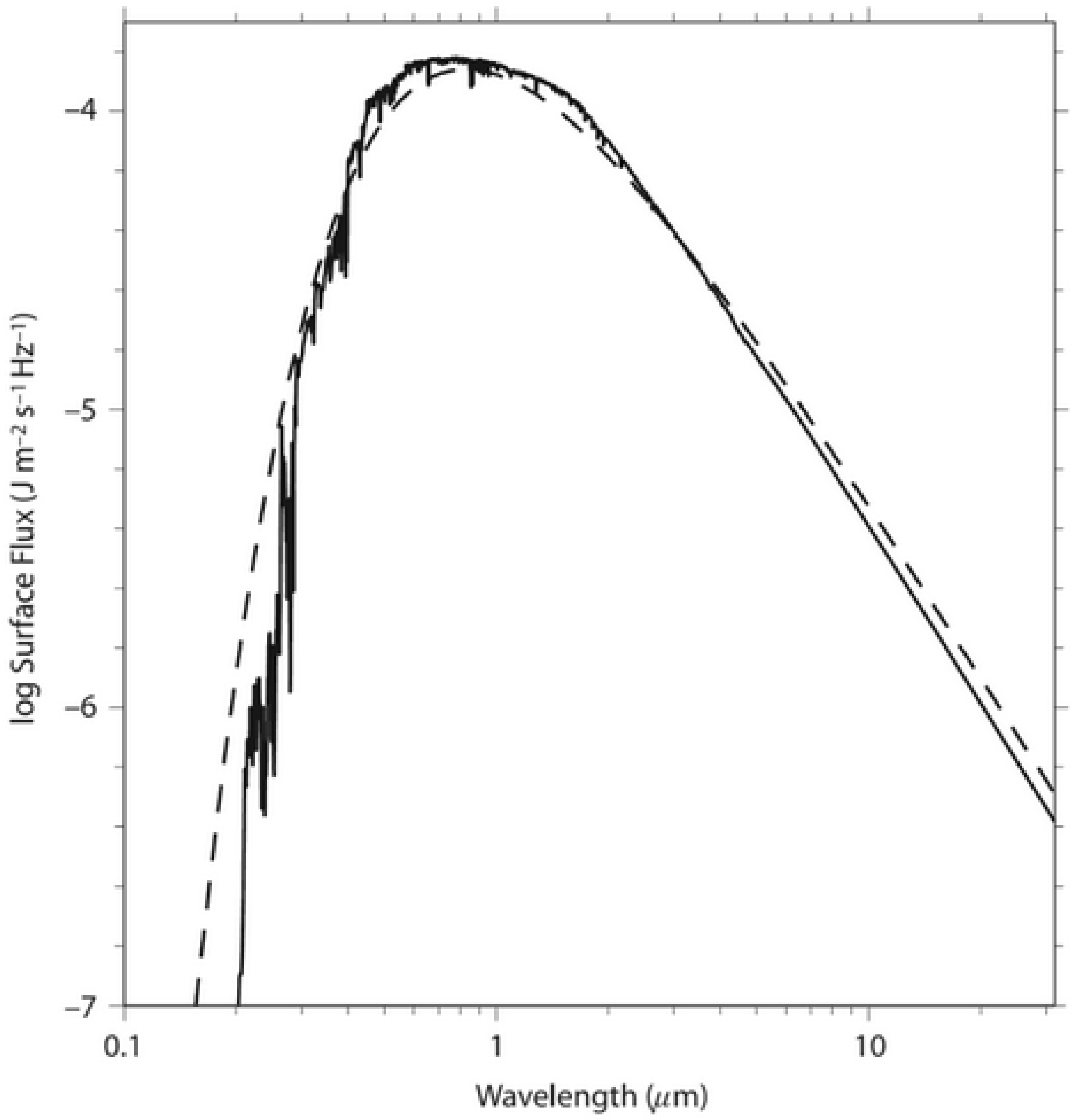
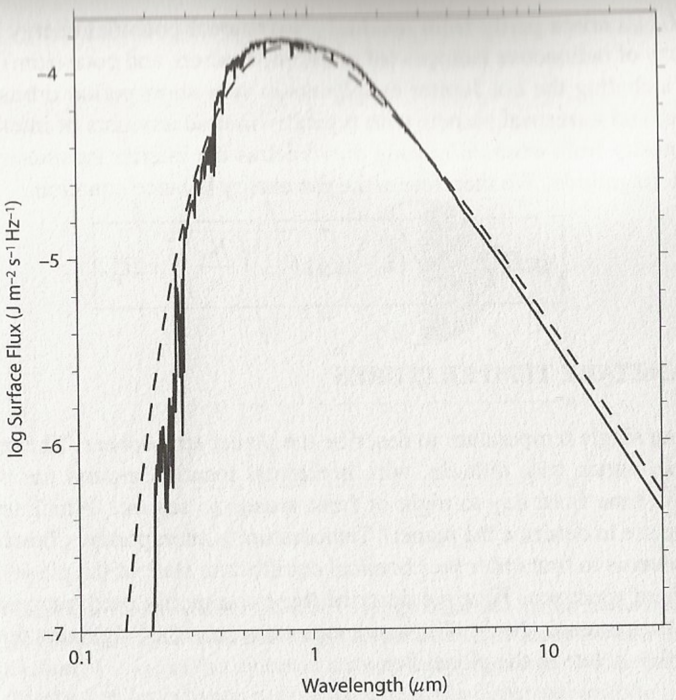


Figure 3.2 Illustration of the effective temperature definition. The theoretical surface flux of a $T_{\rm eff} = 6130$ K G0IV star is shown [1]. A black body with the same 6130 K effective temperature is shown by the dashed curve. The total stellar flux (integrated over all wavelengths) is equivalent to the total black body flux.

Figures 3.2 and 3.3 show the $T_{\rm eff}$ definition applied to two different stars. The first HD 149026 has a stellar flux similar to a black body, but the second, GJ 436, has a flux that departs significantly from a black body flux.

The effective temperature $T_{\rm eff}$ describes the global planet temperature at the altitude where the bulk of the radiation leaves the planet or star. This altitude is sometimes called the planet photosphere or surface, whether or not the altitude is at the planet surface. $T_{\rm eff}$ may be quite different from the surface temperature, as is the case for Venus. Venus's $T_{\rm eff}$ is $\sim$230 K, a value that comes from the cloud tops which obscure the planet's solid surface. Due to a strong greenhouse effect, Venus's surface temperature, at 730 K, is almost 500 K hotter than its $T_{\rm eff}$.

$T_{\rm eff}$ varies widely for planets. The coldest planets in our own solar system, Uranus and Neptune, have $T_{\rm eff} = 59$ K. Jupiter has $T_{\rm eff} = 124$ K, while Earth has $T_{\rm eff} \sim 255$ K. If we could measure $T_{\rm eff}$ for exoplanets, we would expect the hottest hot Jupiters to have $T_{\rm eff} > 2000$ K.



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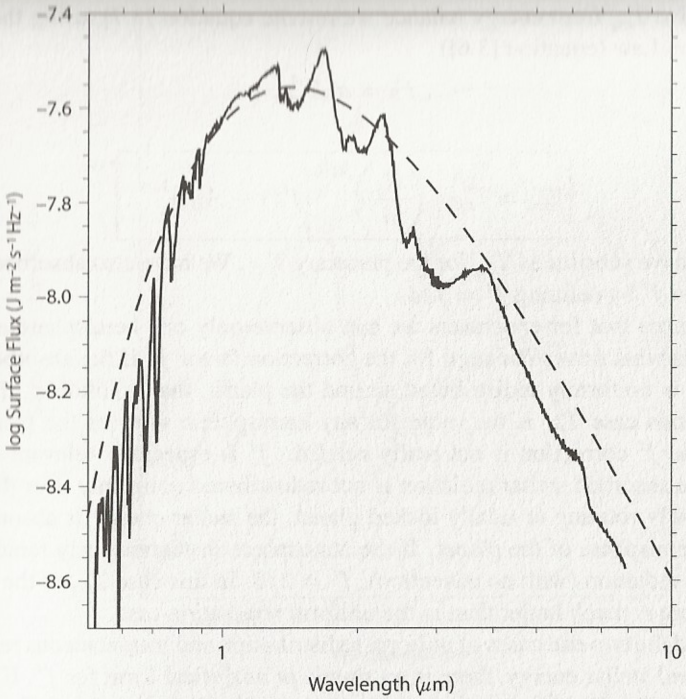

Figure 3.3 Illustration of the effective temperature definition. The theoretical surface flux of a $T_{\rm eff} = 3600$ K M2.5V star is shown [1]. A black body with the same 3600 K effective temperature is shown by the dashed curve. The total black body and stellar flux are equivalent. This star's flux departs from a black body flux; the deep absorption features are due mainly to TiO at visible wavelengths and to H_2O at near-infrared wavelengths.

3.3.2 The Equilibrium Temperature $T_{\rm eq}$

$T_{\rm eq}$ is an estimate of the effective temperature for a planet with no internal luminosity. $T_{\rm eq}$ is a theoretical number and does not refer to any flux measurements of the planet. Physically, $T_{\rm eq}$ is the effective temperature attained by an isothermal planet after it has reached complete equilibrium with the radiation from its parent star. $T_{\rm eq}$ is essential to describe exoplanets, for which it is difficult or impossible to measure the effective temperatures.

