UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2022

Problem Set 1
Due: Friday, 2022/09/02, at the start of class
This problem set is worth **48 points** (+10 possible bonus points).

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, type the Pset, and submit a printed copy.
Recommended tools for typesetting your problem set are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **Flux From A Nearby Star [8 pts].** Alpha Centauri A is one star in the triple-star Alpha Centauri system (1.3 parsecs away), and it is a fairly similar star to our Sun (for this problem, assume it is identical to the Sun).

   (a) Roughly how many times weaker is the stellar flux from this star (F_*) that hits the Earth, relative to the Solar flux from the Sun that reaches the Earth ($F_\odot$)? [4 pts]

   Solution: Eq. 2.19 of Seager's book tells us that the observed flux F_{obs} on Earth depends on the object's radius, distance, and surface flux F_S as:

   $$F_{obs} = \left(\frac{R}{D}\right)^2 F_S, \tag{1}$$

   Based on what the problem tells us, we can assume that the two stars are the same and emit the same amount of energy (i.e., they have the same luminosity L). From Eq. 2.23 of Seager's book, surface flux $F_S = L/4\pi R^2$. Since the two objects also have the same radii, then their surface fluxes are also identical. So the ratio of incident fluxes is

   $$\begin{aligned} \frac{F_*}{F_\odot} &= \left(\frac{D_\odot}{D_*}\right)^2 \\ &= \left(\frac{1\text{AU}}{1.3\text{pc}}\right)^2 \\ &\approx \boxed{1.4 \times 10^{-11}} \end{aligned}$$

   (Note that we were only given two significant figures in the problem statement, so we only include two here in the solution.)

   (b) Roughly how many times weaker is the stellar flux density from this star ($F_{\nu,*}$) that hits the Earth, relative to the Solar flux density from the Sun that reaches the Earth ($F_{\nu,\odot}$)? [4 pts]

   Solution: All of the same arguments made above apply to the flux density, too. So the answer is exactly the same, $\boxed{1.4 \times 10^{-11}}$.

2. **Define Your Terms [10 pts].**

   (a) Explain why the radiation quantity "Intensity" is constant with distance (in empty space), and why this quantity doesn't behave like we're used to when we think about how brightness changes with distance. [5 pts]

   Solution: As one explanation, you saw above that flux and flux density both follow the inverse square law ($1/d^2$) with distance from a source. How does the specific intensity change with distance? The specific intensity can be described as the flux divided by the angular size Ω of the source, or $I_\nu \propto F_\nu/\Delta\Omega$. Again, flux decreases with distance, proportional to $1/d^2$. What about the angular source size? It happens that the source size also decreases with distance – something $2\times$ farther away looks half as tall and half as wide, so its $\Omega \propto 1/d^2$ as well. As a result, the specific intensity (just another name for surface brightness) is independent of distance.

Our intuition is more used to thinking about things looking fainter as they get further away – again, that's true with flux but not with intensity.

(b) Explain the concept of "solid angle," and how it relates to the more usual angles that we learned about in geometry class. [5 pts]

Solution: Wikipedia defines solid angle as "a measure of the amount of the field of view from some particular point that a given object covers. That is, it is a measure of how large the object appears to an observer looking from that point." We measure it in steradians (sr), which act like two-dimensional radians. Just like radians relate to the size of an arc around a circle of radius $r = 1$, sr relate to the total field of view (or "visual area") related to the surface area of a sphere with $r = 1$. So the entire sky in all directions has $\Omega = 4\pi$ sr; the human eye can see a field of view (at varying resolutions) of roughly 4 sr, or roughly one-third of the total area around you.

3. **Radiative Quantities for the Sun and Earth [14 pts].** Consider two astronomical objects: the Earth with $T \approx 300$ K and the Sun with $T \approx 5800$ K:

(a) Use the Stefan-Boltzmann Law to estimate the surface flux (F, in [W m^{-2}]) of each object. [4 pts]

Solution: Since $F = \sigma T^4$ with $\sigma_{SB} \approx 5.67 \times 10^{-8}$ W m^{-2} s^{-1} K^{-4}, the surface fluxes from these two objects should be

$$\boxed{F_\oplus = \left(5.67 \times 10^{-8}\right)(300K)^4 \approx 460 \text{ W m}^{-2}}. \tag{2}$$

and

$$\boxed{F_\odot = \left(5.67 \times 10^{-8}\right)(5800K)^4 \approx 6.4 \times 10^7 \text{ W m}^{-2}}. \tag{3}$$

That's a lot of radiant energy! This is why people can walk around on the Earth, while they would be immediately incinerated at the surface of the Sun.

