UNIVERSITY OF KANSAS
Department of Physics and Astronomy
Astrophysics I (ASTR 691) — Prof. Crossfield — Fall 2026

Take-Home Problem Set 1
Due: Thursday, Sep 24, 2026, at the start of class (by 1500).
This problem set is worth **30 points**.

As always, be sure to: show your work, circle or highlight your final answer, list units, use the appropriate number of significant figures, and submit a physical (printed or handwritten) copy.
Recommended tools for typesetting your problem set are either LibreOffice or the LaTeX typesetting system available either by download at `https://www.latex-project.org/get/` or in online-only mode via, e.g., `https://www.overleaf.com/`.

1. **The Glowing Infrared Sky [30 pts]**. Assume you are observing from a ground-based infrared observatory, when unusual weather conditions result in an atmospheric temperature T that constant from ground level way up until the air is too thin to matter any more.

   (a) The condition of hydrostatic equilibrium in an atmosphere (which we will learn about later) implies that in such an atmosphere, the atmospheric pressure varies as

   $$P(z) = P_{\rm ground} e^{-z/H}, \tag{1}$$

   where z is the altitude above the ground, $H \approx 8$ km is the Earth's typical atmospheric scale height (the e-folding scale), and $P_{\rm ground}$ is the pressure at ground level. Show that if the Earth's atmosphere is an ideal gas, that the number density of particles also scales as $n(z) \propto e^{-z/H}$. [3 pts]

   Solution:
   The ideal gas law for astrophysicists is

   $$P = nk_B T. \tag{2}$$

   If T is constant, then $P \propto n$ and so we have

   $$\boxed{n(z) = \frac{P(z)}{kT} = \frac{P_{\rm g}}{kT} e^{-z/H}}. \tag{3}$$

   (b) Water vapor is one of the dominant infrared opacity sources. Download from `https://crossfield.ku.edu/files/h2o_crosssection_lores.csv` a file containing the cross-section (σ_λ, in cm^2) of an H_2O molecule as a function of wavelength (in μm). Plot σ_λ (use a logarithmic vertical axis, to show the full dynamic range). [4 pts]

   Solution:
   The data are plotted in Fig. 1. You could have used a spreadsheet, but here's the Python code I used instead (it looks a bit more elegant and professional):

```
from matplotlib import pyplot as plt
import numpy as np

dataArray = np.loadtxt('h2o_crosssection_lores.csv', delimiter=',',
    skiprows=1)

wavelength = dataArray[:,0] # in microns
crossSec = dataArray[:,1] # in cm^2

plt.figure()
plt.semilogy(wavelength, crossSec, '-c', linewidth=2)
plt.xlabel('Wavelength [micron]', fontsize=16)
plt.ylabel('Cross Section [cm$^2$]', fontsize=16)
plt.minorticks_on()
plt.title('H$_2$O Cross-Section Data', fontsize=18)
```

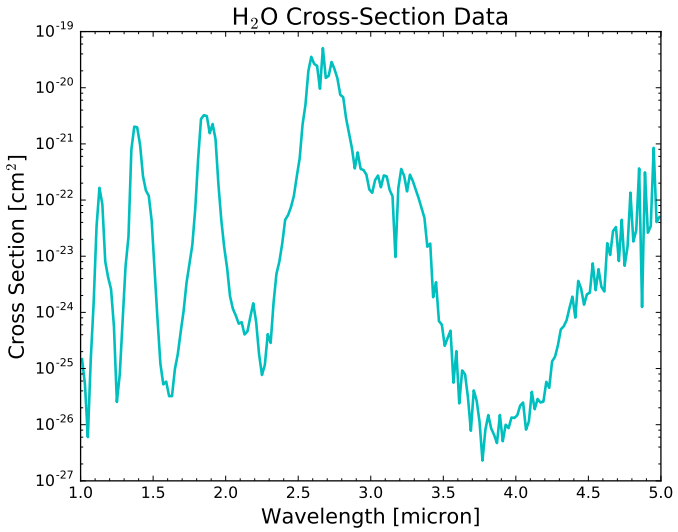


Figure 1: Cross-section data for H_2O, loaded from the provided data file.