$T_{\rm eq}$ can be derived by using the energy balance equation [3.3], that is, by equating the energy emitted by the planet with the energy absorbed by the planet,

$$\frac{4\pi}{f} R_{\rm p}^2 F_{\rm S,p} = (1 - A_{\rm B}) F_{\rm S,*} \left(\frac{R_*}{a}\right)^2 \pi R_{\rm p}^2. \tag{3.7}$$

On the left-hand side we have introduced a parameter f, a correction factor to the term 4π. We use f to enable us to describe the $T_{\rm eq}$ from only one hemisphere of the planet; we shall discuss its meaning and a choice of values later in this subsection.

To derive T_{eq} from energy balance we rewrite equation [3.7], using the Stefan-Boltzmann Law (equation [3.6])

$$F_S = \sigma_R T_{eff}^4, \tag{3.8}$$

to find

$$\boxed{T_{eq} = T_{eff,*}\left(\frac{R_*}{a}\right)^{1/2}\left[f'(1-A_B)\right]^{1/4}.} \tag{3.9}$$

Here we have substituted T_{eq} for the planetary T_{eff}. We have also absorbed a factor of 1/4 into f' by defining $f' = f/4$.

Remember that for exoplanets we can observe only one hemisphere at a time, and this is what drives our need for the correction factor f. If the absorbed stellar radiation is uniformly redistributed around the planet, that is, into 4π, then $f' = 1/4$. In this case T_{eq} is the same for any hemispheric view of the planet—and indeed the f' correction is not really needed. f' is especially relevant for cases where the absorbed stellar radiation is not redistributed uniformly over the planet. For a slowly rotating or tidally locked planet, the stellar energy is absorbed only by one hemisphere of the planet. If the atmosphere instantaneously reradiates the absorbed radiation (with no advection), $f' = 2/3$. In this case T_{eq} of the day-side hemisphere is much hotter than in the uniform reradiation case.

Beyond the two end cases of uniform redistribution and instantaneous reradiation of absorbed stellar energy, there is no simple or analytical form for f'. If we want to estimate the day-side hemispherical temperature, we must use observations or rely on atmospheric circulation models.

We conclude our discussion of f' with the cautionary remark that f has many different definitions in the literature; we emphasize that our choice of $f' = f/4$ as a correction factor to 4π is deliberate, for symmetry with the albedo derivation for scattered light from the dayside of the planet (Section 3.4.3).

How well does the equilibrium temperature T_{eq} approximate T_{eff} for the solar system planets? Using the planets' measured A_Bs, known semimajor axes, and $f' = 1/4$, we see from Table 3.1 that, with the exception of ~~Uranus~~ Jupiter, Saturn, & Neptune, the two temperatures agree to within a few degrees K.

3.3.3 The Brightness Temperature $T_b(\nu)$

The flux of exoplanets at different wavelengths or frequencies is the quantity we can measure at Earth. We have called this $\mathcal{F}_\oplus$, where $\mathcal{F}_\oplus = \frac{R_p^2}{D_\oplus^2} F_S$ for a very distant planet with surface flux F_S. The intuitive familiarity we have with temperature makes it easier to use brightness temperature instead of flux to describe exoplanet atmosphere measurements. Moreover, temperatures are more directly relevant than fluxes for the physics and chemistry of exoplanet atmospheres. Fluxes depend on the planet's distance from Earth and so change from one planet to the next, whereas the temperature of a planet atmosphere is distance independent. Measured exoplanet fluxes are therefore often stated as temperatures.

The brightness temperature $T_b(\nu)$ is defined as the temperature of a black body of the same shape and at the same distance as the planet and with the same flux as the planet *in a specified frequency range*. $T_b(\nu)$ is a frequency-dependent expression of planetary flux,

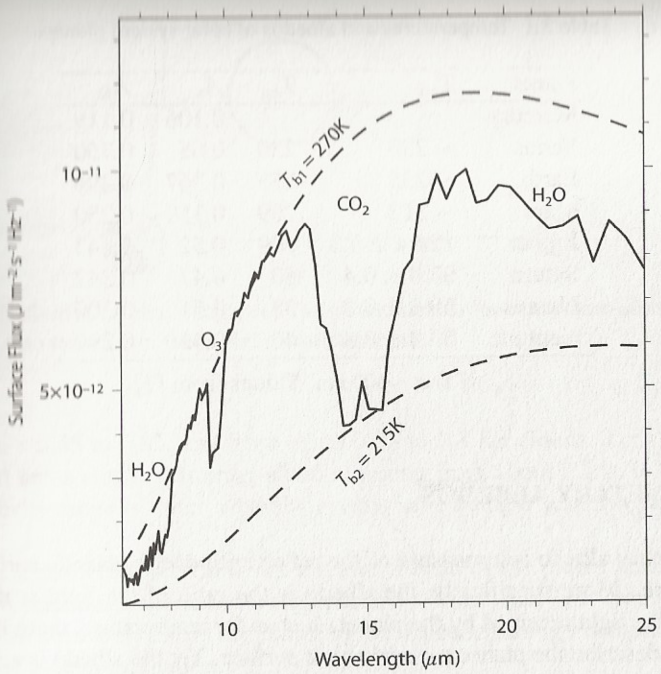

Figure 3.4 Illustration of the brightness temperature definition. The solid curve is Earth's surface flux as measured by the *Mars Global Surveyor* [2]. The dashed curves are black body fluxes with different temperatures, showing that the brightness temperature T_b can vary with wavelength or frequency. From Earth's spectrum we see that $T_b(12\ \mu m) \simeq 270$ K, in contrast to $T_b(14.2\ \mu m) \simeq 215$ K.

$$\boxed{\sigma_R T_b^4(\nu) = F_S(\nu) = \mathcal{F}_\oplus\left(\frac{D_\oplus}{R_p}\right)^2.} \tag{3.10}$$

Here the Stefan-Boltzmann Law (equation [3.6]) is used to convert flux to a temperature. We again note that the brightness temperature assumes the planet behaves as a black body only in the frequency range where the flux is measured. While a black body radiator has a constant $T_b(\nu)$, planets with spectra that depart from a black body have $T_b(\nu)$ that can vary widely with wavelength or frequency (see Figure 3.4).

We emphasize that $T_b(\nu)$ is the only temperature we can currently measure for exoplanets, because for $T_b(\nu)$ flux at only one frequency or wavelength (or frequency or wavelength range) need be observed. To get from a measured $T_b(\nu)$ to the more global T_{eff}, we must resort to a model atmosphere computer code.

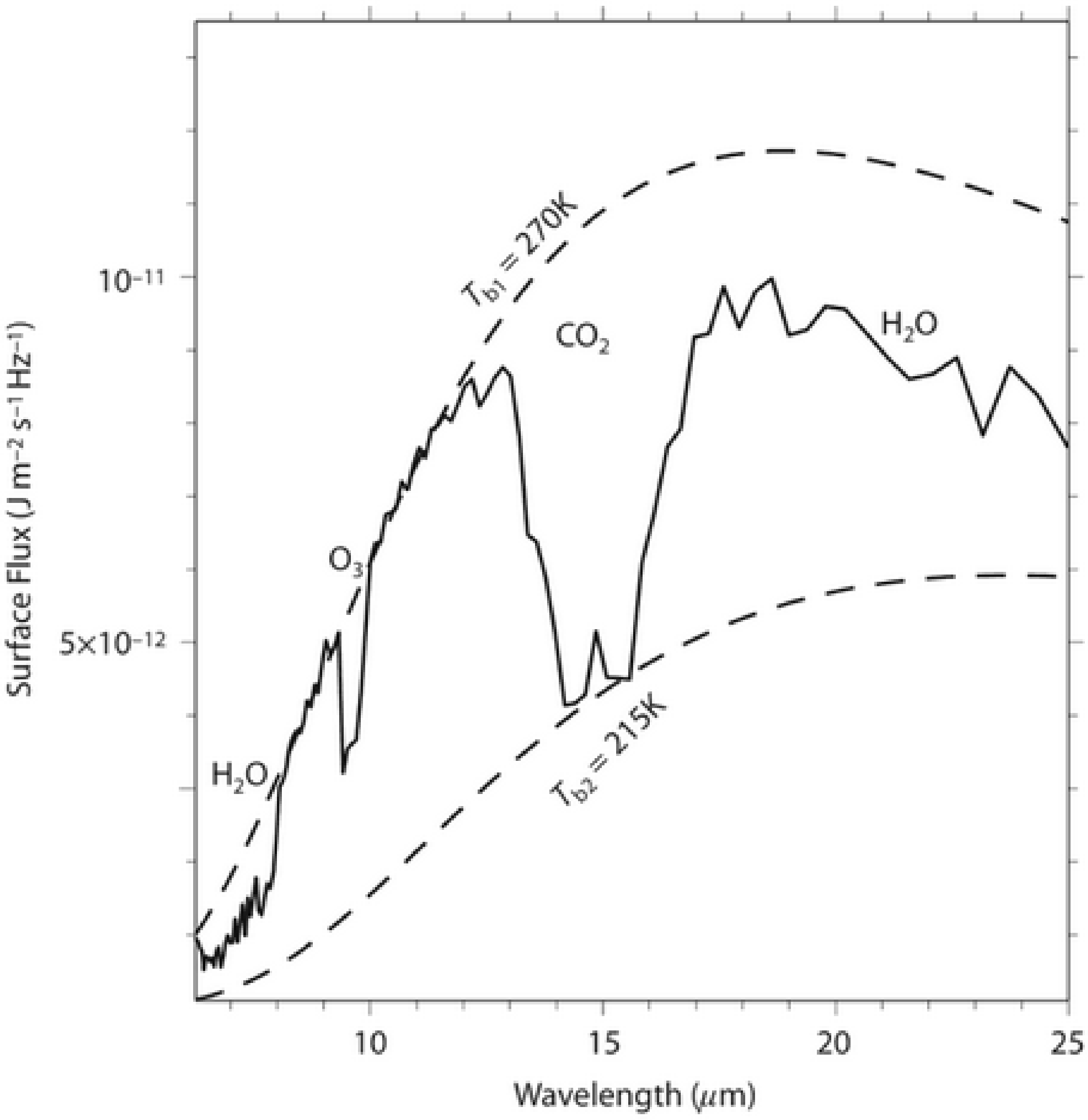

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Table 3.1 Temperatures and albedos of solar system planets.

