Syllabus – Astronomy 278 – Spring 2011

Basic Goals: This course is seminar on Extrasolar Planets. The goal is to understand what we have learned about them and to assess what are observational possibilities during the next ten years or so. Particular emphasis will be placed on rocky planets although gas giants will receive some consideration.

There is a vast relevant literature. Useful books include: *Planetary Sciences* by de Pater & Lissauer and *Planetary Science: The Science of Planets Around Stars* by Cole & Woolfson, *Astrophysics of Planet Formation* by Armitage, *Cosmochemistry* by McSween & Huss, *Exoplanets*, ed. S. Seager

Grading: The course will be graded Pass/Fail. All enrolled students are expected to make a 1/2 hour presentation on a subject of their choice.

Week 1: Protostellar disks, schematic planet formation models

Week 2: Orbits and orbital interactions; resonance and migration

Week 3: Mass/radius relations

Week 4: Interiors/mantles, cores

Week 4: Snow lines/ices and water

Week 5: Atmospheres/winds

Week 6: Surfaces/oceans

Week 7: Satellites/rings

Week 7: Asteroids/comets

Week 8: Elemental compositions

Week 9: Magnetic fields/ionospheres

Week 10: Biosignatures

Lecture 1 – Overview

The goal of this course is to learn about planets and to think about the next steps in improving our understanding of extrasolar planets. The primary focus will be on solid planets, but we will discuss gas giants as well. We will consider planetesimals as well as planets.

The definition of a planet is non-trivial: witness the fate of Pluto. However, by including both planetesimals and planets in the same course, we will sidestep this ambiguity.

Some representative questions to keep in mind:

- How "normal" is the solar system?
- How extensive and massive are extrasolar planetary systems?
- What is the range of masses, compositions, structures, atmospheres, surfaces and orbital parameters of solid planets?

For decades, it was thought likely that planets are common. Most stars are members of binaries and therefore bound systems occur often. Star form from the gravitational collapse of an interstellar cloud of size $\sim$1 pc to an object with a main-sequence radius of $\sim 10^{-8}$ pc. If the cloud has any initial angular momentum or is torqued by some nearby cloud or star as it collapses, then inevitably the protostar flattens and either fragments or forms a disk. Binary systems have much more angular momentum than the solar system. Currently, in the solar system, most of the angular momentum resides the orbital motions of the planets – mainly Jupiter ($2\ 10^{50}$ g cm^2 s^{-1} vs. 3×10^{48} g cm^2 s^{-1} in the Sun[1]). It is most plausible that there is a substantial amount of angular momentum in every collapse event and that interstellar collapse either leads to multiple stars or a single star plus a disk. The fate of most orbiting disk material likely is to form planets. Disks are commonly detected orbiting pre-main-sequence stars.

An important step in learning about extrasolar planetary systems was the discovery of the infrared excesses of many main sequence stars. Two famous examples are Vega and β Pic. It is clear that the orbiting dust is not leftover from the formation episode, but must, instead, be generated by some larger "parent bodies". The lifetime for Poynting-Robertson drag, t_{PR}, for a spherical particle of radius b and density ρ_s in circular orbit or radius D around a star of luminosity L_* is:

$$t_{PR} = \frac{4\,\pi\,b\,\rho_s\,c^2\,D^2}{3\,L_*} \tag{1}$$

Typical grain lifetimes are less than 10^6 yr, much longer than the main-sequence age of the star. There must be some regeneration of the dust. [There may be other paths that limit the grain lifetime; the derived ages are upper bounds.]

[1]The Sun has probably lost most of its initial angular momentum since younger stars are more active and rotate more rapidly than older stars. Nevertheless, it is likely that in the solar system, most of the angular momentum has resided with the planetary motions.

Recognize that the mass of dust, M_{dust}, orbiting at distance D required to produce an infrared excess is bounded by the requirement that enough light from the central star is absorbed and re-emitted. Therefore:

$$M_{dust} \geq \frac{16\,\pi\,L_{IR}\,\rho_s\,D^2\,b}{3\,L_*} \tag{2}$$

Consequently, if t_{disk} is the lifetime of the disk, then the mass of the parent bodies, M_{PB}, that produce the observed dust is:

$$M_{PB} \;>\; \frac{4\,L_{IR}\,t_{age}}{c^2} \tag{3}$$

Masses up to 0.1 $M_{\oplus}$ are derived: coagulation of Earth-mass equivalents of solid material must commonly occur.

There is considerable evidence that pre-main-sequence stars possess disks. The dust mass is most easily measured from the long wavelength flux. That is for a system at distance D at temperature T_{dust} where the opacity (cm^2 g^{-1}) is denoted by χ_ν, then

$$M_{dust} \;=\; \frac{F_\nu\,D^2}{\chi_\nu\,B_\nu(T_{dust})} \tag{4}$$

By observing at long wavelengths, then:

$$B_\nu(T) \approx \frac{2\,k\,T\,\nu^2}{c^2} \tag{5}$$

Therefore, the results are not especially sensitive to the temperature. The largest uncertainty in this analysis is the dust opacity. Often, the results are provided by assuming a gas to dust ratio and then stating a total disk mass. While quite variable, disks masses of 0.01 $M_\odot$ are common.

A key idea is the Minimum Mass Solar Nebula – the least amount required to have produced the planets in the solar system – about 0.01 $M_\odot$. The surface density, Σ of this nebula apparently varied as $D^{-3/2}$. If so, then most of the mass and angular momentum of the disk were carried in the outer regions.

Poynting-Robertson Drag

Here is a simplified, heuristic calculation for the Poynting-Robertson drag time. Consider a spherical grain of radius b and density ρ_S in a circular orbit at distance D around a star of luminosity L_* and mass M_*. The grain absorbs energy with the rate, $\dot{E}$, of:

$$\dot{E} \;=\; \frac{\pi\, b^2\, L_*}{4\,\pi\, D^2} \;=\; \frac{b^2\, L_*}{4\, D^2} \tag{1}$$

The equivalent mass absorption rate is $\dot{E}/c^2$.

In the frame of reference of the grain, the re-radiated energy is $\dot{E}$ and the mass loss equivalent is $\dot{E}/c^2$. In the grain's frame, the grain remains at rest as it radiates. In the frame of reference of the host star, which moves at speed V_{orb} relative to the grain, the momentum flow carried by the mass equivalent of the light must be $(\dot{E}/c^2)\, V_{orb}$ where V_{orb} is the orbital speed of the grain. For circular motion,

$$V_{orb} \;=\; \left(\frac{G\, M_*}{D}\right)^{1/2} \tag{2}$$

The angular momentum of the grain, J, is:

$$J \;=\; \frac{4\,\pi\,\rho_S\, b^3}{3}\, D\, V_{orb} \;=\; \frac{4\,\pi\,\rho_S\, b^3}{3}\, (G\, M_*\, D)^{1/2} \tag{3}$$

The rate of loss of angular momentum is given by:

$$\frac{dJ}{dt} \;=\; -\frac{\dot{E}}{c^2}\, V_{orb}\, D \;=\; -\frac{b^2\, L_*}{4\, c^2}\left(\frac{G\, M_*}{D^3}\right)^{1/2} \tag{4}$$

Therefore:

$$\frac{4\,\pi\,\rho_S\, b^3}{3}\, (G\, M_*)^{1/2}\, \frac{1}{2\, D^{1/2}}\, \frac{dD}{dt} \;=\; -\frac{b^2\, L_*}{4\, c^2}\left(\frac{G\, M_*}{D^3}\right)^{1/2} \tag{5}$$

or collecting terms:

$$D\, \frac{dD}{dt} \;=\; \frac{1}{2}\, \frac{d\, D^2}{dt} \;=\; -\frac{3\, L_*}{8\,\pi\, c^2\, \rho_S\, b} \tag{6}$$

If $D = D_0$ at $t = 0$, then the solution to this differential equation is:

$$D^2 \;=\; D_0^2 \;-\; \frac{3\, L_*\, t}{4\,\pi\, c^2\, \rho_S\, b} \tag{7}$$

The grain reaches the host star when $D = 0$ or

$$t \;=\; \frac{4\,\pi\, D_0^2\, c^2\, \rho_S\, b}{3\, L_*} \tag{8}$$

Lecture 2 – Planetesimal Formation

The widespread presence of debris disks orbiting main-sequence stars argues strongly that the growth of rocky planetesimals is common. The first two steps are described here. We do not consider the growth of planetesimals into planets. Useful presentations are in Armitage's book and Chiang & Youdin (2010, Ann. Rev. Earth & Planetary Science, 38, 493).

1 Dust to Rocks

Planets form in dusty disks that have formed from the gravitational collapse of interstellar matter. Within the interstellar medium, approximately half or more of the atoms heavier than helium are confined to solid dust particles. Nevertheless, It is important to recognize that the Earth is not simply the result of the agglomeration of interstellar particles. Within the Sun, carbon is the second most abundant heavy element after oxygen; $n(\mathrm{C})/n(\mathrm{O})$ = 0.50 (Lodders 2003, ApJ, 591, 1220). Within the interstellar medium, at least half of this carbon is located within solid grains (Zubko et al. 2004, ApJS, 152, 211). However, in bulk Earth where $n(\mathrm{C})/n(\mathrm{O})$ is between 0.004 and 0.009 (Allegre et al. 2001, Earth and Planetary Science Letters, 185, 49). The bulk composition of the Earth is most easily understood as condensation at a temperature between 1100 and 1200 K – with carbon having been previously vaporized.

If the Earth formed anywhere near 1 AU, then the original solids may have formed within an opaque disk. That is, a blackbody with zero albedo at all wavelengths orbiting at distance D from the the Sun whose radius is R_* and temperature T_*, achieves a mean temperature, T, of

$$T \;=\; \frac{1}{2}\left(\frac{R_*}{D}\right)^{1/2} T_* \tag{1}$$

Even if the young sun is more luminous than the current Sun, the temperature at 1 AU is much less than 1000 K. An opaque disk can have a warm interior if there is an energy source (see, for example, Hartmann's *Accretion Processes in Astrophysics*) as occurs if there is viscous dissipation.

The approximate surface density, Σ, of the solar nebula in the neighborhood of the Earth is $M_\oplus/(\pi D^2)$ or about 8 g cm^{-2}. As a first approximation, the opacity, χ (cm^2 g^{-1}) of a spherical grain of radius b is:

$$\chi \;=\; Q\,\frac{\pi b^2}{(4\pi\,\rho_S\,b^3/3)} \;=\; \frac{3\,Q}{4\,\rho_s\,b} \tag{2}$$

In Mie theory, for many materials, we can take $Q \sim 1$ if $b > \lambda/(2\,\pi)$ where λ is the wavelength of light. The vertical optical depth of the disk from the surface to the midplane,

τ_z is

$$\tau_z = \frac{\chi \Sigma}{2} \tag{3}$$

Therefore, for particles with $b \sim 1$ μm, we expect $\tau \sim 10^4$ and the disk is vertically opaque until the particles are typically more than 1 cm in diameter.

In an active disk with accretion rate, $\dot{M}$, it can be shown that the surface temperature, T_{disk}, is:

$$T_{disk} \approx \left(\frac{3\,G\,M_*\,\dot{M}}{8\,\pi\,\sigma_{SB}\,D^3}\right)^{1/4} \tag{4}$$

where σ_{SB} denotes the Stefan-Boltzmann constant. If all the energy is dissipated in the midplane, then the Eddington approximation to the temperature yields a midplane temperature, T_{mid} of:

$$T_{mid} \approx \left(\frac{3\,\tau}{4}\right)^{1/4} T_{disk} \tag{5}$$

In this analysis, the temperature in the midplane achieves some natural maximum. If too hot, the grains sublimate and there is effectively no opacity and the temperature does not continue to rise.

It seems highly likely that solid particles settle into the midplane of a disk. The vertical component of the gravitational acceleration, g_z, is:

$$g_z = \frac{G\,M_*\,z}{D^3} = \Omega^2\,z \tag{6}$$

where Ω is the angular speed of a particle in a circular orbit. For the Earth's orbit, $\Omega = 2.0 \times 10^{-7}$ s^{-1}. The equation of hydrostatic equilibrium for the gas vertical to the disk is:

$$\frac{dp}{dz} = -g_z\,\rho \tag{7}$$

For an isothermal ideal gas where:

$$p = \frac{\rho}{\mu}\,k\,T \tag{8}$$

Then if the midplane density is ρ_0:

$$\rho(z) = \rho_0\,e^{-z^2/(2\,H^2)} \tag{9}$$

where:

$$H = \left(\frac{k\,T}{\mu}\right)^{1/2} \Omega^{-1} \tag{10}$$

Note that for a gas of molecular hydrogen and helium [$\mu = 3.9 \times 10^{-24}$ g]. Therefore, for gas at 300 K at 1 AU, we expect that $H = 5 \times 10^{11}$ cm. Thus $H << D$, and the disk is "thin" and the gas is gravitationally bound. The surface density of the gas disk is:

$$\Omega = \int \rho\,dz = \sqrt{\pi}\,\rho_0\,H \tag{11}$$

If $\Omega = 1000$ g cm^{-2}, then $\rho_0 \approx 10^{-9}$ g cm^{-3} (much less than the density of the Earth's atmosphere).

A solid particle in a disk reaches a terminal velocity to fall into the midplane. There are two cases of drag forces to consider. Either the particle is smaller than the mean free path, x_{mfp}, of a gas particle (Epstein drag) or larger than the mean free path of a gas particle (Stokes drag).

$$x_{mfp} = \frac{\mu}{\rho\,\sigma_{coll}} \tag{12}$$

If σ_{coll} denotes the collisional cross section which we take equal to the geometric cross section of $\sim 10^{-16}$ cm^2, then we find for the disk region near the Earth that $x_{mfp} \approx 40$ cm so that Epstein drag is the appropriate approximation. In the subsonic regime, the drag force, $F_{drag}(v)$ on a solid sphere moving at speed v is:

$$F_{drag}(v) = \frac{4\,\pi}{3}\,\rho_{gas}\,b^2\left(\frac{8\,k\,T}{\pi\,\mu}\right)^{1/2} v \tag{13}$$

We compute the downward drift velocity by setting:

$$F_{drag} = \frac{4\pi b^3\rho_S}{3}\,g_z \tag{14}$$

We define the settling time, t_{set} as z/v and therefore:

$$t_{set} = \frac{\rho_{gas}}{\rho_S}\,\frac{1}{b}\left(\frac{8\,k\,T}{\pi\,\mu}\right)^{1/2}\Omega^{-2} \tag{15}$$

Note that for particles with $b \sim 1$ cm at 1 AU, the settling time is less than 100 year. Although turbulence can return particles to greater elevations, there is a strong tendency for particles to settle quickly. Note also that since different size particles settle at different rates, there is a tendency for many mutual collisions. Since the drift speed is less than 1 m s^{-1}, the particles are likely to stick to each other and grow.