(c) If a fraction f of all particles in the atmosphere are water vapor molecules, what is the number density $n_{\rm H2O}(z)$ – i.e., as a function of altitude? Assume that the atmosphere is uniformly mixed throughout. [3 pts]

Solution:

Since at constant temperature $n \propto P$, and we have

$$n(z) = \frac{P_{\rm g}}{kT} e^{-z/H} \tag{4}$$

this means we should expect to have

$$\boxed{n_{H2O}(z) = \frac{f P_{\rm g}}{kT} e^{-z/H}}. \tag{5}$$

(d) If your observatory is at sea level and f=1% and the atmospheric temperature is $T = 274$ K throughout, what is $\langle n_{H2O} \rangle$, the *average* number density of H_2O molecules between 0–8 km? [3 pts]

Solution:

At sea level $P_g = 10^5$ Pa. So we have

$$n_{H2O}(z) = \frac{f P_{\rm g}}{kT} e^{-z/H} \tag{6}$$

$$= \frac{0.01 \times 10^5 \text{ Pa}}{(1.4 \times 10^{-23} \text{ J K}^{-1})(274 \text{ K})} \exp\left[-\frac{z}{8 \text{ km}}\right] \tag{7}$$

$$= \left(2.6 \times 10^{23} \text{ m}^{-3}\right) \exp\left[-\frac{z}{8 \text{ km}}\right] \tag{8}$$

The average value of e^{-x} over $0 < x < 1$ is 0.63, so the average number density over the altitude range $0 < z < H$ is just

$$\boxed{\langle n_{H2O} \rangle = 0.63 \times \left(2.6 \times 10^{23} \text{ m}^{-3}\right) = 1.6 \times 10^{23} \text{ m}^{-3}}. \tag{9}$$

(e) Now assume that from $z = 0$ to $z = H \approx 8$ km the atmosphere is uniform and homogeneous: with H_2O number density $\langle n_{H2O} \rangle$, temperature $T = 274$ K, and thickness 8 km (with only empty space above; we're simplifying to keep things tractable). Using the downloaded file of σ_λ, calculate and plot both the absorption coefficient α_λ and the optical depth τ_λ as a function of wavelength. [6 pts]

Solution:

We should be starting to get familiar with the relations between σ, α, and τ. We have

$$\alpha_\lambda = n\sigma_\lambda = \sigma_\lambda \left(1.6 \times 10^{23} \text{ m}^{-3}\right) \tag{10}$$

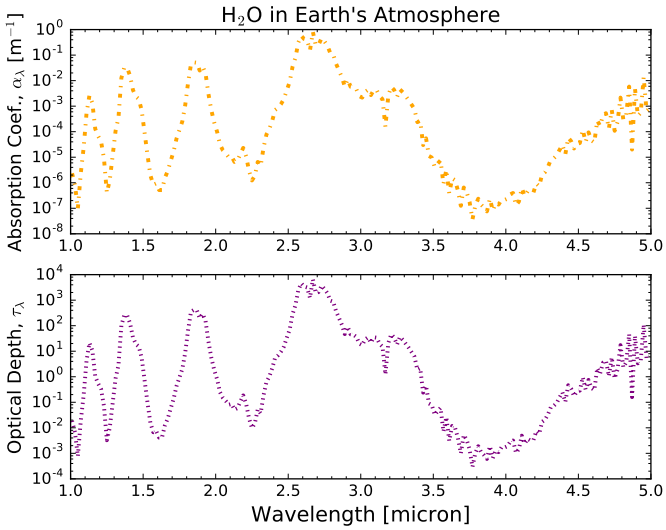


Figure 2: *Left*: Absorption coefficient α_λ and *Right:* total optical depth of the atmosphere, τ_λ. Since we've assumed a homogeneous atmosphere, these (and σ_λ in Fig. 1) look exactly the same (albeit with different units). Nonetheless, note how τ_λ varies strongly with wavelength, from $\ll 1$ to $\gg 1$ depending on the wavelength considered.

and

$$d\tau_\lambda = \alpha_\lambda ds, \tag{11}$$

which for a uniform medium (as in this case) implies that

$$\tau_\lambda = \alpha_\lambda \Delta z = (8000\ \mathrm{m})\,\alpha_\lambda = \left(1.3 \times 10^{27}\ \mathrm{m}^{-2}\right)\sigma_\lambda \tag{12}$$