Planet	T_{eff}	T_{eq}	A_{g}	A_{B}
Mercury			0.106	0.119
Venus	~ 230	230	0.65	0.750
Earth	~ 255	253	0.367	0.306
Mars	~ 212	209	0.150	0.250
Jupiter	124.4 ± 0.3	109	0.52	0.343
Saturn	95.0 ± 0.4	80	0.47	0.342
Uranus	59.1 ± 0.3	58	0.51	0.300
Neptune	59.3 ± 0.8	46	0.42	0.290

$A_{\mathrm{g}}(\nu)$ is at ~500 nm. Values from [3].

3.4 PLANETARY ALBEDOS

The planetary albedo is a measure of the reflectivity of the planet's surface and/or atmosphere. More specifically, the albedo is the ratio of the light scattered by a planet to the light received by the planet. Just as for temperature, there is no single albedo to describe the planet atmosphere or surface. Yet the albedo is an important planet parameter since the planet reflectivity is indicative of cloud or surface conditions. Moreover, the albedo controls the planet's energy balance (equation [3.9]) and its effective temperature. Here we describe four commonly used albedos and the relationship among them: the single-scattering albedo, the geometric albedo, the Bond albedo, and the spherical albedo. We also describe the apparent albedo, a very useful albedo quantity for exoplanets. What makes a planet bright or dark? A planet with a high fractional coverage of very reflective clouds is bright, whereas a planet with no clouds or no atmosphere is typically dark. Venus, for example, is the brightest known planet, scattering 0.75 of the incident energy. This is due to its complete (i.e., 100%) coverage of H_2SO_4 clouds. Earth has highly reflective water liquid or ice clouds, but they cover only about 50% of the surface, and as a consequence Earth scatters only 0.3 of the incident energy. Icy bodies may also be bright, with young ice being more reflective than old ice. Notably, Enceladus, a Saturnian satellite, has water geysers which produce fresh ice, making the satellite very bright. Mercury, in contrast, has no atmosphere, clouds, or ice and is dark, scattering only 0.1 of the incident radiation.

3.4.1 Single Scattering Albedo

The single scattering albedo $\tilde{\omega}$ is the fraction of incident light that is scattered by a given particle in the planetary atmosphere. As an example, the single scattering albedo of a water ice crystal in Earth's atmosphere can be as high as 0.8, while that of a plant leaf at visible wavelengths is closer to 0.1. Single scattering albedos are wavelength dependent. The single scattering albedos of Earth's atmosphere and surface vary widely. At the high end, old to fresh snow single scattering albedos

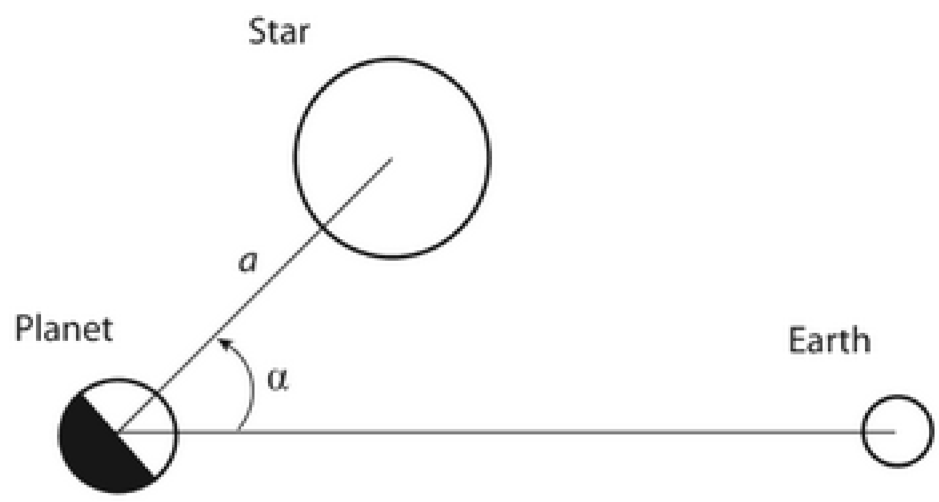

Figure 3.5 Definition of phase angle α. The phase angle is the star-planet-observer angle. Also shown is a, the planet's semimajor axis.

range from $\sim$0.45 to 0.85, and from about 0.35 to 0.8 for clouds. Less reflective, desert sand has a single scattering albedo ranging from about 0.2 to 0.35. At the low reflectivity range is water, whereby oceans have a single scattering albedo less than 0.05.

The single scattering albedo is not a global albedo of the planet atmosphere or surface. Nevertheless, the reflective properties of individual gas, cloud, and surface particles ultimately cause the planet's global albedo through complex multiple scattering of incident radiation.

Important to the overall planet albedo is the directional scattering properties of any individual particle. Few particles in nature, if any, scatter radiation isotropically. Some particles such as ice crystals can have a severe tendency to backscatter incident radiation. This directional scattering property is called the "bidirectional reflection distribution function," often referred to as simply BRDF.

3.4.2 The Phase Angle

Before describing the planetary albedos we first define the planetary phase angle. The phase angle α is the star-planet-observer angle (Figure 3.5). With this definition, $\alpha = 0°$ corresponds to full phase, $\alpha > 170°$ corresponds to a thin crescent phase, and the planet is not at all illuminated at $\alpha = 180°$.

For solar system planets, only Mercury and Venus show all illumination phases as seen from Earth. Outer giant planets show only a few to several degrees in phase angle as seen from Earth. The full phase, or $\alpha = 0$, is a natural configuration for observing solar system planets when the Sun, Earth, and planet are aligned.

For exoplanets, the range of visible planetary phases depends on the orbital inclination of the planet-star system. For example, a planet in a system with orbital inclination of 90° will show all phases as seen from Earth (except phases right near $\alpha = 0$ when the planet is directly behind the star). In contrast, a planet with a "face-on" orbital inclination will have a constant, "half" phase. In contrast to solar system planets, not all exoplanets will be visible at $\alpha \approx 0$; because of the randomness of exoplanet orbital inclinations, $\alpha \approx 0$ may not occur for some planets. In reality, $\alpha \equiv 0$ does not occur for exoplanets because the planet would be directly behind its parent star and not even observable; however, we consider a small α close enough to $\alpha = 0$ for our purposes.

3.4.3 The Geometric Albedo $A_g(\nu)$

3.4.3.1 The Definition of $A_g(\nu)$

The geometric albedo $A_g(\nu)$ is the ratio of a planet's flux at zero phase angle (i.e., full phase) to the flux from a Lambert disk at the same distance and with the same cross-sectional area as the planet (i.e., a surface that subtends the same solid angle). This definition is historical and comes from early observations of solar system planets. To measure the solar system planets' albedo, photographic brightnesses were compared to a photograph of a uniformly illuminated Lambert disk. This relative measurement helped to eliminate instrumental errors. We emphasize that $A_g(\nu)$ is *not* the fraction of incident energy scattered at $\alpha = 0$ (see exercise 3.5).

We derive $A_g(\nu)$ by first considering the numerator in the $A_g(\nu)$ definition: the flux from a planet at zero phase angle as observed at Earth. Choosing a spherical polar coordinate system with latitude θ and longitude ϕ, and considering the flux at Earth $\mathcal{F}_\oplus$ (Chapter 2) we obtain for the flux

$$\mathcal{F}_\oplus(\nu) = \left(\frac{R_\mathrm{p}}{D_\oplus}\right)^2 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} I_{\mathrm{S,scat}}(\theta, \phi, \nu) \cos\phi \cos^2\theta d\theta d\phi. \qquad (3.11)$$

We have used a subscript "scat" on I to indicate that we are considering only scattered radiation (and not radiation that has been absorbed and thermally reradiated). Recall that the subscript S on I indicates flux at the planet's surface.

Next in the geometric albedo derivation we consider the denominator in the $A_g(\nu)$ definition: the flux from a Lambert disk at the same location and of the same cross-sectional area as the planet. We use the equation for flux observed at Earth ($\mathcal{F}$, equation [3.11]) with the surface element of a disk, $r^2 d\theta$. In writing the flux from a Lambert disk, we also relate the emergent intensity to the incoming intensity by $I_{\mathrm{S,scat}}(\theta, \nu) = I_{\mathrm{inc}}(\theta, \nu)$. This relationship comes from the definition of a Lambert surface, in short that intensity is scattered isotropically (Section 2.9). We further assume that the incident intensity is uniform, that is, $I_{\mathrm{inc}}(\theta, \nu) = I_{\mathrm{inc}}(\nu)$. We then have

$$\mathcal{F}_{\mathrm{L\ disk}}(\nu) = \left(\frac{R_\mathrm{p}}{D_\oplus}\right)^2 \int_0^\pi I_{\mathrm{inc}}(\nu) d\theta = \left(\frac{R_\mathrm{p}}{D_\oplus}\right)^2 \pi I_{\mathrm{inc}}(\nu). \qquad (3.12)$$

Now using the definition of A_g—the ratio of the flux from a planet at phase angle zero to the flux from a Lambert disk, that is, the ratio of equations [3.11] and [3.12]—we find

$$\boxed{A_g(\nu) = \frac{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} I_{\mathrm{S,scat}}(\theta, \phi, \nu) \cos\phi \cos^2\theta d\theta d\phi}{\pi I_{\mathrm{inc}}(\nu)}.} \qquad (3.13)$$

We notice something interesting about this expression for $A_g(\nu)$: the denominator can be written as $\pi I_{\mathrm{inc}}(\nu) = \mathcal{F}_{\mathrm{inc}}(\nu)$, the incident flux from the star at the planet's substellar point. So we may also describe $A_g(\nu)$ by the ratio of the planet's scattered flux to the ratio of the incident stellar flux at the substellar point.