2 Rocks to Planetesimals

The largest uncertainty in understanding the formation of planets is how do rocks of perhaps ~1 m in diameter grow into planetesimals larger than 1 km in diameter. While there are likely to be plenty of collisions, the challenge is understanding why the rocks do not all accrete into the Sun.

Assume a disk. The net radial force on an element of gas depends both upon the inward gravitational attraction exerted by the central star and an outward push caused by a pressure gradient which is expected to be negative. This radial drift is superimposed upon the inward radial drift of the gas due to viscosity. For circular motion, we can write that:

$$\frac{V_{orb}^2}{D} = \frac{G\,M_*}{D^2} + \frac{1}{\rho_{gas}}\frac{dp}{dD} \tag{16}$$

For an order of magnitude estimate, we take:

$$\frac{1}{\rho_{gas}}\frac{dp}{dD} \sim \left(\frac{k\,T}{\mu\,D}\right) \tag{17}$$

If the inward radial drift velocity just scales as the difference between the gas circular speed and the rock circular speed, then the time to accrete onto the Sun scales as $D/(V_{rock} - V_{gas})$ which could be shorter than 100 years. For "rocks" that are large enough – bigger than 1 km, the gas drag becomes unimportant and this process does not lead to the infall of the rock.

Various solutions to this problem have been suggested. One idea is that asteroids were "born big" (Morbidelli et al. 2009, Icarus, 204, 558). They write about their model, "This supports the idea that planetesimals formed big, namely that the size of solids in the proto-planetary disk jumped from sub-meter scale to multi-kilometer scale, without passing through intermediate values."

Lecture 3 – Orbits

The Kepler problem for a planet in orbit around a host star is famously solved analytically. The key physical parameters are the total (negative) energy of the planet, E and angular mometum, J. The key orbital parameters are the semi-major axis, a, and the eccentricity, e. If the mass of the host star, M_* dominates over the mass of the planet, M_p, then:

$$a = \left|\frac{G\,M_*\,M_p}{2E}\right| \tag{1}$$

or:

$$E = -\frac{G\,M_*\,M_p}{2\,a} \tag{2}$$

The energy depends only upon the semi-maor axis; not the eccentricity. Also, for an elliptical orbit,

$$e^2 = 1 + \frac{2\,E\,J^2}{M_p^3\,M_*\,G} \tag{3}$$

or:

$$J^2 = M_p^2\,M_*\,G\,a\,(1 - e^2) \tag{4}$$

In many circumstances, we ignore General Relativity and assume that the star is a spherical point mass. There are, however, important exceptions. The orbit of a satellite around the Earth is markedly affected by the Earth's equatorial bulge. Using artificial satellites to measure the higher moments of the gravitational potential of a massive planet like Jupiter is a very valuable constraint on that planet's interior structure.

In the solar system, the orbits or the planets are approximately coplanar and the eccentricites are"small". Note, however, that these parameters change by mutual perturbations. For example, we can write that:

$$e = \sqrt{1 + J^2/J_0^2} \tag{5}$$

where J_0 is the initial angular momentum. If the energy is constant, then

$$\frac{\partial e}{\partial J} = \frac{\sqrt{1 - e^2}}{J_0\,e} \tag{6}$$

If e is small, then even small values of ∂J can lead to appreciable changes in ∂e. This is important for the Milankovich cycles in driving the Earth's climate where e varies from its current value of 0.017 to a low of 0.005 and a high of 0.058 (Varadi et al. 2003, ApJ, 592, 620). Note that the annual insolation, I, onto Earth is:

$$I = \int \frac{L_*}{4\pi\,D^2}\,dt = \frac{L_*}{4\pi\,M_p}\int\frac{d\phi}{J} \tag{7}$$

where ϕ is the angular location of the Earth in polar coordinates and we use:

$$J = M_p D^2 \frac{d\phi}{dt} \tag{8}$$

The transition from stability to chaos in orbital architectures is investigated numerically and described by the Lyapunov time, when separations in phase space can grow exponentially.

Resonances can lead both to stable and unstable orbits. Stable orbits are located at the Lagrangian points. In the restricted three body problem, we assume co-planar motion with two gravitating bodies, M_1 and M_2 moving about the center of mass in circular orbits and separated by each other by a distance, a. There are five locations in the co-rotating frame of reference where the first derivative of the pseudo-potential is zero. body is a test particle. However, three of these locations are saddle points and unstable. Two of these locations are stable when $M_1 > 25$ M$_2$ – and probably, for example, account for the large numbers of Trojan asteroids since the Sun is more massive than Jupiter.

Another important example of a location of stable orbits is the Hill sphere of radius, R_H, where

$$R_H = \left(\frac{M_2}{3(M_1 + M_2)}\right)^{1/3} a \tag{9}$$

All known bound satellites to planets in the solar system lie within this zone which is, effectively, the same as the tidal radius.

An example of an unstable resonance is the Kirkwood gaps in the asteroid belt. There is a lack of asteroids at periods that are harmonic with Jupiter's orbital period. Most notably, for example, there are few asteroids with periods of 3.95 yr or with $a = 2.5$ AU. A plausible description of how this occurs is given by Yoshikawa (1989, A&A, 213, 436). In his calculations, the eccentricity of an asteroid with a semi-major axis of 2.5 AU would be pumped up by Jupiter. This asteroid would then get close enough to Mars to be ejected out of its orbit.

The orbital distribution of asteroids and Kuiper Belt Objects are powerful tools to reconstruct the history of the solar system. While N-body simulations are required for this task, there are few qualitative points we can make. The asteroids are dynamically "hot". That is, the distribution of inclinations of their orbits is much greater than the distribution of the inclinations of the planets. The only plausible explanation for this result is that the asteroids have been dynamically stirred. Similarly, the distribution of inclinations of the Kuiper Belt Objects shows There is both the "classical belt" of objects that are not scattered and the"dynamically" hot population.

Lecture 4 – Migrations

Orbits change because of interactions. A good example to start is the Earth-Moon system. The tides imposed on the Earth by the Moon lead to a spin down of the Earth's rotation – our day becomes longer. As a result, the Earth loses angular momentum and the Moon gains orbital angular momentum. Consequently, the mean distance to the Moon increases with time since:

$$J^2 \;=\; M_{Moon}^2\, M_{Earth}\, G\, a\, (1\,-\,e^2) \tag{1}$$

[Here, the correction for moving to the center of mass is neglected.] The measured mean distance between the Earth and the Moon increases at 3.8 cm yr^{-1} – a rate which can be easily measured with lunar ranging. Since the Earth-Moon distance is 3.84 $\times$ 10^5 km, then at the current rate, even ignoring the increase in torque when the Moon was closer, the Earth and Moon would have been "touching" about 1.0 $\times$ 10^{10} yr ago, barely consistent with the age of the Earth. However, there is no compelling reason to argue that the tidal friction has been constant with time since the surface feature of the Earth have changed because of continental drift.

Ignoring the Moon orbital eccentricity, since e = 0.055, then:

$$\frac{dJ}{dt} \;=\; \frac{M_{Moon}}{2}\left(\frac{G\,M_{Earth}}{a}\right)^{1/2}\frac{da}{dt} \tag{2}$$

Numerically, dJ/dt = 4.5 $\times$ 10^{23} g cm^{-2} s^{-2}. The dissipated tidal energy on the Earth, $\dot{E}$, equals $\dot{J}\omega$ where ω is the angular velocity of the Earth's rotational spin or 7.3 $\times$ 10^{-5} s^{-1}. Consequently, the dissipated tidal energy is 3.3 $\times$ 10^{19} erg s^{-1}.

The discovery of "hot" Jupiters orbiting main-sequence G-type stars lends strong support to the theory that planets migrate. A Jupiter-mass planet with a 4 day period lies at a distance of 7.4 $\times$ 10^{11} cm from the host star. For a disk where Σ varies as $D^{-3/2}$, then the enclosed mass varies as $D_{out}^{1/2}$. Thus, if the disk of a system with a hot Jupiter extended to 100 AU, the enclosed disk mass would have been 0.05 M$_{\odot}$ – larger than most observed disks. It is therefore argued that the planet migrated inwards. The basic idea is that the planet and the disk in which it formed exchange angular momentum and therefore the planet can move either inwards or outwards, depending upon the net transfer. "Type I" migration occurs when a disk maintains its essential character and the planet moves through it. "Type II" migration occurs when the planet opens a gap in the disk.

Consider a highly idealized "impulse approximation" description of type I migration. Consider a blob of gas initially in circular orbit at orbital radius $(b + a)$ beyond the major planet which moves in a circular orbit of radius a. In the frame of reference of the planet, the gas has impact parameter, b, and initial speed Δv and

$$\Delta v \approx \frac{1}{2}\,\Omega_p\, b \tag{3}$$

The component of speed perpendicular to the initial path, $v_\perp$, acquired by the gas blog is:

$$v_\perp \approx \frac{2\,G\,M_p}{b\Delta v} \tag{4}$$

In the planet's frame, the total kinetic energy of the scattered blob is unchanged. Therefore:

$$(\Delta v)^2 \;=\; v_\perp^2 \;+\; (\Delta v \;-\; \delta v_\parallel)^2 \tag{5}$$

Consequently:

$$\delta v_\parallel \approx \frac{v_\perp^2}{2\,\Delta v} \approx \frac{2\,G^2 M_p^2}{b^2 \Delta v^3} \approx \frac{16\,G^2\,M_p^2}{b^5 \Omega_p^3} \tag{6}$$

In the Sun's frame of reference, the blob gains in parallel speed and therefore increases its angular momentum and moves outwards. Consequently, the planet moves inwards. This means that even though the action of gravity is attractive, the planet moves away from the outer ring of material.

The change in angular momentum of the planet, Δj, per unit mass of the gas is:

$$|\Delta j| \;=\; \delta v_\parallel \, a \;=\; \frac{2\,G^2\,M_p^2\,a}{b^2 \Delta v^3} \;=\; \frac{16\,G^2\,M_p^2\,a}{b^5\,\Omega_p^2} \tag{7}$$

The total amount of angular momentum transferred, ΔJ, from a ring of radius b and radial thickness db is $\Delta j\,dm$ where dm denotes the mass in the ring that is scattered and therefore:

$$dm \;=\; 2\pi\,a\,\Sigma\,db \tag{8}$$

where Σ is the surface density of the disk. Let dt denote the time interval that it takes for the ring to pass around the planet in the planet's frame of reference. Then:

$$dt \;=\; \frac{2\pi}{\Omega_p \;-\; \Omega} \tag{9}$$

Because Ω varies as $a^{-1.5}$, and because we wish to compute dt^{-1}, then:

$$dt^{-1} \;=\; \frac{3\,\Omega_P\,b}{4\,\pi\,a} \tag{10}$$

Finally:

$$\frac{d\Delta J}{dt} \;=\; -\frac{\Delta j\,dm}{dt} \;=\; -\frac{24\,G^2\,M_p^2\,a\,\Sigma\,db}{\Omega_p^2\,b^4} \tag{11}$$

Integrating over all possible values of b and assuming a minimum value, b_{min}, then:

$$\frac{dJ}{dt} \;=\; -\frac{d\Delta j\,dm}{dt} \;=\; -\frac{8\,G^2\,M_p^2\,a\,\Sigma}{\Omega_p^2\,b_{min}^3} \tag{12}$$

This expression is a factor of 27 (3^3) greater than in Armitage's book. All I can see is that he uses a value of Δv a factor of 3 larger than I do. However, I cannot understand why that would be the case.

In any case, we can write that:

$$J \;=\; M_p \, a^2 \, \Omega_p \tag{13}$$

In this case, the migration time, $t_{migrate}$, is:

$$t_{migrate} \;=\; \frac{J}{|dJ/dt|} \;=\; \frac{\Omega_p^3 \, a \, b_{min}^3}{8 \, G^2 \, M_p \, \Sigma} \tag{14}$$

With:

$$\Sigma \;=\; \Sigma_0 \left(\frac{a_0}{a}\right)^{3/2} \tag{15}$$

and:

$$\Omega_p \;=\; \left(\frac{G \, M_*}{a^3}\right)^{3/2} \tag{16}$$

and

$$b_{min} \;=\; \epsilon \, a \tag{17}$$

Then:

$$t_{migrate} \;=\; \frac{M_*^{3/2} \, a \, \epsilon^3}{8 \, G^{1/2} \, M_p \, \Sigma_0 \, a_0^{1.5}} \tag{18}$$

While the time scale is sensitive to ϵ, it may be less than 1 Myr and migration can be very important. Note that migration is most important for massive planets.

Lecture 5 – Mass/Radius Relations

A planet's mass and radius can be independently measured and used to infer the mean density and therefore average internal composition. Interpreting the data is not always easy.

One complication is that planets are not spherical. The Earth has a small bulge while Jupiter is notably flattened. However, at the moment, the higher moments of planetary mass distributions cannot be measured, so we focus on estimates of the radius.

For objects whose size is not dominated by gravity, the interpretation of the mean density may not be simple. Consider a simple Young's modulus model. Remember, that Y relates the stress (the force per unit area or p) to the strain, the fractional contraction (dl/l) so that

$$Y = \frac{p}{dl/l} \tag{1}$$

For an isotropic material, the relation between Young's modulus and the bulk modulus, B is:

$$B = \frac{Y}{3(1 - 2\nu)} \tag{2}$$

where ν is (dimensionless) Poisson's ratio. In typical rocks in the Earth, ν ranges between 0.1 and 0.2 (Turcotte & Schubert *Geodynamics*). [Note that values of the bulk modulus can be derived from the binding energy/nucleus and the description of the potential energy of the nuclei within the material.] The ultimate tensile strength of a material is usually much smaller than the Young's modulus which only applies to the linear regime.