(b) Estimate the total luminosity (L, in [W]) of each object. [4 pts]

Solution: Given the surface flux F of an object, we know that its luminosity is given by $L = AF$, where A is its surface area. For a spherical planet or star, we then have:

$$L_\oplus = 4\pi R_\oplus^2 F_\oplus \approx 4\pi\,(6400 \text{ km})^2 \left(460 \text{ W m}^{-2}\right) \approx \boxed{2.4 \times 10^{17} \text{ W m}^{-2}}. \tag{4}$$

and

$$L_\odot = 4\pi R_\odot^2 F_\odot \approx 4\pi\,(700,000 \text{ km})^2 \left(6.4 \times 10^7 \text{ W m}^{-2}\right) \approx \boxed{3.9 \times 10^{26} \text{ W m}^{-2}}. \tag{5}$$

It shouldn't surprise us that the Sun is a lot brighter than a planet!

(c) Use the Planck Blackbody function to calculate and plot the surface brightness (i.e. the intensity, B_ν) for the two objects on the same axes as a function of wavelength. (Note that all plots you make in this class should have the axes and scales labeled, units specified, and can be either linearly or logarithmically scaled on either axis. You can use your favorite program – Python, GNUPlot, Mathematica, or even a spreadsheet program – to make your plots.) For this plot, your plotting range should extend at least from 0.2 to 20 μm, and your Y-axis will need to be logarithmic to show both spectra at the same time. [6 pts]

Solution: Given the Planck function

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT} - 1}, \tag{6}$$

let's define a function for this and then generate the appropriate plot, all using Python:

```
# Import necessary modules
import numpy as np
from pylab import *

# Define a function:
def bnu(T, lam):
```

```
    """Planck function in frequency.

    :INPUTS:
      T : scalar or array
        temperature in Kelvin

      lam : scalar or array
        wavelength in microns [but intensity will be per Hz]

    Value returned is in SI units: W/m^2/Hz/sr
    """
    from numpy import exp

    c  = 299792458 # speed of light, m/s
    h = 6.626068e-34 # SI units: Planck's constant
    k = 1.3806503e-23 # SI units: Boltzmann constant, J/K
    nu = c/(lam/1e6)
    expo = h*nu/(k*T)
    nuoverc = 1./ (lam/1e6)
    return ((2*h*nuoverc**2 * nu)) / (exp(expo)-1)

# Set up wavelength grid:
wavelength_micron = np.linspace(0.1, 50, 10000)

#Calculate blackbodies:
bb_earth = bnu(300, wavelength_micron)
bb_sun = bnu(5800, wavelength_micron)

#Generate figure:
figure()
loglog(wavelength_micron, bb_sun, label='5800 K', color='orange')
loglog(wavelength_micron, bb_earth, label='300 K', color='blue')
xlabel('Wavelength [microns]', fontsize=16)
ylabel('Surface Brightness (Intensity) [W/m2/Hz/sr]', fontsize=12)
xlim(0.1, 50)
ylim(1e-16, 1e-7)
legend()
```

The result is shown in Fig. 1.

4. **Taking it to the Limit [16 pts].** Consider the Planck blackbody function, $B_\nu(T)$.

   (a) Since $e^x \approx 1+x$ when x is small, show that when studying photons at energies much lower than the Wien peak of the blackbody (or equivalently, when $h\nu \ll kT$), the Planck function reduces to the somewhat simpler form $2\nu^2 kT/c^2$. This is called the "Rayleigh-Jeans Limit," and it is usually applicable at radio wavelengths and often in the infrared. [6 pts]

   Solution: Again, we start with

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}. \tag{7}$$

   In the limit that $h\nu \ll kT$, the exponential component goes to:

$$\exp\left(\frac{h\nu}{kT}\right) \to 1 + \frac{h\nu}{kT}, \tag{8}$$

   and so the entire right-hand half of Eq. 7 becomes

$$\frac{1}{e^{h\nu/k_B T} - 1} \to \frac{1}{1 + h\nu/k_B T - 1} \to \frac{1}{h\nu/kT} \to \frac{kT}{h\nu}. \tag{9}$$

This means that the full relation for the Rayleigh-Jeans limit of the Placnk blackbody function then becomes:

$$B_{\nu,RJ}(T) = \frac{2h\nu^3}{c^2}\frac{kT}{h\nu} = \boxed{\frac{2\nu^2 kT}{c^2}}. \tag{10}$$