We pause here to further point out that equation [3.13] is the form to use to compute $A_g(\nu)$ from an exoplanet model atmosphere code output. The model atmosphere code will generate the scattered intensity at the planetary surface as a function of location on the planet surface (here denoted by θ and ϕ).

We now continue to simplify equation [3.13] for a more conceptual understanding. We can rewrite the geometric albedo considering the relationship between the incoming intensity ($I_{\rm inc}$) and the intensity scattered out by the planet ($I_{\rm S,scat}$). We do this by introducing the dimensionless term $p(\Theta)$: the fraction of incident intensity scattered out of the planet into angle Θ. Here $\Theta = 0$ is defined in the direction of the incident intensity (see Figure 5.1). In this description, all of the physics of directional scattering and multiple scattering inside the planet atmosphere is buried in the final form of $p(\Theta)$,

$$I_{\rm S,scat}(\theta, \phi, \nu) = I_{\rm inc}(\theta', \phi', \nu)p(\Theta). \tag{3.14}$$

The terms with primes refer to the direction of incidence and the terms without primes refer to the direction after scattering. We digress to mention that $p(\Theta)$ here is not the same as the dimensionless single scattering phase function $P(\Theta)$ used in later chapters.

We can further simplify the above intensity relationship by assuming that the incident stellar intensity is uniform and noting that the incident stellar intensity is diluted away from the substellar point as

$$I_{\rm inc}(\theta', \phi', \nu) = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}} I_{\rm inc}(\theta'_0, \phi'_0, \nu) = \cos\Theta I_{\rm inc}(\nu). \tag{3.15}$$

See Figure 2.3 to see that $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta$. From Figure 2.3 and exercise 2.4, we also have

$$\cos\Theta = \cos\theta\cos\phi, \tag{3.16}$$

and therefore

$$I_{\rm S,scat}(\theta, \phi, \nu) = \cos\theta' \cos\phi' I_{\rm inc}(\nu) p(\theta, \phi, \theta', \phi'). \tag{3.17}$$

The A_g from equation [3.13] then becomes

$$A_g(\nu) = \frac{\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\rm inc}(\nu)p(\theta, \phi, \theta', \phi')\cos^2\phi'\cos^3\theta' d\theta' d\phi'}{\pi I_{\rm inc}(\nu)}. \tag{3.18}$$

We may cancel out the $I_{\rm inc}(\nu)$ in the numerator and denominator to find

$$\boxed{A_g(\nu) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} p(\theta, \phi, \theta', \phi')\cos^2\phi'\cos^3\theta' d\theta' d\phi'.} \tag{3.19}$$

3.4.3.2 The A_g for a Lambert Sphere

As an example of geometric albedo we will calculate the $A_g(\nu)$ for a Lambert sphere. Recall that a Lambert surface scatters intensity equally in all directions. A Lambert sphere therefore has

$$p(\Theta) = 1. \tag{3.20}$$

Using the A_g definition in equation [3.19],

$$A_{g\text{Lambert}}(\nu) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} \cos^2\phi\cos^3\theta d\theta d\phi = \frac{2}{3}. \tag{3.21}$$

The geometric albedo for a Lambert sphere is $A_g(\nu) = 2/3 < 1$, even though the definition of a Lambert sphere is that all incident radiation is scattered and none is absorbed. This means that for a Lambert sphere one-third of the incident radiation is scattered out of the line of sight.

By way of the $A_g(\nu)$ definition, the geometric albedo may be greater than 1. This is because the geometric albedo is not the fraction of incident radiation scattered at $\alpha = 0$, but the scattered incident radiation compared to the scattered incident radiation from a perfectly diffusing disk at the same location and of the same size. Saturn's icy moon Enceldaus has $A_g = 1.38$ at visible wavelengths [4]. Ice can have a strong backscattering effect, making the scattered radiation greater than the scattered radiation from a diffusing disk.

Measurements of A_g have been elusive for exoplanets. Nevertheless, A_g is still useful for classifying theoretical models.

3.4.4 The Bond Albedo A_B and the Spherical Albedo A_S

A_B is the fraction of incident stellar energy scattered back into space by the planet. A_B includes the radiation scattered into all directions and radiation at all frequencies. By its definition, $A_B \leq 1$.

The spherical albedo $A_S(\nu)$ has the same definition as the Bond albedo but at a specific frequency,

$$A_B = \int_0^\infty A_S(\nu)d\nu. \tag{3.22}$$

We emphasize up front that the Bond albedo, through the definition of the spherical albedo below, is actually weighted by the incident radiation. Both the Bond and spherical albedos therefore depend on the spectrum of the planet's host star.

To derive $A_s(\nu)$ we begin by revisiting the numerator in $A_g(\nu)$ (equation [3.13]), the scattered radiation from the planet at $\alpha = 0$. Recall that the $A_g(\nu)$ numerator is an expression for flux (equation [3.11]). The important point to keep in mind is that at each phase angle α only a portion of the hemisphere facing Earth is illuminated. This is illustrated in Figure 3.6. Using Figure 3.6 we construct a more general form of the light scattered from the planet surface,

$$F_{S,\alpha}(\nu) = \int_{\alpha-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\text{inc}}(\theta',\phi',\nu)p(\theta,\phi,\theta',\phi')\cos(\alpha-\phi')\cos\phi'\cos^3\theta' d\theta' d\phi' \tag{3.23}$$

and we make the assumption that the incident intensity is uniform to write

$$F_{S,\alpha}(\nu) = I_{\text{inc}}(\nu)\Psi_\alpha(\nu). \tag{3.24}$$

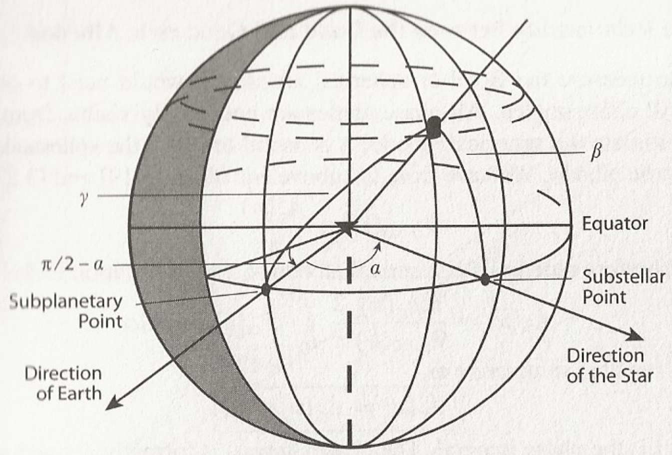

Figure 3.6 Spherical planetary coordinates suitable for deriving the Lambert sphere phase curve. θ is the latitude measured from the intensity equator. ϕ is the longitude measured from the sub-Earth point. α is the phase angle. After [5].

Here Ψ_α is the fraction of scattered radiation in the *direction of the observer* as a function of α.

To account for the total scattered energy in *all directions* we must integrate $F_{B,\alpha}(\nu)$ over all directions. We want to know the total energy per unit time per frequency scattered by the planet compared to the total energy per unit time per frequency incident on the planet. We consider a sphere around the planet with a spherical polar coordinate system. We adopt the angles α as our altitude and β as our azithmuthal angle. We further assume azimuthal symmetry. To find the total scattered energy in all directions, we integrate $F_{S,\alpha}(\nu)$ (defined in equation [3.24]) over $(0 \leq \alpha < \pi)$ and $(0 \leq \beta < 2\pi)$,

$$E_{\text{scat}}(\nu) = I_{\text{inc}}(\nu)\int_0^{2\pi}\int_0^{\pi}\Psi_\alpha(\nu)R_p^2\sin\alpha d\alpha d\beta. \tag{3.25}$$

and because of the azithmuthal symmetry,

$$E_{\text{scat}}(\nu) = 2\pi R_p^2 I_{\text{inc}}(\nu)\int_0^{\pi}\Psi_\alpha(\nu)\sin\alpha d\alpha. \tag{3.26}$$

We now consider the total incident energy on the planet as related to equation [2.31],

$$E_{\text{inc}}(\nu) = \pi R_p^2 \pi I_{\text{inc}}(\nu). \tag{3.27}$$

The spherical albedo is the ratio of scattered energy to incident energy

$$A_S(\nu) = \frac{E_{\text{scat}}(\nu)}{E_{\text{inc}}(\nu)} = \frac{2I_{\text{inc}}(\nu)\int_0^\pi \Psi_\alpha(\nu)\sin\alpha d\alpha}{\pi I_{\text{inc}}(\nu)}, \tag{3.28}$$

$$\boxed{A_{\rm S}(\nu) = \frac{2}{\pi}\int_0^{\pi} \Psi_\alpha(\nu)\sin\alpha d\alpha.} \tag{3.29}$$