Consider a solid column of solid material with density ρ. If z is measured downward from the top, then the pressure as a function of depth is:

$$\frac{dp}{dz} = \rho\, g \tag{3}$$

or:

$$p = \rho\, g\, z \tag{4}$$

The increment of vertical stress, dl/dz at distance z is

$$\frac{dl}{dz} = \frac{p}{Y} = \frac{\rho\, g\, z}{Y} \tag{5}$$

The total vertical contraction, l, of the slab at depth h is:

$$l = \int_0^h dl = \int_0^h \frac{\rho g\, z}{Y}\, dz = \frac{\rho\, g\, h^2}{2\, Y} \tag{6}$$

For rocks in the Earth, a typical value of Y is 10^{11} Pa (Pascal) or about 10^{12} g cm^{-1} s^{-2}. For rocks with a mean density of 3 g cm^{-3}, then $l \sim 0.01\, h$ for a depth of $\sim$ 700 km.

Consider a homogeneous rocky self-gravitating planet. Then

$$g(R) \;=\; G\,\frac{4\,\pi}{3}\,\rho\,R \tag{7}$$

If R_0 is the radius of the planet then the internal pressure is:

$$p = \frac{2\,\pi}{3}\,\rho^2\,G\left(R_0^2\;-\;R^2\right) \tag{8}$$

If T denotes the tensile strength of a material (when the linear approximation for employing Young's modulus fails), then gravity dominates when $p > T$ which occurs at the center of the planet when:

$$R_0 \;>\; \left(\frac{3\,T}{2\,\pi\,\rho^2\,G}\right)^{1/2} \tag{9}$$

Values of T range greatly but 100 MPa or 10^9 g cm^{-1} s^{-2} are common. If so, then self-gravity dominates when R_0 is greater than $\sim$300 km. Interestingly, Baer et al. (2011, AJ, in press) argue that asteroids as large as 300 km in diameter have significant porosities and should be considered as loose aggregates. Larger asteroids have small to negligible porosities.

In general, we can write for the small compression of a material that:

$$\frac{\Delta\rho}{\rho} \;=\; \frac{\Delta p}{B} \tag{10}$$

The compression becomes quite important for p comparable to $0.1B$. Substituting B for T in the above equation, then for $B = 10^{12}$ g cm^{-1} s^{-2} [the bulk modulus in the Earth's crust], we see that compression becomes quite important in the interior of a planet with $R_0 > 3000$ km. Mars with a radius of 3400 km has a mean density of 3.9 g cm^{-3} is not very compressed; Earth with a radius of 6400 km has a mean density of 5.5 g cm^{-3} and is substantially compressed.

The general solution for the density as a function of pressure, p, from initial conditions ρ_0 and p_0 is:

$$\rho \;=\; \rho_0\exp\int_{p_0}^{p}\frac{dp}{B} \tag{11}$$

If we know $B(p)$ then we can compute the mass/radius relationship for a massive planet. The value of B depends upon the material and also the temperature. For low pressures, B is independent of p. As p approaches B, then B becomes larger. As a first approximation:

$$B \;=\; \alpha\,p \tag{12}$$

If so, then for large p,

$$\rho \;=\; \rho_0\left(\frac{p}{p_0}\right)^{1/\alpha} \tag{13}$$

or:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\alpha} \tag{14}$$

This variation of pressure with density is sometimes described as a polytrope where

$$\alpha = 1 + \frac{1}{n} \tag{15}$$

The solution to the mass/radius relationship can be approximated as that for a polytrope. We can show that for a polytrope, the radius, R, varies as M as:

$$R \propto M^{(1-n)/(3-n)} \propto M^{(\alpha-2)/(3\alpha-4)} \tag{16}$$

For the interior of the Earth, $\alpha \approx 4$ or 5. Correspondingly, we expect that R varies as $M^{0.25}$ or $M^{0.27}$.

Valencia et al. (2007, ApJ, 665, 1413) and Sotin et al. (2007, Icarus, 191, 337) have discussed mass/radius relations for rocky planets. For rocky planets more massive than the Earth, they find that R varies as $M^{0.26}$. and $M^{0.27}$, respectively. According to Rogers & Seager (2010, ApJ, 712, 974), the mass and radius of a planet usually is not sufficient to infer the bulk composition – there are too many free parameters. However, with a mass of 4.6 $M_{\oplus}$ and a mean density of 8.8 g cm^{-3}, the bulk composition of Kepler-10b likely is mostly rocky (Batalha et al. 2011, ApJ, 729, 27).

Lecture 6 – Mass/Radius Relations – Giant Planets

The Equation of hydrostatic equilibrium is:

$$\frac{dp}{dr} \;=\; -g\,\rho \;=\; -\frac{G\,M\,\rho}{r^2} \tag{1}$$

The equation that relates the mass in a spherical shell to its density is:

$$\frac{dM}{dr} \;=\; 4\,\pi\,r^2\,\rho \tag{2}$$

A useful first approximation to the Equation of State is that the gas behaves as a "polytrope" of index n such that

$$p \;=\; K\,\rho^{1\,+\,1/n} \tag{3}$$

For example, an ideal monatomic gas has $n = 1.5$ which re-expresses the relationship that p varies as $\rho^{5/3}$.

The above three equations can be re-written. First we use the polytrope relationship for p to write:

$$K\,(1\,+\,1/n)\,\rho^{1/n}\,\frac{d\rho}{dr} \;=\; -\frac{G\,M\,\rho}{r^2} \tag{4}$$

This Equation can be re-written to:

$$(n\,+\,1)\,K\,r^2\,\frac{d\rho^{1/n}}{dr} \;=\; -G\,M \tag{5}$$

Taking the derivative with respect to r of both sides of the above Equation, then:

$$-\frac{dM}{dr} \;=\; -4\,\pi\,r^2\,\rho \;=\; \frac{d}{dr}\left(\frac{(n\,+\,1)\,K}{G}\,r^2\,\frac{d\rho^{1/n}}{dr}\right) \tag{6}$$

It is useful to re-express this Equation in dimensionless coordinates. If ρ_c denotes the central density, then we write:

$$\rho \;=\; \rho_c\,\theta^n \tag{7}$$

and

$$r \;=\; \alpha\,\xi \tag{8}$$

With these substitutions, we derive the Lane-Emden Equation for a polytrope in dimensionless variables:

$$\frac{1}{\xi^2}\,\frac{d}{d\xi}\left(\xi^2\,\frac{d\theta}{d\xi}\right) \;=\; -\theta^n \tag{9}$$

where

$$\alpha \;=\; \left(\frac{(n\,+\,1)\,K}{4\,\pi\,G}\,\rho_c^{([1/n]\,-\,1)}\right)^{1/2} \tag{10}$$

Note that for all values of n, α has the dimensions of a length.

The solution to the differential Equation depends upon the boundary conditions. By definition, we have $\theta = 1$ at $\xi = 0$. Furthermore, by expanding the Lane-Emden Equation, we see that:

$$\frac{d^2\xi}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = -\theta^n \tag{11}$$

Thus at $\xi = 0$, we expect that $d\theta/d\xi = 0$.

The solutions to the Lane-Emden Equation can be found analytically for $n = 0$, 1 and 5. The most interesting case is for $n = 1$. Set $\theta = y/\xi$. In this case, after some algebra, the Equation becomes:

$$\frac{d^2y}{d\xi} = -y \tag{12}$$

With the boundary conditions, then:

$$y = \sin\xi \tag{13}$$

or:

$$\theta = \frac{\sin\xi}{\xi} \tag{14}$$

The radius of the star is found at $\xi = \xi_1 = \pi$

An important set of results is the relationship between the mass and radius of the star. We write that:

$$R = \alpha\,\xi_{max} \propto \rho_c^{(1-n)/(2\,n)} \tag{15}$$

While:

$$M = \int_0^R 4\,\pi\,r^2\,\rho\,dr = \alpha^3\,\rho_c \int_0^{\xi_{max}} 4\pi\,\xi^2\theta^n\,d\xi \tag{16}$$

Therefore:

$$M \propto \rho_c^{(3-3n)/2n+1} \propto \rho_c^{(3-n)/2n} \tag{17}$$

Therefore:

$$R \propto M^{(1-n)/(3-n)} \tag{18}$$

It can be shown that for all fully, ionized, degenerate gas at 0 K,

$$p = \frac{\pi^{4/3}\,3^{2/3}}{5}\,\frac{\hbar^2}{m_e\overline{\mu}^{5/3}}\,\rho^{5/3} \tag{19}$$

where m_e is the mass of an electron and $\rho = \overline{\mu}\,n_e$. Therefore, in this important case $n = 1.5$, and R varies as $M^{-1/3}$. The more massive a star, the smaller it is predicted to be. This relationship holds for white dwarfs.

For giant planets, the internal temperature is important in the Equation of State. A standard reference for the interior conditions is Saumon et al. (1995, ApJS, 99, 713). In the simplest approximation, p varies as ρ^2 and R is independent of M. However, the existence of hot Jupiters shows that this view is too simple. The interiors of giant planets are convective and therefore the matter has a uniform entropy per gram. The value of this entropy depends upon the thermal history of the giant planets. It is important to remember that Jupiter in fact radiates more energy than it receives from the Sun – it is self-luminous.

Fortney, Baraffe & Militzer (2010, astro-ph) have written a recent review addressing these issues. An interesting model to explain the existence of hot Jupiters is the "mechanical greeehouse" of Youndin & Mitchell (2010, ApJ, 721, 1113). A recent unconventional hypothesis is that the hot Jupiters are created during the merger of two close binaries (Martin & Spruit 2011, astro-ph). Batygin et al. (2011, astro-ph) assert that all can be explained by Ohmic dissipation of current loops created in the deep interior as the high speed winds at the surface (containing some ions) move through the planet's magnetic field.The jury is still out as to what is going on!

Because of tidal interactions with their host stars, hot Jupiters are doomed to spiral inwards and be accreted in $\sim 10^9$ yr (Jackson et al. 2009, ApJ, 698, 1357). An interesting prediction of such models is that some relatively old main-sequence stars should be spinning rapidly. For example, the rotational period of a star like the Sun could be decreased from 30 days to 5 days.

Lecture 7 – Interior Temperatures – Rocky Planets

The interior of the Earth is much hotter than its surface. We expect other rocky planets also to hot in their interiors. One goal is to understand how this occurs and its implications.

Within a solid, heat flows by conduction. For a spherically symmetric body in a steady state, we write for the temperature, T, that:

$$\frac{\partial T}{\partial t} = \frac{1}{R^2}\frac{d}{dR}\left(R^2\kappa\frac{dT}{dR}\right) + \frac{Q}{\rho\, c_V} \tag{1}$$

where κ is the thermal diffusivity (cm^2 s^{-1}), Q is the rate of heat production (erg cm^{-3} s^{-1}), ρ is the mass density (g cm^{-3}) and c_V is the specific heat at constant volume (erg g^{-1} K^{-1}). We relate the thermal diffusivity to the thermal conductivity, k, (erg s^{-1} cm^{-1} K^{-1}) by the expression:

$$k = \kappa\, c_V\, \rho \tag{2}$$

Ghosh & McSween (1998, Icarus, 134, 187) note that c_V should be used instead of c_p although numerically, the difference for solids is usually small. This expression is derived from the basic heat conduction equation (see [7] below) and the definition of specific heat as the heat capacity/g and the $dQ = C\, dT$ where C is the heat capacity. The underlying idea is that the change in temperature with respect to time in any shell depends upon both the heating rate within the shell and the net flow of heat through the shell.

In a steady state, we can re-write the above expression as:

$$k\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dT}{dR}\right) = -\rho\, H \tag{3}$$

where the heating H (erg g^{-1} s^{-1}) is:

$$H = \frac{Q}{\rho} \tag{4}$$

The solution to this differential equation in the case that $H\,\rho$ is uniform throughout the planet is:

$$T = -\frac{\rho H}{6\,k}\,R^2 + \frac{C_1}{R} + C_2 \tag{5}$$

where C_1 and C_2 are determined by the boundary conditions. Since the temperature is finite at the origin, we set $C_1 = 0$. If $T = T_0$ at the outer boundary where $R = R_0$, then the solution is:

$$T = T_0 + \frac{\rho H}{6\,k}\left(R_0^2 - R^2\right) \tag{6}$$

In any environment, the heat flux, F (erg cm^{-2} s^{-1}) is:

$$F = -k\,\frac{dT}{dR} \tag{7}$$

Therefore:

$$F \;=\; \frac{1}{3}\,\rho\,H\,R_0 \tag{8}$$

This expression is what we would expect if the total energy flow out of the planet balances the total power generated in the interior:

$$4\,\pi\,R_0^2\,F \;=\; \frac{4\pi}{3}\,R_0^3\,\rho\,H \tag{9}$$

Therefore:

$$\rho\,H \;=\; \frac{3\,F}{R_0} \tag{10}$$

According to Turcotte & Schubert, the mean surface flux of energy is 87 erg cm^{-2} s^{-1}. [The heat flux through the oceans is greater than the heat from through the continental crust; the Earth is a complex environment.] This is approximately 10^{-4} of the solar constant of 1.4×10^6 erg cm^{-2} s^{-1}. The Earth receives much more energy from the Sun than it generates internally. The fraction of the heat flow from the Earth derived from radioactive decays is the "Urey number". Turcotte & Schubert argue that this number is 0.8; other authors prefer 0.5. That is, there could be residual heat trapped within the Earth from its original formation which proceeded by a series of collisions of planetary embryos.

Note that averaged over the Earth, the derived value of H is 7.4×10^{-8} erg g^{-1} s^{-1}. The major sources of radioactive decay in the planet are ^{238}U, ^{235}U, ^{232}Th, and ^{40}K. These elements are probably found mostly in the crust and the mantle; in fact the fraction in the crust is greater than in the mantle. A typical thermal conductivity of rock is 3×10^5 erg cm^{-1} K^{-1}. The maximum temperature computed at the center of the Earth is:

$$T_{max} \approx \frac{F\,R_0}{2\,k} \tag{11}$$

Consequently, the computed temperature at the center of the Earth is 91,000 K. This is much higher than usually modeled for a few reasons. First, as the matter becomes hot enough, it can flow and convection transfers heat more efficiently than conduction. Also, the heating is largely confined to the outer regions of the planet because the radioactive elements are confined to the outer region. One easy way to describe convection is to introduce the Nusselt number which is the ratio of heat flow in convection compared to the heat flow in conduction. Values in the interior of the Earth of the Nusselt number can be about 20 so the actual temperature is much lower than predicted if conduction alone transported heat.