(b) Generate the same plot as in the previous plot (B_ν vs. wavelength), but now using the Rayleigh-Jeans approximation instead. (If you want, you can even overplot all curves on the same axes to compare them). [6 pts]

Solution:

```
# Import necessary modules
import numpy as np
from pylab import *

# Define a function:
def bnu_rj(T, lam):
   """Rayleigh-Jeans approximation to planck function in frequency.

   :INPUTS:
     T : scalar or array
       temperature in Kelvin

     lam : scalar or array
       wavelength in microns [but intensity will be per Hz]

   Value returned is in SI units: W/m^2/Hz/sr
   """
   from numpy import exp

   c  = 299792458 # speed of light, m/s
   k = 1.3806503e-23 # SI units: Boltzmann constant, J/K
   nu = c/(lam/1e6)
   return (2 * nu**2 * k * T)/c**2

# Set up wavelength grid:
wavelength_micron = np.linspace(0.1, 50, 10000)

#Calculate blackbodies:
rj_earth = bnu_rj(300, wavelength_micron)
rj_sun = bnu_rj(5800, wavelength_micron)

#Generate figure:
figure()
loglog(wavelength_micron, rj_sun, '--', label='5800 K (RJ Limit)',
    color='red')
loglog(wavelength_micron, rj_earth, '--', label='300 K (RJ Limit)',
    color='cyan')
xlabel('Wavelength [microns]', fontsize=16)
ylabel('Surface Brightness (Intensity) [W/m2/Hz/sr]', fontsize=12)
xlim(0.1, 50)
ylim(1e-16, 1e-7)
legend()
```

The result is shown in Fig. 1.

(c) Discuss the similarities and differences between the two curves that you plotted. [4 pts]

Solution: The two blackbody curves have the same overall shape, but one is lower and peaks at longer wavelengths. Similarly, the two Rayleigh-Jeans limit curves have the same slope but different vertical offsets.

At very long wavelengths, the Rayleigh-Jeans approximation comes very close to matching the full Planck curve. But at wavelengths comparable to or shorter than wavelength of peak intensity (roughly, for $\lambda \lesssim 2\lambda_{\rm max}$), the approximation breaks down.

So you would almost never use the RJ approximation for studies at, say, ultraviolet or X-ray wavelengths, but it's very frequently used at radio/submm wavelengths and often in the infrared as well.

5. **BONUS [10 pts].** Prove the Wien Law: i.e., show that the wavelength $\lambda_{\rm max}$ for which $B_\nu(T)$ is a maximum is given approximately by $\lambda_{\rm max} T \approx 3000\ \mu$m K.

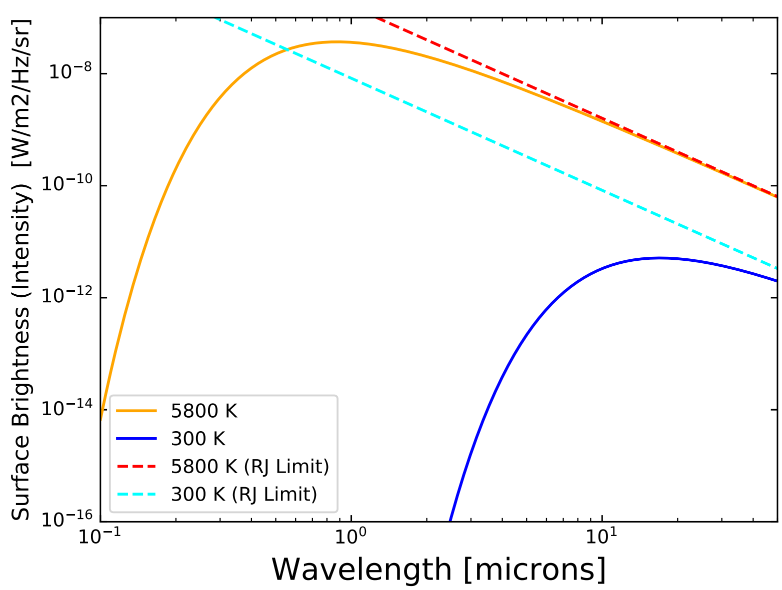

Figure 1: Blackbody spectra (solid curves) and the Rayleigh-Jeans approximations (dashed lines) for the two indicated temperatures.