3.4.5 The Relationship between the Bond and Geometric Albedos

In order to measure the Bond or spherical albedo we would need to observe the planet at all phase angles. All phase angles are not usually visible from Earth. In order to estimate the spherical albedo, it is useful to relate the spherical albedo to the geometric albedo. We have from the above equations [3.19] and [3.23]

$$\Psi_{\alpha=0}(\nu) \equiv \frac{A_{\rm g}(\nu)}{\pi}. \tag{3.30}$$

We may therefore redefine the spherical albedo $A_{\rm S}(\nu)$ in equation [3.29] as

$$A_{\rm S}(\nu) = \frac{\pi A_{\rm g}(\nu)}{\Psi_{\alpha=0}(\nu)}\frac{2}{\pi}\int_0^{\pi} \Psi_\alpha(\nu)\sin\alpha d\alpha, \tag{3.31}$$

which we can also summarize as

$$\boxed{A_{\rm s}(\nu) = A_{\rm g}(\nu)q(\nu),} \tag{3.32}$$

where $q(\nu)$ is the phase integral. The phase integral is formally defined as

$$q(\nu) \equiv \frac{A_{\rm s}(\nu)}{A_{\rm g}(\nu)} = 2\int_0^{\pi} \frac{\Psi_\alpha(\nu)}{\Psi_{\alpha=0}(\nu)}\sin\alpha d\alpha. \tag{3.33}$$

We can more conveniently write

$$\boxed{q(\nu) = 2\int_0^{\pi} \Phi_\alpha(\nu)\sin\alpha d\alpha.} \tag{3.34}$$

$\Phi_\alpha(\nu)$ is the phase function: the fractional scattered flux variation with phase angle normalized to the flux at $\alpha = 0$,

$$\boxed{\Phi_\alpha(\nu) = \frac{\Psi_\alpha(\nu)}{\Psi_{\alpha=0}(\nu)}.} \tag{3.35}$$

The spherical albedo can be estimated from equation [3.33] with both a measured $A_{\rm g}(\nu)$ and a theoretical calculation for $q(\nu)$.

What is the relationship between the geometric and spherical albedos for a Lambert sphere? For a Lambert surface, $\Phi_\alpha(\nu) = \cos\alpha$ and using equations [3.32] and [3.33] we find

$$A_{\rm g}(\nu) = \frac{2}{3}A_{\rm s}(\nu). \tag{3.36}$$

For a Lambert sphere the geometric albedo is lower than the spherical albedo by a factor of 2/3. We can understand this conceptually as follows. For a Lambert sphere all radiation is scattered back to space and, since the spherical albedo includes radiation into all angles $A_s(\nu) \equiv A_{\rm B} = 1$. The geometric albedo, in contrast is the albedo only at phase angle 0 and includes only the radiation backscattered to the observer, so it is less than 1.

A comparison of Bond and total geometric albedos of Jupiter, Saturn, Uranus, and Neptune is shown in Figure 3.7. Here we call "total geometric albedo" the geometric albedo integrated over all wavelengths. We can see that these bodies with deep atmospheres all lie above the line for isotropic scattering (i.e., a Lambert sphere).

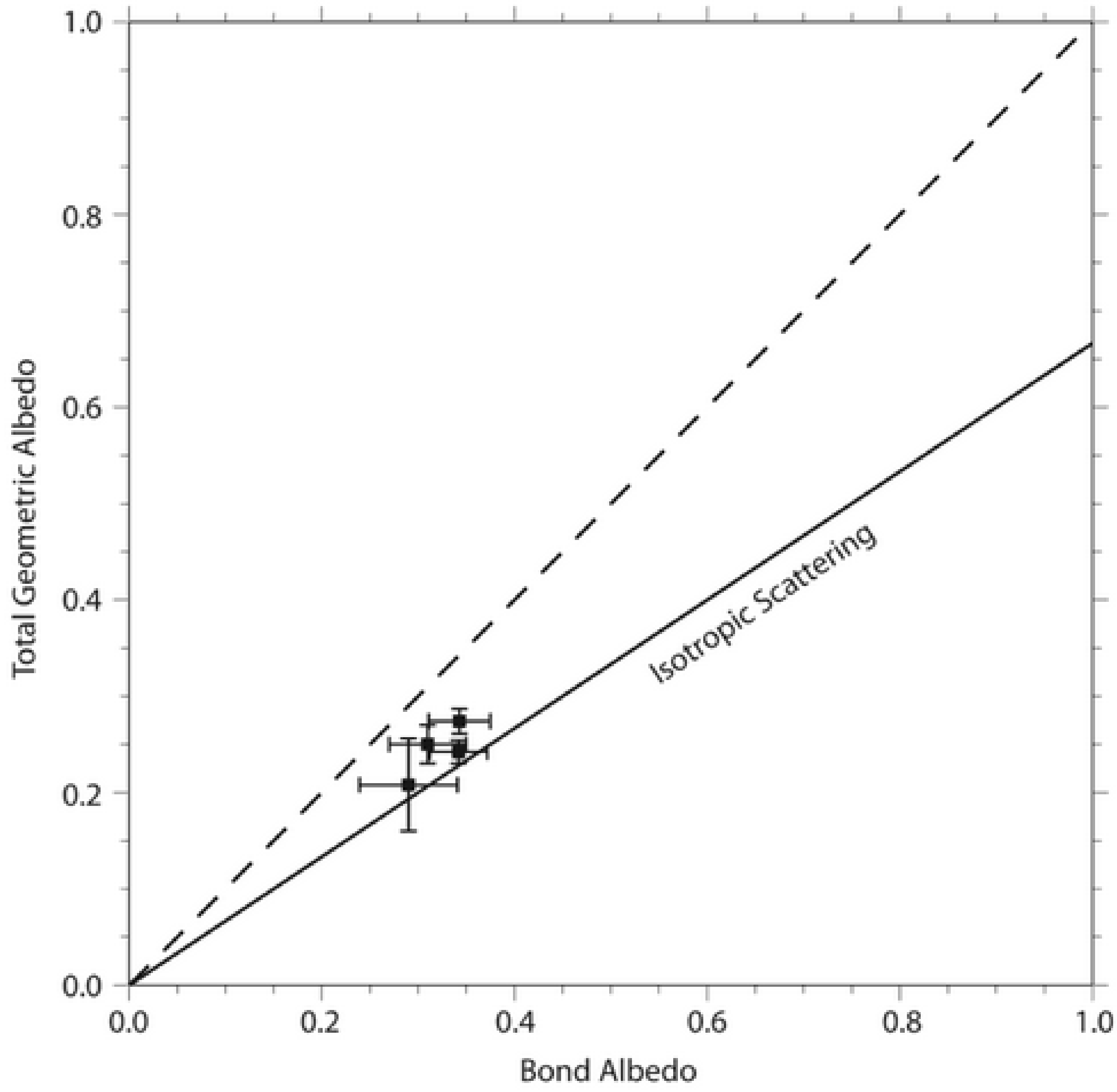

Figure 3.7 Relationship between the Bond albedo ($A_{\rm B}$) and geometric albedo ($A_{\rm g}$) for a Lambert sphere and solar system giant planets. Here we use the "total" $A_{\rm gTot}$ to refer to $A_{\rm g}$ over all visible wavelengths. The points are for Uranus, Neptune, Saturn, and Jupiter (in order of increasing $A_{\rm B}$). The solid line ($A_{\rm gTot}/A_{\rm B} = 0.67$) is for Lambertian isotropic reflectance (i.e., constant for all angles of incidence). The dashed line is the line of equivalence where $A_{\rm gTot} = A_{\rm B}$ (all gas giant planets with deep atmospheres must lie to its right). The wedge between the solid and dashed lines defines a useful limiting region; it bounds the photometric properties of most spherical bodies with deep atmospheres (with, e.g., Rayleigh scattering, clouds, dust, etc.). Figure adapted from [6]. Solar system data points are referenced in [7].

3.4.6 The Apparent Albedo

The geometric, spherical, and Bond albedos are not very satisfactory for exoplanets because none can be measured. The geometric albedo is defined at full phase ($\alpha = 0$), a configuration where the planet is behind the star and not observable. The spherical and Bond albedos are also not measurable because all phase angles are not accessible for exoplanets. With the direct imaging technique, many phase angles where the planet is projected too close to the star will not be observeable. Moreover, unless the planet orbit is edge-on to our line of sight the planet will not go through all phases as seen from Earth. We therefore turn to an albedo definition that is more appropriate for exoplanets.

The apparent albedo can be defined as the ratio between the scattered flux emerging from the planet in the direction of the observer and the scattered flux from a perfectly reflecting Lambert sphere with the same size as the planet, at the same phase angle, and at the same distance from the observer. The apparent albedo can be understood as a planet's reflectance in the direction of the observer at a given time. The apparent albedo is the only albedo quantity that can potentially be derived from a single observation, for a planet of known size. The apparent albedo is a function of the phase angle and frequency. We leave it as an exercise to formulate an equation for the apparent albedo.