Both these data are plotted in Fig. 2. Since we've just taken the cross-sectional data in Fig. 1 and multiplied it by a few constants, we shouldn't be surprised that it looks the same! Here's the Python code I used to plot it:

```
from matplotlib import pyplot as plt
import numpy as np

dataArray = np.loadtxt('h2o_crosssection_lores.csv', delimiter=',',
    skiprows=1)

wavelength = dataArray[:,0] # in microns
crossSec = dataArray[:,1] # in cm^2

alpha = crossSec * 1e-4* 1.6e23 # convert from cm2 to m2; result in 1/m
tau = 8000. * alpha # optical depth, unitless

plt.figure()
ax1= plt.subplot(2,1,1)
ax2= plt.subplot(2,1,2)
ax1.semilogy(wavelength, alpha, '-', color='orange', linewidth=2)
ax2.semilogy(wavelength, tau, '-', color='purple', linewidth=2)
ax2.set_xlabel('Wavelength [micron]', fontsize=16)
ax1.set_ylabel('Absorption Coef., $\\alpha_\lambda$ [m$^{-1}$]', fontsize=14)
ax2.set_ylabel('Optical Depth, $\\tau_\lambda$', fontsize=14)
ax1.minorticks_on()
ax2.minorticks_on()
ax1.set_title("H$_2$O in Earth's Atmosphere", fontsize=16)
```

(f) If we make the reasonable assumption that the atmosphere is in LTE and our telescope is pointing at an empty patch of space overhead, calculate and plot (vs. wavelength) the intensity I_λ radiated by the warm

sky. (This sky emission will consequently be an annoying source of background 'noise' in any infrared astronomical measurements). On the same graph, overplot the intensity $B_\lambda(T)$ that would be emitted by a T=274 K blackbody. (Hint: apply the general solution to the radiative transfer equation!) [5 pts]

Solution:

We know from the readings and the course notes that if the source function S_λ is constant, then the general solution to the equation of radiative transfer is

$$I_\lambda(\tau) = I_{\lambda,0}e^{-\tau} + S_\lambda\left(1 - e^{-\tau}\right) \tag{13}$$

Since we're assuming here a constant density, constant temperature, and otherwise homogeneous atmosphere then it's a safe bet that S_λ is constant too.

Furthermore, we know that when a situation is in LTE, we can safely assume that $S_\lambda = B_\lambda(T)$, i.e. the Source function is the Planck blackbody function.

Further-furthermore, if our telescope is pointing toward an empty patch of space then $I_{\lambda,0} = 0$ [1]. So we expect

$$I_\lambda(\tau) = B_\lambda\left(1 - e^{-\tau}\right) \tag{14}$$

with τ calculated in the previous part of the problem.

The result is plotted in Fig. 3. The Python code I used to do this is below.

```
from matplotlib import pyplot as plt
import numpy as np

dataArray = np.loadtxt('h2o_crosssection_lores.csv', delimiter=',',
    skiprows=1)

wavelength = dataArray[:,0] # in microns
crossSec = dataArray[:,1] # in cm^2

alpha = crossSec * 1e-4* 1.6e23 # convert from cm2 to m2; result in 1/m
tau = 8000. * alpha # optical depth, unitless

# Define a Blackbody function:
def blam(T, lam):
   """Planck function in wavelength.

   :INPUTS:
    T : scalar or array
      temperature in Kelvin

    lam : scalar or array
      wavelength in microns [but intensity will be per Hz]

   Value returned is in semi-cgs units: erg/s/cm^2/micron/sr
   """
   from numpy import exp

   c = 29979245800. # cm/s
   nu = c/(lam/1e4)
   h = 6.626068e-27 # cgs units
   k = 1.3806503e-16 # cgs units
   expo = h*nu/(k*T)
   nuoverc = 1./ (lam/1e4)
   return ((2*h*nuoverc**2 * nu)) / (exp(expo)-1) * (nu / lam)
```

[1] This isn't ever precisely true since the Cosmic Microwave Background – the faded afterglow of the Big Bang – is in all directions. But at these wavelengths it isn't a big concern.