For a homogeneous planet, we can compute the maximum temperature as:

$$T_{max} \;=\; \frac{\rho H\,R_0^2}{6\,k} \tag{12}$$

If R_0= 100 km, $\rho = 3$ g cm^{-3} and H and k taken from the current Earth, then $T_{max} = 13$ K and radioactive heating is not important. For larger objects, the heating could be important. Also, for younger objects where the radioactive decays occurred more frequently, there could be more heating.

A slightly more sophisticated thermal model is to assume all the radioactive heating is confined to a shell with radius between R_b and R_0. We can take $dT/dR = 0$ at $R = R_b$ so that there is no net flux of heat at this boundary. In this case,

$$C_1 \;=\; -\frac{R_b^3 \rho\, H}{3\, k} \tag{13}$$

and

$$C_2 \;=\; T_0 \,+\, \frac{\rho\, H\, R_0^2}{6\, k} \,+\, \frac{\rho\, H\, R_b^3}{3\, k\, R_0} \tag{14}$$

Therefore in the interval $R_b < R < R_0$:

$$T(R) \;=\; T_0 \,+\, \frac{\rho\, H}{6\, k}\left(R_0^2 \,-\, R^2\right) \,+\, \frac{\rho\, H\, R_b^3}{3\, k}\left(\frac{1}{R_0} \,-\, \frac{1}{R}\right) \tag{15}$$

for $R < R_b$, then in this approximation the temperature is constant. The flow of energy out of the planet can be used to fix $\rho\, H$ by conservation of energy:

$$4\,\pi\, R_0^2\, F \;=\; \frac{4\,\pi}{3}\,\rho\, H\,\left(R^3 \,-\, R_b^3\right) \tag{16}$$

or:

$$\rho\, H \;=\; \frac{3\, F\, R_0^2}{R_0^3 \,-\, R_b^3} \tag{17}$$

In this case, the maximum temperature is reached at $R = R_b$ where

$$T_{max} \approx \frac{F\, R_0}{2\, k}\left(\frac{R_0(R_0^2 \,-\, R_b^2)}{R_0^3 \,-\, R_b^3} \,+\, 2\frac{R_b^3}{R_0^3 \,-\, R_b^3}(1 \,-\, \frac{R_0}{R_b})\right) \tag{18}$$

In the approximation that the heating is confined to a thin shell so that $R_b = R_0 \,-\, \Delta R$, then after some algebra:

$$T_{max} \approx \frac{F\, \Delta R}{2\, k} \tag{19}$$

Lecture 8 – Differentiation

A remarkable feature of the Earth is that it is layered with a core, mantle and crust. This differentiation requires that the Earth be hot enough that matter becomes fluid and the heavier material can sink to the center. Depending upon the amount of early radioactivity, differentiation may occur in bodies as small as perhaps 300 km in radius. That is, the current heating rate within the Earth is $H = 7.4 \times 10^{-8}$ erg g s^{-1}. However, if the half-life of a radioactive isotope is $t_{1/2}$, then the past concentration, C, varies as:

$$C = C_0 \exp\left(\frac{t \ln 2}{t_{1/2}}\right) \tag{1}$$

if t is measured backwards from now. Using the numbers in Turcotte & Schubert, the heating rate at $t = 4.5$ Gyr was 3.0×10^{-7} erg g s^{-1}, about a factor of 4 greater than the current heating. In the early Earth, the unstable isotopes with the shorter half-lives, ^{235}U and ^{40}K were most important; now ^{238}U and ^{232}Th are more important.

One source of uncertainty in the early history of planetesimals is their amount of ^{26}Al with $t_{1/2} = 7.2 \times 10^5$ yr. Because Al composes 1.5% of the mass of the Earth (Allegre et al. 2001, Earth Planet Sci. Lett., 185, 49), the presence of even a small fraction of Al in the radioactive form can be important. Because ^{26}Al is inferred to have been present in meteorites, it could have been quite important in the thermal balance of material in the early Solar System. The ^{26}Al may have been produced by a nearby supernova and deposited within the solar nebula.

The usual story for the formation of the Earth's core is that once the interior of the planet melted, it was possible for fluid flows to proceed and the denser material sank to the interior (see McDonough 2003, in Volume 2 of Treatise on Geochemistry). By mass, the Earth currently is about 33% iron. The core/mantle boundary is at about $R = 3500$ km, somewhat more than half of the Earth's radius. With a density of 7.9 g cm^{-3}, metallic iron is much denser than the mean density of the Earth and therefore it naturally dominates the core. The iron in the Earth's crust is largely oxidized into minerals; the iron in the interior core also is not pure. (Steel is iron with impurities selected to alter various properties.) The composition of the core is still somewhat unknown.

One argument for the existence of a dense core is the Earth's moment of intertia. It is only about $0.33\,M\,R^2$ instead of $0.4\,M\,R^2$ as would be expected for a homogeneous object.

Evidence for the core comes from seismology, the planets moment of inertia, modeling of the Earth's magnetic field and indirect methods such as studying meteorites. For example, the argument that $\sim$10% of the core is nickel comes both from cosmic abundances and the widespread presence of nickel in iron meteorites. A key seismological tool is the distinction between the S (shear waves) and P waves (compressional waves). The wave velocities are:

$$V_P = \left(\frac{K_m + \frac{4}{3}\mu_{rg}}{\rho}\right)^{1/2} \tag{2}$$

where K_m is the bulk modulus, ρ is the density and μ_{rg} is the shear modulus. Also:

$$V_S \;=\; \left(\frac{\mu_{rg}}{\rho}\right)^{1/2} \tag{3}$$

The shear modulus of the liquid core is 0 although shear waves can be reflected off the core and there is a conversion of S waves into P waves. Note that

$$\mu_{rg} \;=\; \frac{3\,K_m(1\,-\,2\nu)}{2(1\,+\,\nu)} \tag{4}$$

where ν is Poisson's ratio. At the surface of the Earth, V_P is about 8.1 km s^{-1} while V_S is about 4.5 km s^{-1}; both speeds are much greater than the speed of sound in the air.

The density in the core ranges from 9.9 g cm^{-3} to 13.1 g cm^{-3}. These densities are considerably greater than the density of iron at ambient pressure. However, they are about 5-10% smaller than expected for the internal pressure derived from seismology. Consequently, it is presumed that there must be some additional light elements within the core. The light elements must be able to "dissolve" into iron at high pressure. The current candidates are hydrogen, oxygen, carbon, silicon and sulfur. Unfortunately, the iron-rich meteorites were not formed at sufficiently high pressure to display the appropriate contamination.

An essential chemical classification scheme is an element's affinity in the solid phase with other elements. The usual groupings are lithophiles (that bond readily with oxygen and are concentrated in the mantle and crust), siderophile (readily bond with iron and are found in the core), chalcophiles (which bond with sulfur and are distributed throughout the planet) and atmophiles which remain gaseous and remain in the atmosphere. According to McDonough, some of the most prominent siderophile elements are Fe, Co, Ni, V, Cr and P. chalcophile elements include S, Cu, Sc and Pb.

This qualitative scheme can be made more exact with detailed thermodynamic considerations. Petrology is the study of the composition, structure and origin of rocks. It is of course an enormously complex subject on which we can only touch very briefly. Rocks are composed of minerals. Some of the most important minerals are silicates (pyroxene and olivine) and oxides. The thermodynamic constraint is that the Gibbs free energy as a function of temperature, pressure and elemental composition is minimized. An essential set of tools are the phase diagrams of the different minerals that are being considered. Some of the most important curves are the *liquidus* and *solidus*. A complex mixture can be partially liquid. These two lines define the regime in the phase diagram where the material is either purely liquid or purely solid. The oxygen fugacity, a measure of the "effective pressure" is also key.

The timing of core formation can be derived from the W (atomic number 74) -Hf (atomic number 72) system where ^{182}Hf decays to ^{182}W with a half-life of 9 Myr. [There is an intermediate step of ^{182}Ta which has a half-life of 114 days.] Hf is lithophile and resides in the mantle while W is siderophile and resides in the core. Because the mantle is found to be ^{182}W rich, this means that the core formed early; perhaps within 30 Myr. [t The essentially stable isotopes of W are ^{180}W – measured half-life of 1.88×10^{18} yr, ^{182}W, ^{183}W, ^{184}W and ^{186}W. This result also allows us to date the event for the formation of the Moon as some time after the core formed since the Moon appears to have relatively little iron. The mean density of the Moon is 3.3 g cm^{-3}, the mean density of the uncompressed Earth is 4.4 g cm^{-3}. The resolution of this discrepancy is that the Moon was created after the core was formed.

According to Palme & O'Neill (2003, in Volume 2 of Treatise on Geochemistry), we expect that all the dominant elements in the mantle combine with oxygen in the forms MgO, SiO_2, Al_2O_3, CaO and FeO.

The crust is chemically very distinct from the mantle. The crust contains "incompatible" elements – species which do not blend into the mantle. Thus, the crust is lighter than the mantle and very enriched in Si relative to Mg. Other elements particularly enriched in the crust are K, Li, Al and Sr. An interesting footnote is that Ir is generally depleted in the crust. However, there are strata where Ir is enriched – a strong signature for the collision event that led to the extinction of the dinosaurs.

Lecture 9 – Snow Lines and Ice

The composition of a solid planet depends upon the elemental abundances in the disk where it forms and the local temperature. The Earth is largely composed of material that condensed at between 1100 K and 1200 K. However, in the outer disk, the temperature was much cooler and ice was incorporated into planets such as Uranus and Neptune.

The behavior of oxygen is key because with "cosmic abundances", there is a substantial excess over what can be contained within minerals. That is, using the abundances from Lodders (2003, ApJ, 591, 122) where Si is used as a fiducial, there are 14.1 O atoms for every Si atom. Since O can be contained within MgO, SiO_2, Al_2O_3, FeO and CaO, the total number of oxygen atoms locked into these elements is 4.0. Consequently, 10.1 O atoms can be bound with H to form water. Just considering the mass of material, we expect that there could be about as much mass in water as in heavy minerals.

The standard thermodynamic calculations show that at temperatures below 1000 K, oxygen, carbon and nitrogen are largely contained with H_2), CH_4 and NH_3 (see, for example, Sharp & Burrows 2007, ApJS, 168, 140). At higher temperatures CO is a major constituent. Of course there is no guarantee that the gas achieves thermodynamic equilibrium. Nevertheless as a first expectation, we imagine that for $T < 1000$ K, a large amount of water is present in nebular gas.

There is excellent evidence that water is widespread in the outer solar system. The mean densities of Uranus and Neptune are 1.3 g cm^{-3} and 1.6 g cm^{-3}, respectively. These densities are too high for low-mass planets composed of H and He and too low for rocky materials. The argument is strong that they a large fraction of their mass is H_2O. It is also assumed that CH_4 and NH_3 also are major constituents. If so, this may be an indication that the planet accumulated much of its material as gas rather than solids since CH_4 and NH_3 are substantially more volatile than H_2O. Because their atmospheres are mostly H and He, we cannot directly measure the bulk composition of the planets. Leger et al. (2004, Icarus, 169, 499) postulate the existence of "ocean" planets which are largely composed of H_2O. They allow for NH_3 and CO_2 but exclude CH_4. Comets outgas large amounts of H_2O but, apparently, relatively little CH_4. Therefore, there appear to be regimes in the solar system where water but not methane condensed into solids.

At the surface of the Earth, the pressure is usually sufficiently high that local thermodynamic equilibrium can be assumed for the computation of the fraction of water that is contained within ice. However, in protostellar disks, the density can be orders of magnitude lower and a kinetic treatment for the fraction of water that adheres to solids may be more appropriate. According to Leger et al. (1985, A&A, 144, 147) the rate of evaporation, R, (molecules cm^{-2} s^{-1}) at low temperature, T, is approximately given by:

$$R = \frac{p_0}{(2\,\pi\,\mu\,k\,T)^{1/2}}\exp(-\Delta H/T) \tag{1}$$

where μ is the molecular weight, p_0 a pressure and ΔH the binding energy divided by Boltzmann's constant. For H_2O, we can take $p_0 = 2.5 \times 10^{13}$ erg cm^{-3} and $H = 6070$

K. Sasselov & Lecar (2000, ApJ, 528, 995) adopt $T = 170$ K as the boundary for the snow line. This choice is somewhat arbitrary and depends upon the ambient temperature. Lecar et al. (2006, ApJ, 640, 1115) develop a more sophisticated description of the snow line, but in practice, it is not much different from the previous model. Also, ΔH depends upon the composition of the surface; this value is chosen for water binding to water. However, the energy barrier that water must overcome to sublimate is so substantial that it is plausible that a temperature near 170 K is useful to adopt. Note that for CH_4 ΔH is 1110 K, much lower than the value for H_2O and therefore CH_4 rapidly sublimates as a much lower temperature than H_2O. The "snow line" temperature for CH_4 might be near 31 K. An interesting application of this low temperature for the sublimation of CH_4 is the variation in the atmosphere of Pluto. The eccentricity of Pluto's orbit is so large that there are substantial variations in the expected surface temperature and this can lead to large changes in the amount of atmospheric methane (see, for example, Trafton & Stern 1983, ApJ, 267, 872).

It must be recognized that the ices that are formed likely are not pure. A molecule embedded in another solid is a clathrate. Important clathrates are NH_3 and CH_4.

The temperature of an isolated black body irradiated by the Sun is:

$$T = \left(\frac{R_*}{2\,D}\right)^{1/2} T_* \tag{2}$$

where D is the distance to the Sun which is assumed to have an effective temperature T_* and a radius R_*. For the Solar System, $T = 170$ K is currently found at $D = 2.7$ AU, in the middle of the asteroid belt. According to this formulation, asteroids could be water rich. On the basis of its low density, Ceres, the largest asteroid is thought to be $\sim$25% water by mass, but other asteroids with higher mean densities may have much smaller fractions of water.