3.5 PLANET-STAR FLUX RATIOS

An estimate of the flux ratio of the planet and star is useful for exploring whether a particular planet and star combination can be detected with a given telescope. For the planet-star flux ratio estimate, we assume that the stars and planets radiate as black bodies. Figure 3.8 shows planet-star flux ratios for the Sun and solar system planets, as well as for a hypothetical hot Jupiter. The planets have two peaks in their spectra. The short-wavelength peak is due to starlight scattered from the planet atmosphere, and therefore has a flux peak at the same wavelength as the star. For each planet, the long-wavelength flux peak is from the planet's thermal emission and peaks at the planet's characteristic temperature. The high contrast at all wavelengths of the planet-to-star flux ratio is the primary challenge for detecting radiation from exoplanets.

We recall that the planet flux at Earth is (equation [2.19])

$$\mathcal{F}_{\oplus,p} = F_{\mathrm{S,p}} \left(\frac{R_{\mathrm{p}}}{D_{\oplus}} \right)^2, \tag{3.37}$$

where the subscript $\oplus$ refers to a flux measurement at Earth and the subscript S refers to flux at the planet's surface. Similarly, the star's flux at Earth is

$$\mathcal{F}_{\oplus,*} = F_{\mathrm{S},*} \left(\frac{R_*}{D_{\oplus}} \right)^2, \tag{3.38}$$

To find the planet-star flux ratio, we divide the planet flux at Earth by the star flux at Earth,

$$\boxed{\frac{\mathcal{F}_{\oplus,\mathrm{p}}}{\mathcal{F}_{\oplus,*}} = \frac{F_{\mathrm{S,p}} R_{\mathrm{p}}^2}{F_{\mathrm{S},*} R_*^2}.} \tag{3.39}$$

Note that we can use a similar equation for frequency-dependent flux.

3.5.1 Thermal Emission Flux Ratio

We can estimate the thermal emission planet-star flux ratio on the illuminated side of the planet by equating the planet's thermal emission with the absorbed stellar radiation, in a rearranged form of energy balance described in equation [3.7],

$$\boxed{\frac{\mathcal{F}_{\oplus,\mathrm{p}}}{\mathcal{F}_{\oplus,*}} = \frac{F_{\mathrm{S,p}} R_{\mathrm{p}}^2}{F_{\mathrm{S},*} R_*^2} = \left(\frac{R_{\mathrm{p}}}{a} \right)^2 \frac{f}{4} (1 - A_{\mathrm{B}}) .} \tag{3.40}$$

The apparent albedo can be defined as the ratio between the scattered flux emerging from the planet in the direction of the observer and the scattered flux from a perfectly reflecting Lambert sphere with the same size as the planet, at the same phase angle, and at the same distance from the observer. The apparent albedo can be understood as a planet's reflectance in the direction of the observer at a given time. The apparent albedo is the only albedo quantity that can potentially be derived from a single observation, for a planet of known size. The apparent albedo is a function of the phase angle and frequency. We leave it as an exercise to formulate an equation for the apparent albedo.

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$$\mathcal{F}_{\oplus,*} = \mathcal{F}_{\mathrm{S},*} \left(\frac{R_{*}}{D_{\oplus}}\right)^2, \tag{3.38}$$

To find the planet-star flux ratio, we divide the planet flux at Earth by the star flux at Earth,

$$\boxed{\frac{\mathcal{F}_{\oplus,\mathrm{p}}}{\mathcal{F}_{\oplus,*}} = \frac{\mathcal{F}_{\mathrm{S,p}} R_{\mathrm{p}}^2}{\mathcal{F}_{\mathrm{S},*} R_{*}^2}.} \tag{3.39}$$

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$$\boxed{\frac{\mathcal{F}_{\oplus,\mathrm{p}}}{\mathcal{F}_{\oplus,*}} = \frac{\mathcal{F}_{\mathrm{S,p}} R_{\mathrm{p}}^2}{\mathcal{F}_{\mathrm{S},*} R_{*}^2} = \left(\frac{R_{\mathrm{p}}}{a}\right)^2 \frac{f}{4} (1 - A_{\mathrm{B}}).} \tag{3.40}$$

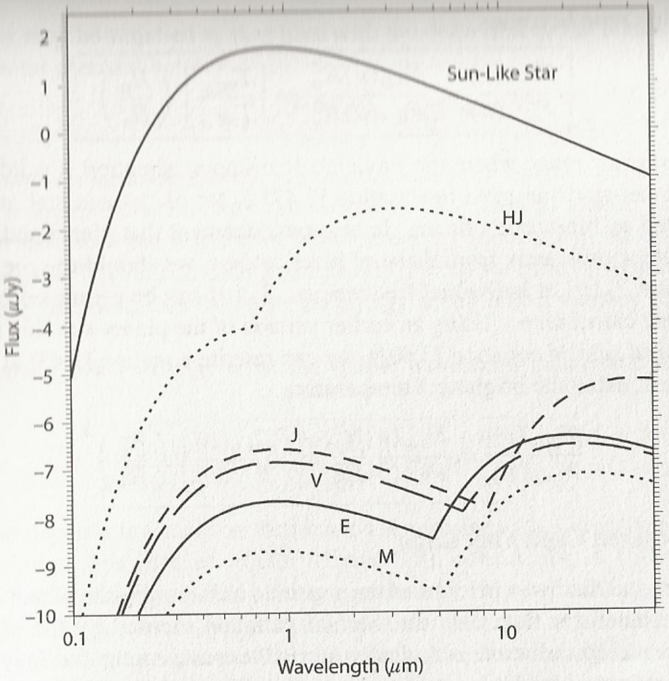

Figure 3.8 The approximate spectra (in units of 10^{-26} W m^{-2} Hz^{-1}) of some solar system bodies as seen from a distance of 33 light years. The Sun is represented by a black body of 5750 K. The planets Jupiter, Venus, Earth, and Mars are shown and are labeled with their first initial. A representative hot Jupiter exoplanet is also shown.

Here we recall that f is a correction factor to 4π to describe the unknown redistribution of absorbed stellar radiation. We further note that the fluxes in the above equation are total fluxes, integrated over all frequencies.

A more useful planet-star flux ratio estimate is one that is frequency dependent, because telescope observations of exoplanets are limited to a fixed frequency or wavelength range. We may approximate the star and planet surface fluxes with black body fluxes $\pi B(T, \nu)$, where T is the effective temperature of the star or the equilibrium effective temperature of the planet,

$$\frac{\mathcal{F}_{\oplus,\mathrm{p}}(\nu)}{\mathcal{F}_{\oplus,*}(\nu)} = \frac{\mathcal{F}_{\mathrm{S,p}}(\nu) R_{\mathrm{p}}^2}{\mathcal{F}_{\mathrm{S},*}(\nu) R_{*}^2} = \left[\frac{e^{h\nu/kT_{\mathrm{eff},*}} - 1}{e^{h\nu/kT_{\mathrm{eq,p}}} - 1}\right] \left(\frac{R_{\mathrm{p}}^2}{R_{*}^2}\right). \tag{3.41}$$

At mid-infrared wavelengths we may make a further simplification for the planet-star flux ratio. This simplification is based on the point that the black body fluxes for reasonable planet and star temperatures are both in the Rayleigh-Jeans region of the black body spectrum, $h\nu/kT \ll 1$. Under this approximation, $e^{-x} \approx 1 - x$, and the black body flux (using equation [2.34]) becomes

$$\pi B(T, \nu) \approx \pi \frac{2\nu^2}{c^2} kT, \tag{3.42}$$

and the flux ratio becomes

$$\frac{\mathcal{F}_{\oplus,\mathrm{p}}(\nu)}{\mathcal{F}_{\oplus,*}(\nu)} = \frac{F_{\mathrm{S,p}}(\nu)R_\mathrm{p}^2}{F_{\mathrm{S},*}(\nu)R_*^2} = \left[\frac{T_{\mathrm{eq,p}}}{T_{\mathrm{eff},*}}\right]\left(\frac{R_\mathrm{p}}{R_*}\right)^2 \quad (3.43)$$

in the frequency range where the Rayleigh-Jeans approximation is valid.

The planet-star flux ratio in equation [3.43] is for planet and star atmospheres represented by black body fluxes. In acknowledgement that planet and star fluxes can deviate significantly from those of black bodies, we should use the brightness temperature $T_\mathrm{b}(\nu)$ at individual frequencies. $T_\mathrm{b}(\nu)$ can be estimated from model atmosphere calculations. Using an earlier version of the planet-star flux ratio (from the left-hand side of equation [3.40]), we can rewrite equation [3.43] at individual frequencies, using the brightness temperature,

$$\frac{\mathcal{F}_{\oplus,\mathrm{p}}(\nu)}{\mathcal{F}_{\oplus,*}(\nu)} = \frac{F_{\mathrm{S,p}}(\nu)R_\mathrm{p}^2}{F_{\mathrm{S},*}(\nu)R_*^2} = \left[\frac{T_{\mathrm{b,p}}(\nu)}{T_{\mathrm{b},*}(\nu)}\right]\left(\frac{R_\mathrm{p}}{R_*}\right)^2. \quad (3.44)$$

3.5.2 Scattered Light Flux Ratio

The planet-star flux ratio in terms of the scattered radiation by the planet is different from the planet-star flux ratio for thermal radiation emitted by the planet. The planet's scattered radiation is typically at visible wavelengths but may extend to much longer wavelengths for colder planets (see Figure 3.8). In this subsection we will concern ourselves with the planet-star flux ratio at $\alpha = 0$.