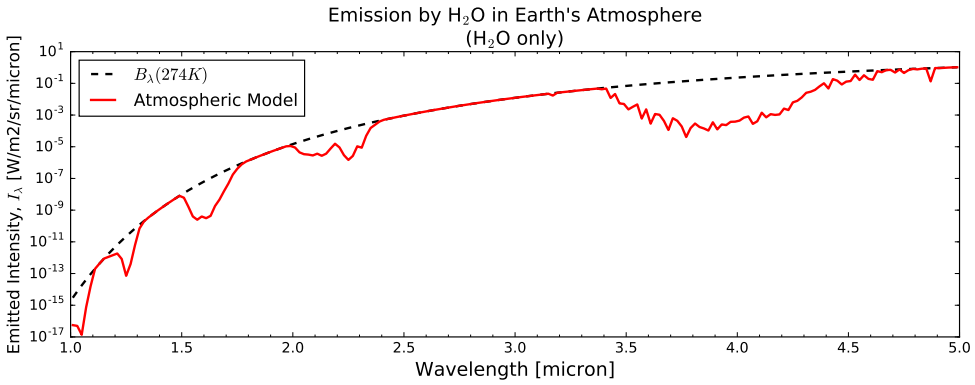


Figure 3: Infrared surface brightness of the empty night sky as seen from the ground, assuming a simple homogeneous model *and* that the atmosphere contains only H_2O.

```
planck274 = blam(274., wavelength)
emission = planck274 * (1. - np.exp(-tau))

plt.figure()
semilogy(wavelength, emission, '-r', label='H2O-only atmospheric model')
semilogy(wavelength, planck274, '--k', label='$B_\lambda (274\ K)$')
ylabel('Sky Brightness, $I_\lambda$ [erg/cm2/s/sr/micron]', fontsize=14)
xlabel('Wavelength [micron]', fontsize=14)
plt.title("Sky Brightness = Emission of Earth's Atmosphere", fontsize=16)
```

(g) Compare your plot with that shown in Fig. 4 — it has slightly different vertical units, but fundamentally it's still I_λ. What features look similar between your plot and this one? What looks different (ignore the difference in units)? Discuss and explain the similarities and differences. [6 pts]

Solution:

Both the real, observed sky brightness in Fig. 4 and our simple model of it in Fig. 3 have much stronger emission at longer wavelengths and weaker emission at shorter λ. Both show a sort of flat, sloping 'plateau' from roughly 2.5–3μm and maybe another at roughly 4.5μm; from our model, we can recognize those flat

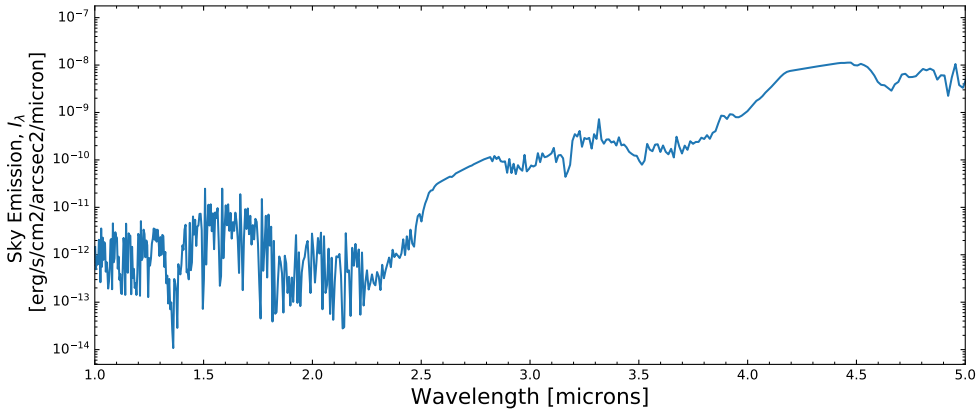


Figure 4: Observed emission spectrum of the glowing infrared night sky over Maunakea Observatory. The vertical units may look a bit unusual, but fundamentally it's still just I_λ.

regions as wavelengths where the Earth's atmosphere is almost totally opaque and so the sky emission is just blackbody-like.

On the other hand, the detailed shapes of the larger, smoother 'chunks' that have been eaten out of the blackbody spectrum look different in the two spectra. This presumably is because the Earth's atmosphere is *not* a vertically uniform slab, nor is it made solely of H_2O. Furthermore, the real model shows lots of quickly-varying features from 1–2μm; these are narrow molecular emission lines in the Earth's atmosphere not caused by H_2O, so they don't show up in our model.