In the outer region of a nebula where ice condenses onto solids, it is possible that the disk is flared. In this case, if vertically isothermal, the temperature in a passive disk is:

$$T = \left(\frac{1}{7}\right)^{2/7} \left(\frac{R_*}{D}\right)^{3/7} \left(\frac{2kT_*R_*}{GM_*\mu}\right)^{1/7} T_* \tag{3}$$

This expression predicts a snowline inside of 1 AU – an unrealistic result.

If the disk is active, then the simple estimate for the vertically averaged temperature is:

$$T \approx \left(\frac{3\,G\,M_*\,\dot{M}}{8\,\pi\,\sigma_{SB}\,D^3}\right)^{1/4} \tag{4}$$

Note that for an environment where $\dot{M} = 10^{-8}$ $M_\odot$ yr^{-1}, T falls below 170 K at D much less than 1 AU. Lecar et al. argue that the disk is vertically opaque and that therefore the midplane temperature where the dust accumulated is much hotter than the surface temperature. In this case, the snow line can be extended to beyond 2 AU.

Lecture 10 – Volatiles at Rocky Planets

As a first approximation, the Earth is composed of elements that condense into solids between 1100 K and 1200 K. The question then naturally arises as to why there are any volatiles present in the planet. The usual explanation is that the Earth acquired a thin veneer of volatiles as accretion during its final phases of evolution.

The volatile inventory of the Earth (van Thienen et al. 2007, Space Science Reviews 129, 167) includes H_2O, C and the noble gases. The ocean contains 1.4×10^{24} g of H_2O, about 0.02% of the total mass of the planet. However, there is certainly some internal water in the mantle and crust – estimates range from 2 to 5 oceans. The water is contained in minerals (such as clays and serpentine [one example is lizardite = $Mg_3Si_2O_5(OH)_4$] the state rock of California!); it is not separately bound in underground pools. There may also be a large amount of water contained within the core, but this is not known.

The total mass of the Earth's atmosphere is 5.1×10^{21} g, about 0.4% of the mass of the ocean. The atmosphere is composed of volatiles: mostly N_2 and O_2. It is usually argued that the O_2 is biogenic. The mass of carbon in the atmosphere – largely in the form of CO_2 is quite small [although rising!] – about 8×10^{17} g. There is a comparable amount of carbon in the living biomass and more in the soils. However, the fraction of carbon dissolved into the ocean is substantial, about 3.6×10^{19} g. Carbonate minerals (such as $CaCO_3$) are probably the largest reservoir of carbon in the mantle There maybe a significant but unknown amount of carbon in the Earth's core. Perhaps 10^{-4} of the mass of the Earth is carbon (Palme & On'Neill), but this is very uncertain.

One argument for a large amount of carbon within the Earth is that the atmosphere of Venus has about 1.3×10^{23} g of carbon, mainly in the form of CO_2. It would seem plausible that the Earth possesses as much carbon as does Venus.

The noble gases He, Ne, Ar, Kr and Xe have a distinctive chemistry since they do not become chemically incorporated into minerals. The source of these elements in the different planets is still uncertain (Pepin 2006, Earth and Planetary Science Letters, 252, 1; Mousis et al. 2010, ApJ, 714, 1418). While ^{4}He is produced during radioactive decay, ^{3}He is not. However, there is some ^3He in terrestrial rocks. Possibly this is primordial, but it is also imaginable that there has been some accretion from the solar wind. That is, the surface of the Moon has a significant amount of ^{3}He from accreted solar wind material. In bulk Earth, Ne and ^{36}Ar are 1.1×10^{-11} and 3.9×10^{-11} by mass (Allegre et al. 2001, Earth and Planetary Science Letters) while in the Sun, these elements are 4.2×10^{-5} and 4.1×10^{-6}, respectively. [Note that there is about 4.7×10^{19} g of Ar in the Earth's atmosphere. This argon is largely ^{40}Ar which results from the radioactive decay of ^{40}K. Approximately 11% of all ^{40}K decays into ^{40}Ar. Since the estimated mass of ^{40}K in the Earth is 1.3×10^{20} g (Turcotte & Schubert), then with a half-life of 1.25 Gyr, we expect that the initial mass of ^{40}K was 1.7×10^{21} g. Thus radioactive decay produced 1.7×10^{20} g; approximately 1/3 of this Ar currently is in the Earth's atmosphere. This calculation shows that degassing from the Earth's interior, largely by volcanoes, must be efficient.

The source of the Earth's water is thought to be accretion of material that was formed in the outer solar system. Since the Earth is thought to have accumulated from collisional build-up, it is natural to imagine that a small fraction of the material originated beyond the snow line which may have been at 5 AU (Encrenaz 2008, Ann. Rev. Astr. Ap., 46, 57). Currently, the Earth accretes about 10^3 g s^{-1} in Interplanetary Dust Particles (Love & Brownlee 1993, Science, 262, 50). It is likely that these particles originate in the zodiacal cloud and drift inwards under the action of the Poynting-Robertson effect. That is, the dust production rate in the zodiacal cloud is about 3×10^6 g s^{-1} (Fixsen & Dwek 2002, ApJ, 578, 1009). The average particle size in this cloud is about 30 μm in radius. When it has an aphelion of 1 AU, the probability that a grain will collide with the Earth is approximately $2R_E/(2\pi D)$ where R_E and D are the radius of the Earth and the Astronomical Unit, respectively. The number of complete orbits completed by the grain, n_{orbit}, before it drift inwards is given by the ratio of t_{PR} to the orbital period, P.

$$\frac{t_{PR}}{P} = \left(\frac{4\pi\, D^2\, c^2\, \rho_S\, b}{3\, L_*}\right) \left(\frac{4\pi^2\, D^3}{G\, M_*}\right)^{-1/2} \tag{1}$$

The fraction of these orbits which intercept the Earth is $2R_E/D$ Therefore, the fraction of inward drifting grains which are accreted by the Earth, f, is:

$$f = \left(\frac{R_E}{\pi D}\right) \left(\frac{t_{PR}}{P}\right) \left(\frac{2\, R_E}{D}\right) = \frac{4}{3\,\pi}\, R_E^2 \left(\frac{G\, M_*}{D^3}\right)^{1/2} \frac{c^2\, \rho_S\, b}{L_*} \tag{2}$$

Thus $f \sim 10^{-4}$, somewhat smaller than we might expect. However, we did not include gravitational focusing or a distribution of particle sizes and densities.

An interesting unsolved question is the origin of the zodiacal light dust. Over the years it is uncertain whether it is largely cometary or asteroidal. A recent interesting paper is by Nesvorny et al. (2010, ApJ, 713, 816) who argue that the dust results from the spontaneous disruption of Jupiter family comets. If correct, this process could be important for debris disks in general.

The current accretion rate is much too small to account for the mass of volatiles on the Earth. However, we expect that in the past, the mass flux at the Earth could have been much greater. For example, the asteroid belt almost certainly was much more massive in the past. If so, then the Earth may have accreted many of the ‘lost" asteroids. If these asteroids had an appreciable abundance of volatiles, this could explain the reservoirs of this material on the Earth.

An argument in favor of the view that the Earth accreted its volatiles is that Mars apparently has proportionally even less water than the Earth. That is, because the Earth is more massive than Mars, gravitational focusing is much more important, and the Earth could have accreted more material. The argument about Mars is uncertain. Their oxygen isotope ratios demonstrate that the SNC meteorites arise from Mars (Clayton, 1993, Ann. Rev. Earth Planetary Science, 21, 115). The argument is that Cl has a low abundance and that the Cl in rocks can trace the initial H_2O (Deibus & Wanke 1987, Icarus, 71, 225).

Lecture 11 – Planetary Atmospheres

Planetary atmospheres are characterized by many properties: temperature, composition, variation and evolution. The atmosphere is often the most directly observable part of the planet.

The escape velocity, V_{esc}, from a planet of mass M, radius R and mean density ρ is:

$$V_{esc} \;=\; \left(\frac{2\,G\,M}{R}\right)^{1/2} \;=\; \left(\frac{8\,\pi\,G\,\rho}{3}\right)^{1/2} R \tag{1}$$

Evidently, small bodies do not retain atmospheres as well as large ones. Also, for a thermal distribution, the light elements have greater mean speeds than do the lighter elements. Consequently, it is easier for lighter gases to escape from a planet. The Earth does not retain H and He for geological times.

There are some intermediate objects such as the Moon and Mercury with very tenuous gaseous envelopes (surface number densities near 10^3 to 10^4 cm^{-3} or mass densities near). In these environments, the mean free path of a gas atom may be 10^{11} to 10^{12} cm, much greater than the radius of the object. The motions are therefore ballistic and unlike the Earth's atmosphere, collisions among the atoms and molecules are not important. The source of these "atmospheres" is thought to be sputtering by the solar wind. That is, solar wind particles are incident at $\sim$600 km s^{-1}; the escape velocity from the Moon is 2.3 km s^{-1}. A "modest" transfer of energy from the wind to the surface can eject material. The solar wind loss rate is about 2×10^{-14} $M_\odot$ yr^{-1} or about 10^{12} g s^{-1}. The particle flux, F, at the Moon is about 4×10^{-16} g cm^{-2} s^{-1}. In the simplest model, the "atmosphere" density, ρ_{atm} is:

$$\rho_{atm} \;=\; \frac{Y\,F}{V_{atm}} \tag{2}$$

where Y is the sputtering yield and V_{atm} is the velocity of the atoms flowing outwards. With $Y = 0.01$, and for $V_{atm} = V_{esc} = 2.3$ km s^{-1}, the predicted density is much less than observed. Consequently, to understand the presence of these very tenuous gaseous envelopes, we must consider the very low speed tail of the distribution of the sputtered material.

The structure of conventional atmospheres is conventionally described by temperature, composition and pressure. From hydrostatic equilibrium, we write:

$$\frac{dp}{dz} \;=\; -\rho\,g \tag{3}$$

where g is the gravitational acceleration. For an isothermal ideal gas where:

$$p \;=\; \frac{\rho}{\mu}\,k\,T \tag{4}$$

Then:

$$\rho \;=\; \rho_0\,e^{-z/H} \tag{5}$$

where the scale height, H, is:

$$H = \frac{k\,T}{\mu\,g} \tag{6}$$

Note that we expect that H is much less than the radius of the planet. Note also that the column of material in the atmosphere can be determined from the pressure at the ground. From hydrostatic equilibrium, we see that:

$$\int_0^\infty \rho\,dz = \frac{p_0}{g} \tag{7}$$

Atmospheres are unlikely to be isothermal. If adiabatic and purely gaseous, then:

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma}\frac{g\,\mu}{k} \tag{8}$$

where γ is the ratio of specific heats. If other processes are important in the energy budget, such as condensation of water vapor, then the temperature gradient is altered.

We describe the flow of photons in an atmosphere with the formalities of radiative transfer. One useful simpification is the two stream approximation. The incident light has intensity I_0. Consider a plane parallel atmosphere where z and τ are measured downward from the top. In such an atmosphere, we can write that

$$dx = -\frac{dz}{\cos\theta} = -\frac{dz}{\mu} \tag{9}$$

Consequently, along each direction defined by θ, the equation of transfer becomes:

$$\mu\frac{dI}{d\tau} = I - (1 - a)\,B - a\,J \tag{10}$$

where a is the single particle albedo, B the Planck function and J the mean intensity. Consider now the situation where we only consider scattering; there is no thermal emission so that $B = 0$. In the two-stream approximation, we only consider light that moves in the directions $\mu = \pm 1$. We define I_+ as the upward intensity and I_- as the downward intensity. We can write that

$$J = \frac{1}{2}\,(I_+ + I_-) \tag{11}$$

Each stream of radiation must satisfy the transfer equation so that

$$\frac{dI_+}{d\tau} = I_+ - a\,J \tag{12}$$

and

$$\frac{dI_-}{d\tau} = -I_- + a\,J \tag{13}$$

It is useful to define the scaled flux, H, such that

$$H = \frac{(I_+ - I_-)}{2} \tag{14}$$

By summing the equations, we find that:

$$\frac{dI_+}{d\tau} + \frac{dI_-}{d\tau} = 2\frac{dJ}{d\tau} = 2H \tag{15}$$

Thus:

$$\frac{dJ}{d\tau} = H \tag{16}$$

Subtracting the two equations, we find that:

$$\frac{dI_+}{d\tau} - \frac{dI_-}{d\tau} = 2\frac{dH}{d\tau} = 2J - 2aJ \tag{17}$$

Thus:

$$\frac{dH}{d\tau} = J(1 - a) \tag{18}$$

We can eliminate H by writing:

$$\frac{d^2J}{d\tau^2} = \frac{dH}{d\tau} = J(1 - a) \tag{19}$$

The solution to the differential equation is:

$$J = C_1 e^{\sqrt{1-a}\tau} + C_2 e^{-\sqrt{1-a}\tau} \tag{20}$$

where C_1 and C_2 are constants to be determined by the boundary conditions. In order to keep the mean intensity well bounded within the atmosphere, we take $C_1 = 0$. We find from our solution that:

$$\frac{dJ}{d\tau} = H = -\sqrt{1-a}\, C_2\, e^{-\sqrt{1-a}\tau} \tag{21}$$

At the surface of the atmosphere where $\tau = 0$, $I_- = I_0$ as the incident radiation.

$$I_-(0) = I_0 = J(0) - H(0) \tag{22}$$

Therefore:

$$C_2 + \sqrt{1-a}\, C_2 = I_0 \tag{23}$$

Therefore:

$$C_2 = \frac{I_0}{1 + \sqrt{1-a}} \tag{24}$$

Therefore:

$$J = \frac{I_0}{1 + \sqrt{1-a}} e^{-\sqrt{1-a}\tau} \tag{25}$$

while

$$H = \frac{dJ}{d\tau} = -\frac{I_0\sqrt{1-a}}{1+\sqrt{1-a}}\, e^{-\sqrt{1-a}\tau} \tag{26}$$

Note that the negative value of H implies a net scaled flux downwards into the atmosphere. We can also write that

$$I_- = J - H = I_0\, e^{-\sqrt{1-a}\tau} \tag{27}$$

While

$$I_+ = J + H = I_0 \frac{1-\sqrt{1-a}}{1+\sqrt{1-a}}\, e^{-\sqrt{1-a}\tau} \tag{28}$$

At the top of the atmosphere where $\tau = 0$, the planetary albedo is:

$$\frac{I_+(0)}{I_-(0)} = \frac{1-\sqrt{1-a}}{1+\sqrt{1-a}} \tag{29}$$

An essential property of an atmosphere is its opacity which depends upon the composition. Often, one of the largest uncertainties is the role of dust, aerosols and/or droplets in clouds. If the optical opacity is very low while the infrared opacity is high, then we expect a greenhouse situation. That is, in the extreme case, the planetary atmosphere resembles a stellar atmosphere; there is effectively a net flux from below and a temperature gradient such that the atmosphere is cooler near the top. Since the "top" of the atmosphere must re-radiate into space the incoming energy, then the "bottom" of the atmosphere is warmer than the "top" which must attain the equilibrium temperature in the absence of an atmosphere.