We again start with the planet-star flux ratio equation [3.40]. Because we are now interested in the scattered stellar flux instead of the absorbed stellar flux, we replace the term that describes the absorbed and reradiated stellar radiation on the illuminated hemisphere,

$$\frac{f}{4}(1 - A_\mathrm{B}), \quad (3.45)$$

with a term to describe the scattered radiation on the illuminated hemisphere,

$$\frac{g}{4}A_\mathrm{s}(\nu). \quad (3.46)$$

We have used the variable g to replace f. Like f, g is a correction factor to 4π—specifically, g describes the solid angle into which the incident stellar radiation is reradiated, compared to 4π. To explain the need for g in more detail, consider that one hemisphere of the planet is illuminated by the star. Is the incident radiation scattered equally into $4\pi R_\mathrm{p}^2$? No, because if the reradiation is scattered back toward the observer, it cannot be advected around the planet.

In equation [3.46] we use the spherical albedo $A_\mathrm{s}(\nu)$ instead of the Bond albedo A_B because we are interested in a wavelength-dependent flux ratio. Finally, for simplicity we assume monochromatic scattering. The planet-star flux ratio at visible wavelengths, including all of the above, is

$$\frac{\mathcal{F}_{\mathrm{p},\oplus}(\nu)}{\mathcal{F}_{*,\oplus}(\nu)} = \frac{F_{\mathrm{p,S}}(\nu)R_\mathrm{p}^2}{F_{*,\mathrm{S}}(\nu)R_*^2} = \frac{R_\mathrm{p}^2}{a^2}\left[\frac{g}{4}\right]A_\mathrm{s}(\nu). \quad (3.47)$$

Many readers may be satisfied to stop here with the definition of the scattered light correction factor (derived below)

$$g = \frac{4A_\mathrm{g}(\nu)}{A_\mathrm{s}(\nu)}. \quad (3.48)$$

= 8/3 for Lambertian sphere (Eq. 3.36)

It is a *very common misunderstanding* in this derivation of visible-wavelength flux ratio to not recognize that this flux ratio involves the spherical albedo $A_\mathrm{s}(\nu)$—a quantity that describes the fraction of incident energy scattered in all directions. For the planet-star flux ratio at planet full phase ($\alpha = 0$) we need to replace $A_\mathrm{s}(\nu)$ with a quantity that describes the scattered energy at full phase, namely, related to $A_\mathrm{g}(\nu)$.

With equation [3.48] we may write the visible wavelength planet-star flux ratio as

$$\frac{\mathcal{F}_{\mathrm{p},\oplus}(\nu)}{\mathcal{F}_{*,\oplus}(\nu)} = \frac{F_{\mathrm{S,p}}(\nu)R_\mathrm{p}^2}{F_{\mathrm{S},*}(\nu)R_*^2} = A_\mathrm{g}(\nu)\frac{R_\mathrm{p}^2}{a^2}. \quad (3.49)$$

The flux ratio depends inversely on the planet's semimajor axis. Let us consider the visible-wavelength flux ratio of Jupiter (Figure 3.8), considering $A_\mathrm{g}(500\ \mathrm{nm}) = 0.5$, $a = 5.2$ AU, and $R_\mathrm{p} = R_\mathrm{J}$. Using equation [3.49] we find $\mathcal{F}_{\mathrm{p},\oplus}/\mathcal{F}_{*,\oplus} = 4.4 \times 10^{-09}$. Scaling this to a planet at 0.05 AU, the flux ratio is 10^4 times higher, or 4.4×10^{-4}. The actual expected flux ratio of a hot Jupiter orbiting its star at 0.05 AU (or a corresponding few-day period) is not so high compared to Jupiter because Jupiter is bright from its icy clouds of ammonia and water, whereas the hot Jupiters are dark due to lack of bright clouds.

We now return to the correction factor g. This correction factor is the most critical step in the derivation of the scattered light planet-star flux ratio at phase angle zero. g describes the ratio of energy emitted at zero phase angle to energy emitted at all phase angles. From equation [3.29] for scattered energy in all directions and using a form of equation [3.24] for scattered energy in the direction of the observer, we therefore have

$$\frac{g}{4\pi} = \frac{E_{\mathrm{scat},\alpha=0}}{E_\mathrm{scat}} = \frac{\Psi_{\alpha=0}(\nu)}{2\pi\int\Psi_\alpha(\nu)\sin\alpha d\alpha}. \quad (3.50)$$

We can use our previous definition of Φ and $q(\nu)$ to find

$$\frac{g}{4\pi} = \frac{1}{\pi q(\nu)} = \frac{A_\mathrm{g}(\nu)}{\pi A_\mathrm{s}(\nu)} \quad (3.51)$$

and simplify to

$$g = \frac{4A_\mathrm{g}(\nu)}{A_\mathrm{s}(\nu)}. \quad (3.52)$$

This derivation of g is the final step in finding the scattered light planet-star flux ratio given in equation [3.49].

3.5.3 Transmission Flux Ratio

During an exoplanet transit, stellar radiation is *transmitted* through the planet atmosphere, picking up some spectral features from the planet atmosphere. We call this the transmit transmission spectrum. In practice the planet transit transmission spectrum is not measured alone. Astronomers measure the total flux from the planet and star during transit (the "in-transit flux") and the flux outside of transit (the stellar flux or the "out-of-transit" flux). Dividing these two quantities gives us the in-transit to out-of-transit flux ratio, what we simply call the transmission flux ratio. Here we will estimate the magntiude of the transmission flux ratio.

We begin with the in-transit flux by using the derivation of the planet flux at Earth in Section 2.5 (and equations [2.16] and [2.19]). For the planet, we use R_p to mean the radius where the planet is opaque at all wavelengths of interest and A_H to mean the additional radial extent of the planetary atmosphere (in the same units as R_p).

We consider the angles subtending the star, planet, and planet atmosphere:

$$\begin{aligned}\mathcal{F}_{\text{in trans},\oplus}(\nu) &= \int_0^{2\pi}\int_{(R_p+A_H)/D_\oplus}^{R_*/D_\oplus} I_*(\vartheta,\phi,\nu)\cos\omega\sin\omega d\omega d\phi \\ &+ \int_0^{2\pi}\int_0^{R_p/D_\oplus} I_p(\vartheta,\phi,\nu)\cos\omega\sin\omega d\omega d\phi \\ &+ \int_0^{2\pi}\int_{R_p/D_\oplus}^{(R_p+A_H)/D_\oplus} I_{atm}(\vartheta,\phi,\nu)\cos\omega\sin\omega d\omega d\phi. \qquad (3.53)\end{aligned}$$

The intensity passing through the opaque part of the planet, I_p, is zero. We follow Section 2.5 and divide by the observed stellar flux $\mathcal{F}_{*,\oplus}(\nu) = F_{S,*}(R_*/D_\oplus)^2$ (from equation [2.19]) to find

$$\boxed{\frac{\mathcal{F}_{\text{in trans},\oplus}(\nu)}{\mathcal{F}_{*,\oplus}(\nu)} = 1 - \left(\frac{R_p}{R_*}\right)^2 + \frac{2R_pA_H}{R_*^2}\left(1 - \frac{F_{S,p,\text{trans}}(\nu)}{F_{S,*,\text{trans}}(\nu)}\right).} \qquad (3.54)$$

Here we have used the subscript "in trans, ⊕" to refer to the overall flux measured at Earth during transit, the subscript "S, p, trans" to refer to just the flux from the planet atmosphere, and "$S_{*,\text{trans}}$" to refer to radiation from the star incident on the planet atmosphere. We have assumed that $A_H \ll R_p$, and $F_{S,p,\text{trans}}$ is further described in Section 6.4.1. We emphasize that the choice of A_H is not important as long as it is large enough. For example, beyond some atmospheric height A_H, there is no atmosphere left to attenuate the stellar radiation, $F_{S,p,\text{trans}}/F_{S,*,\text{trans}} = 1$, and there is no contribution to the in-transit flux.

How can we estimate the magnitude of the transmission flux signal? We will consider the last term on the right-hand side of the above equation—this is the planet-to-star transmission flux ratio. We will estimate the maximum $\mathcal{F}_{S,p,\text{trans}}$ as if the atmosphere *annulus* were completely opaque to the stellar radiation, that is, at some wavelengths no radiation is transmitted; all radiation is blocked by the atmosphere, $\mathcal{F}_{S,p,\text{trans}} = 0$. We have for the planet-to-star transmission flux ratio

$$\frac{\mathcal{F}_{p,\oplus,\text{trans}}(\nu)}{\mathcal{F}_{*,\oplus}(\nu)} = \frac{2R_pA_H}{R_*^2}\left(1 - \frac{F_{S,p,\text{trans}}(\nu)}{F_{S,*,\text{trans}}(\nu)}\right) \approx \frac{2R_pA_H}{R_*^2}. \qquad (3.55)$$

As an example, let us consider a short-period Jupiter transiting a sun-sized star. The hot Jupiter is composed primarily of H and He, and with $T_{eq} \simeq 1000$ K the scale height (equation [9.11]) is $H \simeq 500$ km. Typically, we may take the planet atmosphere to be $5 \times H$, that is, $A_H = 5H$. The resulting planet annulus compared to star area is 1.5×10^{-3} (i.e., the last term in the above equation [3.54]). This number may seem very small. In fact, one part per thousand is a relatively large number in the field of exoplanet atmosphere detection. We can see this by comparing to the planet-to-star flux ratios earlier in this section. In contrast to the short-period hot Jupiters, for a colder planet such as a Jupiter analog (a Jupiter-sized planet in a Jupiter-like orbit with $T_{eq} \sim 120$ K) with $H = 24$ km, the estimated transmission spectrum strength is only 7×10^{-5}.

The measurement of an exoplanet spectrum for hot Jupiters has been a successful technique for bright star targets, with sodium, water vapor, methane, and other gases identified in this way.