Lecture 12 – Planetary Winds

Gases escape from planets. Although variable and sensitive to the level of solar activity, the Earth loses hydrogen at a rate of about 1.5×10^8 atoms cm^{-2} s^{-1} or a total of about 10^3 g s^{-1} (see, for example, Pierrard 2003, Planetary Space Science, 51, 319). . The escape rate of hydrogen from Mars is comparable to this value. While the current rate of escape from the Earth is not important for our overall water budget, hydrogen escape from Mars is important in the water budget of this smaller, drier planet. Extrasolar planets can have much greater wind rates. The hot Jupiter HD 209458b is thought to be losing hydrogen at a rate of 10^{10} g s^{-1}, but the interpretation of the data is controversial (see Murray-Clay et al. 2009, ApJ, 693, 23 for a long discussion on this point. There is some question whether hot Jupiters can be substantially evaporated.

There are a variety of mechanisms which can lead to escape.

1 JEAN'S ESCAPE

The simplest case of escape to consider is Jean's escape. In this case, we do not consider the gas as a fluid but rather as an ensemble of individual particles. This is justified as long as the mean free path is comparable or larger than the other dimensions of the system – such as the local scale height of the atmosphere. Let n_0 denote the density at the base of the "exosphere". We require that:

$$n_0 \leq \frac{1}{\sigma H} \tag{1}$$

where H is the scale height of the atmosphere and σ is the collisional cross section which is taken as 3.3×10^{-15} cm^2 (Hunten, 1982, Planetary Space Science, 30, 773). Let v_e denote the escape velocity so that:

$$v_e \;=\; \left(\frac{2\,G\,M_p}{R_p}\right)^{1/2} \tag{2}$$

where m_p and R_p are the mass and radius of the planet, respectively. Assume a gas of temperature T composed of particles of mass m which obey a Maxwellian distribution of speeds, $f(v)\,dv$. If we define the Z axis along the radial direction, then we describe the rate of escaping particles, F_z, as the flux of particles with speeds greater than v_e which is determined by:

$$F_z \;=\; n_0\,v_z \;=\; n_0\,v\,\cos\theta \tag{3}$$

where v_z must be positive and also v must be greater than v_e.

We can write that:

$$f(v)\,v^2\,dv\,\sin\theta\,d\theta\,d\phi \;=\; \left(\frac{m}{2\pi\,k\,T}\right)^{3/2} v^2\,\exp\left(-\frac{m\,v^2}{2\,k\,T}\right)\,dv\,\sin\theta\,d\theta\,d\phi \tag{4}$$

We can integrate over azimuth to find that:

$$f(v)\, v^2\, dv\, \sin\theta\, d\theta \;=\; 2\pi\, \left(\frac{m}{2\pi\, k\, T}\right)^{3/2}\, v^2\, \exp\left(-\frac{m\, v^2}{2\, k\, T}\right)\, dv\, \sin\theta\, d\theta \tag{5}$$

We write that:

$$F_z \;=\; n_0 \int_{v_e}^{\infty} \int_0^{\pi/2} v\, \cos\theta\, f(v)\, dv\, \sin\theta\, d\theta \tag{6}$$

Integrating over angle, then:

$$F_z \;=\; n_0\, \pi\, \left(\frac{m}{2\, \pi\, k\, T}\right)^{3/2} \int_{v_e}^{\infty} v^3\, \exp\left(-\frac{m\, v^2}{2\, k\, T}\right)\, dv \tag{7}$$

Evaluating this integral, then:

$$F_z \;=\; n_0\, \left(\frac{k\, T}{2\, \pi\, m}\right)^{1/2}\, \exp\left(-\frac{m\, M_p\, G}{R_p\, k\, T}\right)\, \left(1 \,+\, \frac{m\, M_p\, G}{R_p\, k\, T}\right) \tag{8}$$

The predicted escape rate depends critically upon the amount of very high altitude atomic hydrogen.

2 HYDRONAMIMC ESCAPE

Consider an isothermal stellar wind from a planet of masss M_p. Let $\dot{M}$ be the mass loss rate, T the gas temperature, μ the mean atomic weight, p, the pressure, ρ the density, v the fluid velocity and r the radial coordinate. In a steady state, the equation of continuity gives:

$$\dot{M} \;=\; 4\pi\, r^2\, v\, \rho \tag{9}$$

Newton's second law for a fluid gives:

$$v \frac{dv}{dr} \;=\; -\frac{G M_p}{r^2} \;-\; \frac{1}{\rho} \frac{dp}{dr} \tag{10}$$

We can also use the ideal gas law,

$$p \;=\; \frac{\rho k T}{\mu} \;=\; \rho\, a^2 \tag{11}$$

where we define the "isothermal" sound speed, a, as

$$a \;=\; \left(\frac{kT}{\mu}\right)^{1/2} \tag{12}$$

Since the gas is isothermal, we may write that:

$$\frac{1}{\rho}\frac{dp}{dr} \;=\; \left(\frac{kT}{\mu}\right)\frac{1}{\rho}\frac{d\rho}{dr} \;=\; a^2\frac{1}{\rho}\frac{d\rho}{dr} \tag{13}$$

From the equation of continuity, we can write that:

$$\frac{1}{\rho}\frac{d\rho}{dr} \;=\; -\frac{1}{v}\frac{dv}{dr} \;-\; \frac{2}{r} \tag{14}$$

Thus:

$$v\frac{dv}{dr} \;=\; -\frac{GM_p}{r^2} \;+\; a^2\left(\frac{1}{v}\frac{dv}{dr} \;+\; \frac{2}{r}\right) \tag{15}$$

Therefore, collecting terms:

$$\frac{1}{v}\frac{dv}{dr} \;=\; \frac{\left(\frac{2a^2}{r} \;-\; \frac{GM_p}{r^2}\right)}{(v^2 \;-\; a^2)} \tag{16}$$

The "sonic point" occurs when both the numerator and denominator equal zero. At the critical point,

$$v \;=\; a \tag{17}$$

At this critical point, r_S, is given by the expression:

$$r_S \;=\; \frac{GM_p}{2a^2} \tag{18}$$

Thus thus the sonic point is controlled by the ratio of the potential energy/particle vs. the kinetic energy/particle.

At the critical point, we can find from De l'Hopital's rule that at the critical point all evaluated at $r = r_S$ that:

$$\frac{1}{v}\frac{dv}{dr} \;=\; \frac{\left(-\frac{2a^2}{r_S^2} \;+\; \frac{2GM_p}{r_S^3}\right)}{\left(2\,v\,\frac{dv}{dr}\right)} \tag{19}$$

Thus at the critical point, we find that

$$\left(\frac{dv}{dr}\right)^2 \;=\; \left(-\frac{a^2}{r_S^2}\right)\left(1 \;-\; \frac{GM_p}{a^2\,r_S}\right) \tag{20}$$

Thus, at $r = r_S$,,

$$\frac{dv}{dr} \;=\; \pm\frac{2a^3}{GM_p} \tag{21}$$

The solution to the fluid flow equation (7) is found by integrating and given by the expression:

$$\frac{v^2}{2} \;=\; \frac{GM_p}{r} \;+\; a^2\left(\ln\frac{v}{a} \;+\; 2\ln\frac{r}{r_S}\right) \;+\; constant \tag{22}$$

The constant is evaluated by using $v = a$ at the critical point, $r = r_S$. Therefore, using $GM_p = 2a^2 r_S$, we have that:

$$constant = -\frac{3a^2}{2} \tag{23}$$

Therefore, we can re-write the above to find:

$$\left(\frac{v}{a}\right) \exp\left(-\frac{v^2}{2a^2}\right) = \left(\frac{r_S}{r}\right)^2 \exp\left(\frac{3}{2} - \frac{2\,r_S}{r}\right) \tag{24}$$

We have therefore solved for v as a function of r. Very approximately, as $r \to \infty$, then the solution to this equation is

$$\left(\frac{v}{a}\right) \approx 2\sqrt{\ln\frac{r}{r_S}} \tag{25}$$

Consider the photosphere of the planet where $r = r_p$ and the gas is moving slowly so that $v << a$. At this radius, we can define the escape velocity from the star, v_{esc} as

$$v_{esc}^2 = \frac{2GM_p}{r_p} \tag{26}$$

In other words:

$$r_S = \frac{r_p\,v_{esc}^2}{4\,a^2} \tag{27}$$

Then near the star, we have $v = v_p$

$$\left(\frac{v_p}{a}\right) \approx \left(\frac{v_{esc}}{2a}\right)^4 \exp\left(-\frac{v_{esc}^2}{2a^2} + \frac{3}{2}\right) \tag{28}$$

We can write that the total mass loss rate, $\dot{M}$ is given by the conditions at the base of the outflow:

$$\dot{M} = 4\pi r_*^2 \rho_* v_* = 4\pi r_*^2\,\rho_*\,a\left(\frac{v_{esc}}{2a}\right)^4 \exp\left(-\frac{v_{esc}^2}{2a^2} + \frac{3}{2}\right) \tag{29}$$

If the base of the outflow has density ρ_p, then from the equation of continuity

$$v\rho r^2 = v_p \rho_p r_p^2 \tag{30}$$

Or:

$$\frac{r_p^2 v_p}{r^2 v} = \frac{\rho}{\rho_p} \tag{31}$$

From above, then:

$$\frac{\rho}{\rho_*} = \frac{v_p r_p^2}{a r_S^2} \exp\left(-\frac{1}{2}\left(\frac{\rho_p r_p^2 v_p}{\rho r^2 a}\right)^2\right) \exp\left(-\frac{3}{2} + \frac{2r_S}{r}\right) \tag{32}$$

Using the values for v_p and R_S given above, this expression can be rewritten as:

$$\frac{\rho}{\rho_p} = \exp\left(-\frac{1}{2}\left(\frac{\rho_p r_p^2 v_p}{\rho r^2 a}\right)^2 - \frac{GM_*}{a^2}\left(\frac{1}{r_*} - \frac{1}{r}\right)\right) \tag{33}$$

Note that in a purely hydrostatic environment that:

$$\frac{\rho}{\rho_p} = \exp\left(-\frac{GM_*}{a^2}\left(\frac{1}{r_p} - \frac{1}{r}\right)\right) \tag{34}$$

3 NONTERMAL ESCAPE

Thermal escape is probably not the dominant path by which hydrogen leaves the Earth. More importantly, there are a variety of nonthermal escape paths. For example, charge exchange with solar wind protons can lead to escape of the newly-formed ion if this ion is created near a magnetic pole.

The escape velocity from the Earth of 11.2 km s^{-1} implies a kinetic energy of 0.66 eV. Since the binding energies of molecules typically are much greater than this energy, then dissociation of a molecule can often produce atoms which can escape. For example:

$$h\nu + H_2O \rightarrow H + OH \tag{35}$$

can result in H atoms with energies well in excess of that required to escape. Dissociative recombination is another way in which an atom can acquire sufficient energy to escape.

Jeans escape and this molecular processes are atom-specific. In a hydrodynamical outflow, all the gas is entrained and selective elements are not depleted. Each planet may have a separate history.

Lecture 13 – Planetary Surfaces

Planetary surfaces have been very well studied in the solar system; obviously we know much less about the surfaces in extrasolar systems. Although it will be difficult, there are approaches that we can adopt to hope to study the planets.

One key parameter is a planet's albedo. It is usual to distinguish between the geometric albedo and the Bond albedo. The geometric albedo, p, is defined at zero phase angle and is the percent of incident light that is reflected directly backwards compared to what would be reflected by an idealized flat diffuse scattering surface. One can think of the geometric albedo as the efficiency of "head-on" reflection. The Bond albedo, A, is the fraction of all incident light that is scattered into all directions. In the solar system, the Bond albedo of atmosphere-less objects range from high values over 0.9 for Enceladus to much lower values such as 0.07 for the average of the Moon. The albedo is a function of the surface material – water ice, for example, is very highly reflective. The phase function of an observation is quite important. The Moon, for example, displays the opposition effect. Because of multiple scatterings in the regolith (or surface layer) the brightness at opposition (Full Moon) is appreciably greater than expected for an ideal, smooth, reflector.

Light which is not scattered is absorbed and heats the surface of the planet. By comparing the infrared re-emission with the optical reflection, it is possible to measure the albedo of a small object which cannot be resolved – such as most asteroids.

A measurable quantity is the day-night variation of a surface. For thermal conduction, if y is measured vertically into the surface, we can write that:

$$\frac{\partial T}{\partial t} \;=\; \kappa\,\frac{\partial^2 T}{\partial y^2} \tag{1}$$

where κ is the thermal diffusivity which has units of cm^2 s^{-1}. If P is the daily period, then the depth of penetration of thermal variations is given by $(\kappa\, P)^{1/2}$. For example, the thermal diffusivity of ice is about 10^{-2} cm^2 s^{-1} so that at night on the Earth, the depth at which the temperature might vary in a snow bank is about 30 cm. There are, of course, a wide variety of solutions to the heat conduction equation – depending upon the boundary conditions and thermal history.

Turcotte & Schubert describe a similarity solution to the boundary value problem where a planet at a uniform temperature, T_1 "instantaneously" acquires an external boundary temperature T_0. They show that the time, t, that this transition has occurred can be derived from the equation,

$$t \;=\; \frac{(T_1\,-\,T_0)^2}{\pi\kappa(\partial T/\partial y)_0^2} \tag{2}$$

This expression is of considerable historic interest because Lord Kelvin used it to estimate the age of the Earth to equal 65 Myr with a mean surface temperature gradient of 25 K/km and a thermal diffusivity of 0.01 cm^2 s^{-1}. However, this calculation relied upon using T_1

$\approx$ 2000 K which seems (to me) to be much too low given the energy dissipated in forming the Earth.

It is important to recognize that solid materials have "colors" and spectral features and therefore it is possible to sense remotely their compositions. The simplest measure is the albedo as a function of wavelength. With higher spectral resolution features can be detected. A good example is the classes of asteroids where there is a progression from S to C to D type (see Gradie & Telesco 1982, Science, 216, 1405) as a function of semi-major axis. The solid state features in solid state spectra can be used to infer the presence of different minerals and ices.

The surface composition of a planet can be affected by interactions of the solids with its atmosphere and oceans. Examples are weathering and the presence of iron oxides on the surface of Mars.

It must be recognized that planetary surfaces are inhomogeneous. There can be both external effects – such as impacts – and internal effects – such as plate tectonics. An interesting manifestation onto the surface of interior structure is volcanism. On the Earth, for example, the rate of volcanic release of H_2O is $\sim 5 \times 10^6$ g s^{-1} (van Thienen et al. 2007, Space Science Review, 129, 167). Io, the innermost Galilean satellite of Jupiter, has volcanoes because of heating driven by internal tides. When active, Mars is observed to have a surface venting of 600 g s^{-1} of CH_4 (Mumma et al. 2009, Science, 323, 1041). but this detection is controversial (Zahnle et al. 2011, Icarus, 212, 493).

Lecture 14 – Oceans

Large collections of liquids may be very important in a wide variety of planets. Liquid water is particularly of interest since it is a suitable environment for life as we know it.

1 WATER

The Earth, of course, has oceans with a total mass of 1.4×10^{24} g or 0.02% of the total mass of the planet. The amount of internal water may be much greater, but it is not in liquid form. There is no obvious reason why the Earth has its current mass in water. If, for example, delivered by asteroids and/or comets, the amount of surface water could have been much greater. Callisto, the outermost Galilean moon of Jupiter, has a total mass of 1.1×10^{26} g and half of this mass is likely water. If one asteroid even a small fraction of the size of Callisto impacted the Earth, then we could have a substantially larger amount of water. The oceans are not pure H_2O, but instead have dissolved impurities ("salt"), about 3.5% by mass. The two most important elements are Na and Cl.

There is good evidence that Mars once had substantial amounts of liquid water which is no longer present. For example, spectroscopy at 1.9 μm and 3 μm by orbiting spacecraft demonstrates the presence of phyllosilicates at the surface (Bibring et al. 2005, Science, 307, 1576). These minerals are produced by water reacting with rocks ("serpentization") under pressure and at elevated temperature. There is likely substantial subsurface ice (Boynton et al. 2002, Science, 297, 818). The argument follows from γ-ray spectroscopy. When cosmic rays strike the atmosphere and surface of Mars, they generate neutrons from other nuclei. The neutrons then lose energy by collisions and excite the targets into excited nuclear levels which de-excite by emission of gamma rays. Once they are moving sufficiently slowly, the neutrons are captured and the capturing nucleus can be excited thus producing more gamma rays. There is also a leakage of neutrons into the atmosphere and space. The modeling is complex but it is argued that the upward flux of neutrons and the spectrum of the emitted gamma rays can be used to map the distribution of hydrogen in the upper layers (1 m or less) of the surface. The data show that subsurface hydrogen is largely confined to the polar regions which are expected to be especially cold and able to retain ice. Consequently, evidence is strong that the upper 1 m of Mars has buried ice at the poles. There are surface geological features such as gullies and valley networks that indicate the past presence of flowing water and cracks that

Theoretically, there is a boundary layer near the surface of Mars where ice is stable. The interior is presumed heated by radioactive decays and ice is unstable. However, this layer is perhaps as thick as 10 km and there could therefore be a substantial amount of subsurface water. One somewhat speculative possibility is that early Mars had a substantial amount of water and the greenhouse effect was sufficiently effective to maintain surface liquid water (Pollack et al. 1987, Icarus, 71, 293). With time, the water would be lost in a wind, and the planet became drier and colder. There are other models for the water history of Mars.

The likelihood that liquid water was widespread in ancient Mars immediately raises the question of whether life formed on that planet. This question is still unanswered.

There is strong evidence that Europa, the third Galilean satellite of Jupiter, has a subsurface ocean (Kivelson et al. 2000, Science, 289, 1340). The Galileo spacecraft has a magnetometer which measured the direction and magnitude of the local magnetic field. As Europa orbits Jupiter and passes through Jupiter's magnetic field, there are currents induced in Europa. These currents in turn produce a time-varying magnetic field from Europa. By mapping the permanent field from Jupiter and Europa, it is possible to estimate the induced field on Europa. The magnitude of the induced field depends upon Europa's electrical conductivity. The measured "high" conductivity is most easily understood if there is liquid water with dissolved ions rather than solid rocks.

Leger et al. (2004, Icarus, 169, 499) have presented the idea of an "ocean planet". Such planets might exist if they are formed beyond the snow line and then migrate inwards (see also Kuchner 2003, ApJ, 596, L105). The depth of the ocean depends upon the thermal history of the planet and location. However, oceans as deep as 100 km are possible. [Note that convection dominates the thermal profile of the Earth's oceans which are mostly colder at the bottom than at the surface. This is completely different from the thermal profile in the solid crust which is warmer at depth.]

2 ORGANICS

Cassini radar mapping of Titan shows flat regions of low reflectivity with nearby channels mainly at the poles. [Direct optical imaging of the surface is not possible because of aerosols in the atmosphere.] Morphologically, these structures appear as lakes and they are assumed to be composed largely of methane (CH_4) and ethane (C_2H_6) which are expected to be liquid at the temperature and pressure at the surface of Titan (Stofan et al. 2007, Nature, 445, 61). The methane is relatively volatile and should evaporate and re-cycle through the atmosphere. The ethane is probably largely liquid. There are "dry channels" at the equator which may be produced by transient flows and methane rain (Turtle et al. 2011, Science, 331, 1414).

The atmosphere of Titan is largely N_2 but minor species include H_2 and many different organics (Yung et al. 1984, ApJS, 55, 465). The organics are processed at the surface and the lifetime of CH_4 is much less than the age of the solar system. Likely, there is replenishment of surface methane by volanoes.

3 "LAVA-OCEANS"

Normally rocky planets may have liquid surfaces if they are sufficiently hot (Leger et al. 2011, Icarus, 213, 1). At the subsurface location of a tidally locked planet in a short period orbit, the surface temperature can exceed 2400 K. At this temperature, most solids melt and the surface becomes liquid. As yet, there is no direct observational confirmation of this theory.

Lecture 15 – Moons and Rings

An essential feature of solar system planets is their moons. To date, no moon has been firmly identified orbiting any extrasolar planet.

1 Our Moon

Our Moon has an identical oxygen isotope fractionation patter as the Earth; a result that lends strong support to the hypothesis that the Moon formed from a giant impactor hitting the Earth (Wiechert et al. 2001, Science, 294, 345). The Moon must have formed after the Earth's core was formed since the mean density of the Moon (3.34 g cm^{-3}) is considerably less than the mean density of the uncompressed Earth of 4.4 g cm^{-3}. Thus, bulk Moon likely is low in iron.

The formation of the Moon by a giant collision may have been somewhat accidental. That is, the Earth was formed by the assembly of planetary embryos and mutual collisions. However, the collision to create a Moon may have been fine-tuned (Canup & Asphaug 2001, Nature, 412, 708). With an orbital speed of 30 km s^{-1} and an escape speed of 11 km s^{-1}, most collisions do not simultaneously remove a large amount of material from the surface which however remains gravitationally bound and can re-assemble into a new self-gravitating object. The absence of large moons orbiting Mercury, Venus and Mars is consistent with this view.

Moon-creating events in extrasolar planetary systems ought to be detectable by the production of large amounts of warm dust such as may have occurred around HD 23514, a solar-type star in the Pleiades (Rhee et al. 2008, ApJ, 675, 777). The WISE data are ideal for investigating this possibility. There are 188 G-type main-sequence stars within 25 pc of the Sun (Greaves & Wyatt 2003, MNRAS, 345, 1212). WISE ought to be able to detect a large infrared excess for any G-type star within 200 pc of the Sun. In this volume, there are $\sim 10^5$ main-sequence G stars and the rare ones with a large excess will be identified.

The Moon has only 0.0123 the mass of the Earth. The orbital angular momentum of the Moon is about 2.9×10^{41} g cm^2 s^{-1}, about a factor of four larger than the Earth's rotational angular momentum of 7.1×10^{40} g cm^2 s^{-1}. An interesting dynamical consequence of the Moon is that it stabilizes changes in the Earth's obliquity that would occur from perturbations (Laskar et al. 1993, Nature, 361, 615). An interesting consequence is that the Earth's climate – variable as it is (with Ice Ages, for example) – would display much greater variations if it did not possess the Moon.

2 Moons in the Outer Solar System

Jupiter, Saturn, Uranus and Neptune all have moon systems, each of which is about 10^{-4} of the mass of the orbited planet. This is not extremely different from the ratio of the summed mass of the planets orbiting the Sun. The suggestion is that the moons formed in disks similar to the presolar nebula.

The Galilean satellites of Jupiter display a progression in mean density: Io (3.53 g cm^{-3}), Europa (3.02 g cm^{-3}) Ganymede (1.94 g cm^{-3}) and Callisto (1.85 g cm^{-3}). Curiously, while Callisto has the highest fraction of ice, it has the lowest geometric albedo by far. The usual explanation is that the fraction of ice to rock increases outwards. The disk where the moon formed may have had its own snowline (Canup & Ward 2002, AJ, 124, 3404). Such scenarios may be relevant to all moon systems, but this is highly uncertain. As far as known, all the other large moons in the solar system have densities less than 2 g cm^{-3} and therefore have substantial amounts of ice. Why do Saturn and Neptune have one dominant moon (Titan and Triton, respectively) while Jupiter has 4 large moons as does Uranus (Ariel, Umbriel, Titania and Oberon)? Titan, the most massive moon, with 1.3 $\times$ 10^{26} g is much larger than the total of the entire asteroid belt of 3.6 $\times$ 10^{24} g (Krasinsky et al. 2002, Icarus, 158, 98).

3 Extrasolar Moons

An interesting feature about Moons is their projected area compared to that of their host planet. For the Moon/Earth system this fraction is 7.1%. For the major moons for the outer planets, this fraction is 0.39%, 0.20%, 0.30% and 0.32% for Jupiter, Saturn, Uranus and Neptune, respectively. Except for Moon/Earth, the Moons display a relatively small surface area and therefore may be difficult to detect photometrically in extrasolar planetary systems.

Another method to detect moons might be by eclipse timing. If the center of mass of a planet/moon is sufficiently offset from the center of mass of the planet, then there could be detectable variations in a planet's occultation. To date, no moon has been identified with this approach. It would seem that the Kepler data base would be the best place to search for moons with this technique (Szabo et al. 2006, A&A, 490, 395).

Another possibility would be to identify orbitally modulated nonthermal radio emission analogous to the Io/Jupiter system (Bastian et al. 2000, ApJ, 545, 1058). Gravitational microlensing also might work (Liebig & Warmbsganss 2010, A&A, 520, 68).

4 Rings

The outer planets of the solar system possess rings; Saturn's is the most famous. Because rings have such large areas, they are much easier to detect photometrically than moons (Barnes & Fortney 2004, ApJ, 616, 1193). The mass in rings can be very small and still detectable. That is the mass of Saturn's rings may be only 3 $|times$ 10^{22} g (Esposito 2010, Ann. Rev. Earth Planetary Sci., 38, 383) – much less than the mass of Titan.

Lecture 16 – Minor Planets and Comets

The solar system has a population of minor planets – asteroids, Kuiper Belt Objects and comets in a wide range of dynamically stable locations. These objects are often described as planetesimals that never were incorporated into planets. Dusty debris disks around main-sequence stars are the consequence of the erosion of a population of minor planet parent bodies. Therefore, the high frequency of debris disks orbiting main-sequence stars implies that populations of minor planets are common (Wyatt 2008, Ann. Rev. Astr. Ap., 46, 339).

1 Main Belt Asteroids

The main belt orbits mainly between 2.1 AU and 3.3 AU. The total mass is perhaps 3.6 $\times$ 10^{24} g (Krasinksy et al. 2002, Icarus, 158, 98). The three most massive asteroids with diameters larger than 500 km (Baer et al. 2011, AJ, 141, 143) are Ceres (9.5 $\times$ 10^{23} g, ρ = 2.1 g cm^{-3}), Pallas (3.2 $\times$ 10^{23} g, ρ = 2.6 g cm^{-3}) and Vesta (2.6 $\times$ 10^{23} g; ρ = 3.3 g cm^{-3}). Therefore, the three largest asteroids possess over 40% of the mass. The dispersion in mean density implies a dispersion in bulk composition since there is no reason to imagine that these objects have much porosity. According to the tabulation of Binzel et al. (2000, in Allen's Astrophysical Quantities, 4th edition), their Bond albedos are 0.037, 0.055 and 0.20 for Ceres, Pallas and Vesta, respectively. They therefore have very different surface compositions. The relatively low mean density of Ceres is usually taken to mean that it has a substantial amount of internal water (Thomas et al. 2005, Nature, 437, 224).

There is strong photometric, dynamic and imaging evidence that Vesta is the parent body of the eucrite, diogenite and howardite meteorites (Binzel & Xu 1993, Science, 260, 186; Thomas et al. 1997, Science, 277, 1492). We therefore have laboratory samples of the outer potions of Vesta. Evidence for differentiation is displayed because in the howardites, Si is appreciably more abundant than Mg and Fe – as found in the Earth's crust but not in bulk Earth (Duke & Silver 1967, Geochimica et Cosmochimica Acta, 31, 1637).

The asteroid belt is dynamically "hot". That is, the eccentricities and inclinations are much greater than for planets. For example, Ceres, Pallas and Vesta have orbital inclinations of 10.6°, 34.8° and 7.1°. Except for Mercury which has an orbit with an inclination of 7.0°, all the planets have appreciably smaller inclinations Their orbital eccentricities of the three largest asteroids are 0.078, 0.234 and 0.091 for Ceres, Pallas and Vesta, respectively. For comparison, except again for Mercury with e = 0.21 and Mars with e = 0.093, the planets orbit the Sun with smaller eccentricities. The Kirkwood Gaps show that the asteroid belt has been dynamically stirred and depleted.