3.6 PLANETARY PHASE CURVES

An exoplanet goes thorough illumination phases as seen from Earth. These phases are akin to the phases the moon goes through. We are interested not only in the flux ratio at $\alpha = 0$ (i.e., full phase) but also the flux ratio as a function of α. In this section we discuss the planet phase curves, building on previous equations in this chapter.

3.6.1 Visible-Wavelength Phase Curves

We first recall the normalized phase function equation [3.35]:

$$\Phi_\alpha(\nu) = \frac{\Psi_\alpha(\nu)}{\Psi_{\alpha=0}(\nu)}, \qquad (3.56)$$

where $\Psi_\alpha(\nu)$ describes the fraction of scattered radiation at any phase angle and is defined in equation [3.23],

$$\Psi(\alpha,\nu) = \int_{\alpha-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{inc}(\theta',\phi',\nu)p(\theta,\phi,\theta',\phi')\cos(\alpha-\phi')\cos\phi'\cos^3\theta' d\theta' d\phi'. \qquad (3.57)$$

There is one case where the phase curve can be analytically derived, that of a Lambert sphere. We use the definition of a Lambert sphere $p(\theta,\phi,\theta',\phi') = 1$ (introduced in section 3.4.3) and assume that the incident radiation is uniform, thus taking I_{inc} out of the integrand in equation [3.57]. We then compute the integral in equation [3.57] normalized as shown in the phase curve definition in equation [3.56] to derive the frequency-independent phase curve of a Lambert sphere

$$\boxed{\Phi_\alpha = \frac{1}{\pi}\left[\sin\alpha + (\pi - \alpha)\cos\alpha\right].} \qquad (3.58)$$

The Lambert sphere phase curve is shown in Figure 3.9.

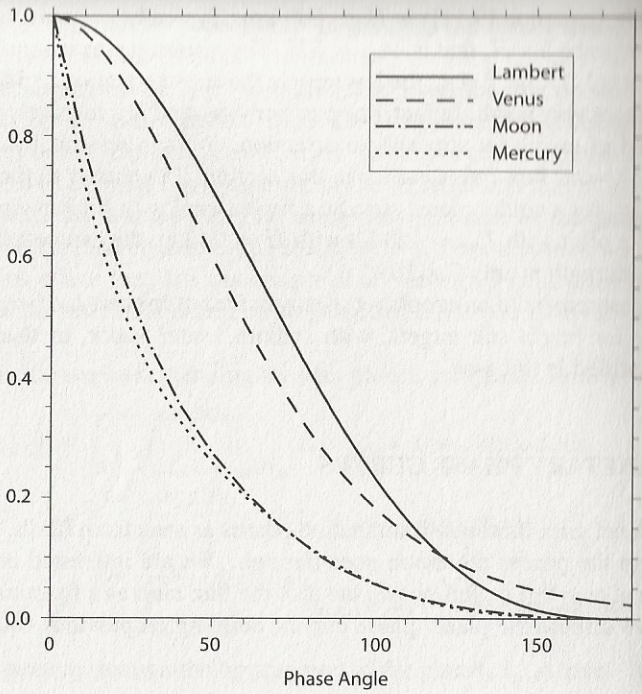

Figure 3.9 Visible-wavelength illumination phase curves for Venus, Mercury, and the Moon. A Lambert sphere phase curve is shown as the rightmost solid black line. From formulae in [3].

Visible-wavelength phase curves of the Moon, Mercury, and Venus are also shown in Figure 3.9. We can see that the atmosphereless bodies Mercury and the Moon have phase curves that are very different from a Lambert sphere.

The phase curve in equation [3.58] is normalized to 1. If we want to compute the planet-star flux ratio at all phases we may use this equation in conjunction with the planet-star flux ratio at $\alpha = 0$ (equation [3.49]):

$$\frac{\mathcal{F}_{\mathrm{p},\oplus,\alpha}(\nu)}{\mathcal{F}_{*,\oplus}(\nu)} = A_{\mathrm{g}}(\nu)\frac{R_{\mathrm{p}}^2}{a^2}\Phi_\alpha(\nu). \tag{3.59}$$

Let us consider the exoplanet example of Earth orbiting the Sun. At $\alpha = 0$, with $A_{\mathrm{g}}(500\text{–}800\ \mathrm{nm}) = 0.2$, $a = 1.0$ AU, and $R_{\mathrm{p}} = R_\oplus$, $\mathcal{F}_{\mathrm{p},\oplus,\alpha}(\nu)/\mathcal{F}_{*,\oplus}(\nu) = 2.1 \times 10^{-10}$. Earth is almost ten billion times fainter than the Sun. Recall that $\alpha = 0$ corresponds to an exoplanet's position directly behind its star as seen from Earth. If we want to know the planet-star flux ratio at the planet-star maximum separation, we must take $\alpha = 90°$. Approximating Earth as a Lambert sphere (equation [3.58]), the planet-star flux ratio is 1.2×10^{-10}, a factor of π lower than the zero-phase-angle value.

IR-Wavelength Phase Curves

An exoplanet may have phase variation at infrared wavelengths from thermal emission, and not just at visible wavelengths due to scattered stellar radiation. Thermal phase variation may arise for an exoplanet whose thermal emission is dominated by reemitted absorbed stellar radiation rather than any internal energy. One class of planets that fits this condition is the tidally locked hot exoplanets. These hot Jupiters, hot Neptunes, and hot Earths have semimajor axes smaller than 0.05 AU and are strongly heated by their parent stars. Tidally locked (or synchronously rotating) planets present the same face to the star at all times (just as the moon presents the same face to the Earth). The permanent day-night sides set up a temperature gradient which may or may not be evened out by atmospheric circulation. The topic of how absorbed stellar energy is circulated in the atmosphere of a tidally locked planet is a complex one. We address it further in Chapter 10.

We limit ourselves to the simplest case: a hot tidally locked planet where the incident stellar energy is absorbed in one layer, instantaneously heats that surface element, and is reemitted. We can solve this problem in analogy with the Lambert sphere scattered light flux ratio developed in section 3.6.1. For a Lambert sphere, the incoming intensity is scattered isotropically. In our ideal case the incident energy heats a surface element and is instantaneously reemitted. Because thermal emission is isotropic, this situation is akin to a Lambert sphere. The normalized phase curve in thermal emission at infrared wavelengths will therefore follow equation [3.58].

Beyond this simplest case, if the exoplanet thermal flux distribution on the planet surface is available from a model atmosphere computation, the thermal IR phase curve can still be calculated from the planet-star flux ratio, where the planet's phase-angle-dependent flux is

$$\mathcal{F}_{\alpha,\mathrm{thermal}}(\nu) = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_{\alpha,\mathrm{thermal}}(\theta,\phi,\nu)\cos\phi\cos^2\theta\, d\theta\, d\phi. \tag{3.60}$$

Here the subscript "thermal" refers to the thermal infrared component of the flux or intensity.

3.7 SUMMARY

We have taken a first, signficant step towards understanding exoplanetary atmospheres. We described the different definitions of temperature, comparing the definitions by using data on stars and planets. We connected temperature to planetary albedo, via energy balance. A careful derivation of the geometric albedo was compared with a derivation of the Bond albedo, in order to understand the historical definitions and their present relevance to actual exoplanet observations. We derived basic planet-to star flux ratios, using the temperatures and albedos. These planet-to-star flux ratios are useful not just for a basic understanding of planetary radiance properties, but to estimate the feasibility of exoplanet discovery and study with ground- or space-based obserations.

REFERENCES

For further reading

A thorough description of planet albedos is:

- Chapter 9 in Sobolev, G. G., 1975, *Light Scattering in Planetary Atmospheres* (New York: Pergamon Press).

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EXERCISES

1. a) Compare the incident energy from stellar radiation to the internal energy for a hot Jupiter exoplanet. Use $T_{\rm eq} = 1600$ K for the planet's equilibrium temperature. Use $T_{\rm eff} = 120$ K for the planet's interior energy (equivalent to Jupiter's interior energy). b) Compare the Earth's received energy from the Sun to its interior energy. The former can be taken as 270 K and the latter is known to be 44×10^{12} W.

2. Derive the radiation constant $\sigma_{\rm R}$ in the Stefan-Boltzmann Law equation [3.6] by integrating the black body over frequency as shown in equation [3.4]. Hint: use the identity

$$\int_0^\infty du \frac{u^3}{e^u - 1} = \frac{\pi^4}{15}. \tag{3.61}$$

3. Derive $T_{\rm eq}$, the equilibrium temperature (equation [3.9]) for a planet on an eccentric orbit.

4. Find five different exoplanets with published $T_{\rm b}(\nu)$. Compute $T_{\rm eq}$ for these planets and compare it to the measured $T_{\rm b}(\nu)$.

5. Show that the fraction of incident energy per unit time scattered at $\alpha = 0$ is $E_{\rm frac} = \frac{1}{\pi} A_{\rm g}$.

6. Show that for a plane Lambert surface, $\Phi_\alpha = \cos\alpha$. Over what angular range is this valid?

7. What is the planet-star flux ratio of a Lambert sphere at $\alpha = 90°$? Use equation [3.58]. Describe conceptually why the flux ratio at $\alpha = 90°$ is less than 1/2 of the flux ratio at $\alpha = 0°$.

8. How does the Lambert sphere phase curve equation [3.58] depend on orbital inclination?

9. Write down an equation for the apparent albedo, introduced in Section 3.4.6. How does the apparent albedo relate to the Bond and spherical albedos?