The size distribution, for the number of asteroids of radius R is usually approximated (de Pater & Lissauer, Planetary Science) as:

$$n(R)\,dR \;=\; n_0\,R^{-3.5}\,dR \tag{1}$$

There are in fact deviations from this expression although it qualitatively conveys the result that most mass is carried in the larger objects and most area in the smaller bodies. As far as I know, the size distributions for the different families of asteroids follow this more general pattern, but I have not found an analysis of this topic.

2 Other Asteroids

Besides the main-belt, there are other populations of asteroids. One of the most studied systems is the Near Earth Objects since they pose a potential danger to us. There are also the Trojans in stable orbits.

3 Kuiper Belt

The Kuiper Belt is more massive than the asteroid belt by a factor of approximately 20 (Bernstein et al. 2004, AJ, 128, 1364). The most massive members are Pluto (1.3×10^{25} g) and Eris (1.7×10^{25} g) (Brown & Schaller 2007, Science, 316, 1585). While most of the mean densities are thought to be representative of largely icy bodies, Quaoar has a mean density of 4.2 g cm^{-3} and a mass of 1.6×10^{24} g (Fraser & Brown 2010, ApJ, 714, 1547). The semi-major axis of the Kuiper Belt Objects range from about 40 AU to 50 AU. Sedna has a semmi-major axis of 490 AU and clearly has a different dynamic history. There are various sub-populations: classical disk, resonant, and scattered disk.

4 Comets

Comets are composite bodies in the outer solar system which traditionally are viewed as about 50% ice and 50% rock. The "active area" of a comet's nucleus is the region covered by ice from which sublimation occurs because of solar heating. H_2O composes most of the ice, but some comets display activity so far from the Sun that it is likely that other ices such as CO_2 also are present.

There are several populations of comets. Long period comets (those with periods greater than 200 years) originate from the Oort cloud (Francis 2005, ApJ, 635, 1348) which has a total of 5×10^{11} comets with an average mass of 2×10^{17} g implying a total mass near 10^{29} g. Therefore, the Oort cloud is the largest mass reservoir of minor planets in the solar system. The arrival rate of dynamically new comets (those with e near 1) is about 0.8 AU^{-1} yr^{-1} with a flat distribution in AU. The arrival directions are approximately isotropic.

During its first passage through the inner solar system, the comet's motion is perturbed by Jupiter and other planets. As a consequence a small amount of energy is either added or subtracted to the comet. If added, the comet becomes unbound and never returns. If subtracted, the comet returns but with an appreciably smaller eccentricity (Wiegert &

Tremaine 1999, Icarus, 137, 84). During each return the comet's orbit is perturbed again, and there is an orbital evolution. There are fewer multi-visting comets than expected on the basis of these models. This is known as the "fading problem"; comets disappear faster than predicted.

Short period comets have periods less than 200 years. There are two sub-families: Jupiter family comets with periods less than 20 years and Halley family comets with periods between 20 and 200 years. The Jupiter period comets have relatively low inclinations and may originate in the Kuiper Belt or from the Trojans of Neptune (Sheppard & Trujillo 2006, Science, 313, 511).

No interstellar comets have been detected although they would be easily recognized as such from their orbits. Because the formation of Oort clouds is inefficient, the absence of interstellar comets strongly suggests that our own Oort cloud is unusual (see Jura 2011, AJ, 141, 155).

The Tisserand parameter, T, is nearly constant over many orbits. If a_J denotes Jupiter's semi-major axis, then

$$T \;=\; \frac{a_J}{a} \,+\, 2\left(\frac{a}{a_J}(1\,-\,e^2)\right)^{1/2}\,\cos i \tag{2}$$

Asteroids typically have $T > 3$ while comets have $T < 3$. Note that e and i can individually change substantially as in a Kozai resonance.

Active comets lose both gas and dust. The gas is excited by fluorescence. Strong molecular bands from C_2, OH and CN are some of the most salient emissions (A'Hearn et al. 1995, Icarus, 118, 223). The usual picture is that parent molecules (such as H_2O) sublimate off the cometary nucleus. Daughter molecules are formed; mainly by photo-dissociation such as:

$$h\nu \,+\, H_2O \rightarrow OH \,+\, H \tag{3}$$

The dust from comets also scatters sunlight. Because the dust and gas disperse from the nucleus at $\sim$1 km s^{-1}, a comet can develop a long tail. Large comets such as Hale-Bopp (with a radius of between 14 and 21 km, Weaver et al. 1997, Science, 275, 1900) can be very bright. In fact, in extrasolar planetary systems, a bright comet can be brighter than the Earth (Jura 2005, AJ, 130, 1261).

Observational Characterization of Super-Earths

June 3, 2011

Super Earths are a proposed class of planets with bulk properties intermediate between solar system rocky and ice giant planets; no such objects exist in the solar system, so they are a new type of object with as-yet-undetermined properties (see the table included with this document).

The phrase "super-Earth" is of uncertain origin, though its etymology is clear. The earliest reference I can find is a 2001 AAS poster abstract (The Extra-Solar Planet Imager [ESPI]); several papers in the middle of that decade suggest that planets as massive as 14 Earth masses could be considered super-Earths if they are rocky. More recent references that show the term gaining widespread acceptance in the field are are Valencia et al. 2007 (ApJ 656:545) and Fortney et al. 2007 (ApJ 659:1661); both set an upper limit of 10 Earth masses, while Fortney suggests a lower mass limit of 5 Earth masses. This was straightforward enough at the time because all these objects (discovered via radial velocity surveys) had measured minimum masses. In the current age of Kepler the definition of a super Earth becomes more complicated because we will often have planetary radii but no masses, especially for low-mass planets in long-period orbits. The IAU has yet to weigh in, but upper limits of 2-4 Earth radii have been suggested. Lower limits on mass or radius remain to be determined, but will presumably be set larger and more massive than the Earth.

Some of these so-called super Earths lend themselves to straightforward interpretation: some seem to be quite rarified, with densities < 1 g cm^{-3} (Kepler-11 d,e,f) and so are likely to be gas-dominated. On the other hand some have densities ranging from 7-16 g cm^{-3} (55 Cnc e, CoRoT-7b, Kepler-9d and -10b) – these are presumably dominated by rocks and "terrestrial" maeterials and have little or no volatile content. At least two have intermediate densities: Kepler-11b (4 g cm^{-3}, similar to Mars but on a planet 25 times more massive) and GJ 1214b (2 g cm^{-3}). It's too early to tell whether we have a dichotomous set of objects or just a continuum.

Defining super Earths by either mass or radius seems to work equally well, since smaller planets seem to be lighter. A better metric may turn out to be a combination of these parameters, since it seems likely (to me) that a rarified H-dominated planet, even one of 3 Earth masses, sounds more like some sort of mini-Neptune than a super-Earth.

The most intermediate of these planets are both especially interesting, and especially frustrating, because their interior compositions are poorly constrained by measurements of mass and radius. GJ 1214b, at 2 g cm^{-3}, is currently the best example of this; interior modeling suggests the planet probably has a large gas component, but this could be either a H/He mix (< 5 % by mass) on a "standard" rocky core or a steam/vapor envelope on a world that is 50% H_2O (Rogers & Seager 2010, ApJ 716:1208; Nettelmann et al. 2011, ApJ 733:2); the former paper notes that the bulk parameters are only 2 sigma discrepant from models wholly lacking an atmosphere. Atmospheric characterization is the most promising way to probe the nature of this planet, which has been the study of many (and ongoing) observational campaigns.

There are two main ways to probe the atmosphere of an exoplanet: (1) measure emission by looking for the flux decrement when the planet passes behind its star (secondary eclipse), and (2) measure absorption by looking for the variation with wavelength of the transit depth (transmission spectroscopy). The latter quantity, in a much-simplified case, can be estimated as $(R_P/R_*)^2 \frac{T_P}{T_*}$. In the case of absorption, one probes the limb of the planet and the signal (Miller-Ricci et al. 2009) is proportional to HR_P/R_*^2 (where H is the atmospheric scale height $k_B T_P/[\mu g_P]$), or $\frac{1}{R_*^2}\frac{T_P}{\rho_P \mu}$. So both measures prefer large, hot planets around small stars; absorption tends to become more preferred as planets get smaller, less massive, cooler, and to have atmospheres of lower mean molecular weight (μ). Clouds or haze (see Sing et al. 2011) can confuse the issue further and are still poorly understood in these objects.

Because GJ 1214b orbits an M dwarf it's too cool (¡600 K) to expect to see much emission short of 5 microns – so the hope is to use transmission spectroscopy. The planet's transit depth is about the same in the optical (Bean et al. 2010), J band (Croll et al. 2011), and IRAC 1 and 2 channels (Desert et al. 2011), which would suggest either an atmosphere in chemical equilibrium with high μ, a low-μ atmosphere out of equilibrium, a planet devoid of atmosphere, or an opaque cloud deck. Ks-band transits (Croll et al. 2011) are significantly deeper (on 4 separate occasions!), which strongly argues for an atmosphere with low μ; taken together this all suggests a H-dominated atmosphere depleted in methane relative to equilibrium expectations. We (Crossfield et al., in press) looked at GJ 1214b with NIRSPEC, and detected no variation of transit depth with wavelength; consistent with other results, but NIRSPEC isn't an ideal instrument for these observations (ask me why MOSFIRE will be much, much better!). Methane depletion has also been claimed in the warm Neptune-mass planet GJ 436b (which also orbits an M dwarf), so maybe we're starting to see a trend. More Spitzer and HST observations are in the works as well, so stay tuned for more on GJ 1214b and future super Earths – whatever they are.

Table 1: Super Earths with Measured Masses and Radii

Name	M/M_E	R/R_E	ρ (g cm^{-3})	g (m s^{-2})	$T_{eq,approx}$ (K)	K mag	R_P/R_*
Kepler-11 f	2.3 ± 1.7	2.55 ± 0.24	0.7 ± 0.6	3.2 ± 2.4	680	12.2	0.023
Kepler-11 b	4.4 ± 2.1	1.92 ± 0.19	3.3 ± 1.9	10 ± 5	1100	12.2	0.017
Kepler-10 b	4.6 ± 1.2	1.377 ± 0.003	9.4 ± 2.5	22 ± 6	2600	9.5	0.012
Kepler-11 d	6.2 ± 2.5	3.34 ± 0.31	0.9 ± 0.4	4.9 ± 2.1	860	12.2	0.027
GJ 1214 b	6.5 ± 0.9	2.577 ± 0.088	2.0 ± 0.4	8.7 ± 1.4	660	8.8	0.116
CoRoT-7 b	7.3 ± 1.4	1.633 ± 0.088	8.9 ± 2.2	24 ± 5	2200	9.8	0.019
Kepler-11 e	8.5 ± 2.2	4.41 ± 0.42	0.5 ± 0.2	3.9± 1.3	780	12.2	0.036
55 Cnc e	8.6 ± 0.6	1.6-2.1	4.5-11	19-32	2500	4.0	0.014
Earth	1	1	5.5	9.8	330		0.009
Uranus	14.5	4.0	1.3	9	76		0.038
Neptune	17	3.9	1.8	11	61		0.036

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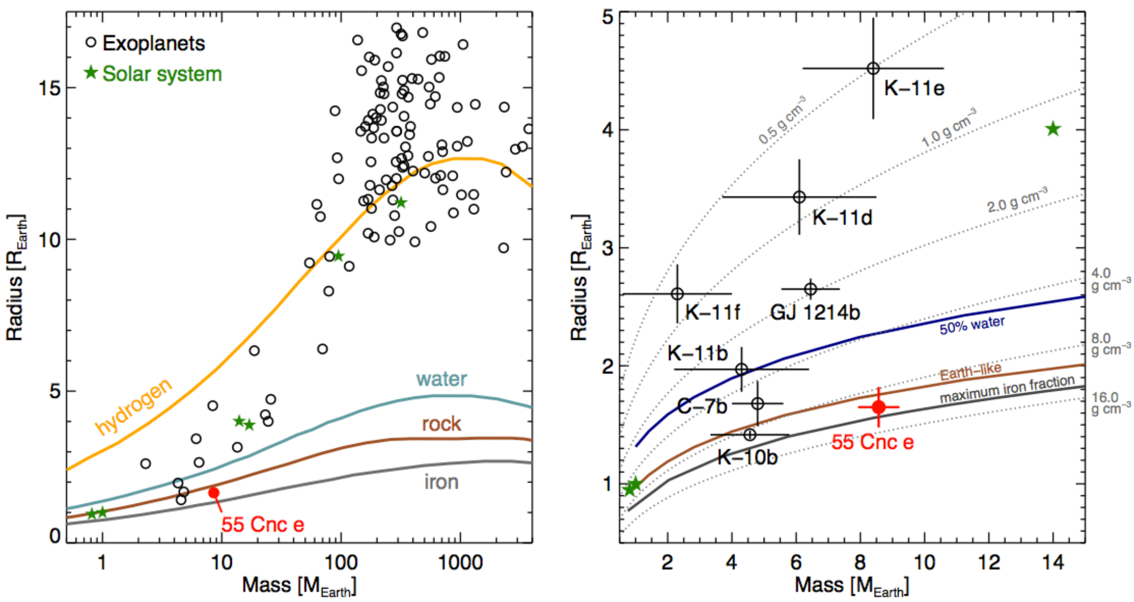

FIG. 3.— **Masses and radii of transiting exoplanets.** Open circles (blue) are previously known transiting planets. The filled circle (red) is 55 Cnc e. The stars (green) are Solar system planets, for comparison. *Left.*—Broad view, with curves showing mass-radius relations for pure hydrogen, water ice, silicate ($MgSiO_3$ perovskite) and iron, from Figure 4 of Seager et al. (2007). *Right.*—Focus on super-Earths, showing contours of constant mean density and a few illustrative theoretical models: a "water-world" composition with 50% water, 44% silicate mantle and 6% iron core; a nominal "Earth-like" composition with terrestrial iron/silicon ratio and no volatiles (Valencia et al. 2006, Li & Sasselov, submitted); and the maximum mantle stripping limit (maximum iron fraction, minimum radius) computed by Marcus et al. (2010).

Figure 1: