UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 1
Due: Thursday, February 13, 2025, in class
This problem set is worth **74 points**.

1. **Blackbody radiation [16 pts].** The Planck radiation spectrum is given by

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \quad (\text{erg cm}^{-2}\ \text{s}^{-1}\ \text{Hz}^{-1}\ \text{steradian}^{-1}),$$

per unit frequency.

   (a) **Wavelength spectrum [4 pts].** Show by explicit calculation that the equivalent Planck radiation spectrum per unit *wavelength* is given by

$$B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1} \quad (\text{erg cm}^{-2}\ \text{s}^{-1}\ \text{cm}^{-1}\ \text{steradian}^{-1}),$$

   starting from the expression for B_ν.

   (b) **Stefan-Boltzmann law. [4 pts]** Derive the Stefan-Boltzmann law ($F = \sigma T^4$) by integrating the Planck blackbody spectrum over all wavelengths or frequencies. (Note that there is an extra factor of π to convert from brightness per unit solid angle to total brightness, so that $F = \pi \int B_\nu d\nu = \pi \int B_\lambda d\lambda$.) You may use the fact that

$$\int_0^\infty \frac{u^3}{e^u - 1} du = \frac{\pi^4}{15}.$$

   Give an expression for the Stefan-Boltzmann constant σ in terms of fundamental physical constants, and check its numerical value and units, $\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}.

   (c) **Wavelength of radiation peak. [4 pts]** Derive the Wien displacement law, which relates the wavelength of the radiation at the peak of the Planck function B_λ to the temperature: $T\lambda_{\max} = 0.29$ cm K. [When you differentiate to find the maximum of B_λ, you will obtain a nonlinear equation of the form $5(1-e^{-y}) - y = 0$ which you can solve numerically.]

   (d) **Frequency of radiation peak. [4 pts]** Repeat the previous part, but this time find the relation between the *frequency* at the peak of the Planck function B_ν and the temperature: $\nu_{\max}/T = 5.9 \times 10^{10}$ Hz K^{-1}. For a given temperature T, does the photon energy corresponding to $\nu_{\max}$ agree with that for $\lambda_{\max}$ in the previous part? Should they agree? Explain.

2. **Angular diameters and effective temperatures [10 pts].**

   (a) Show that if you can measure the bolometric flux F and the angular diameter ϕ of a star, then you can determine the effective temperature T_{eff} even if you do not know the distance to the star. Note, "bolometric" means "integrated over all frequencies." [4 pts]

   (b) In one recent application of this technique, astronomers used optical interferometry to measure the angular diameters of both stars in the binary system β CrB. The results were 0.699 ± 0.017 mas for star A, and 0.415 ± 0.017 mas for star B, where "mas" means milli-arcseconds. The bolometric (total wavelength-integrated) *apparent* magnitudes of stars A and B are 3.87 ± 0.05 and 5.83 ± 0.10, respectively. The bolometric *absolute* magnitude of the Sun is 4.75, and the effective temperature of the Sun is 5777 K. Use this information to calculate the effective temperatures of stars A and B. You need not calculate the uncertainties (though if you care to try, it wouldn't hurt). [6 pts]

   (In case you are curious to learn more, the reference is Bruntt et al. 2010, *Astron. & Astrophys.*, 512, 55.)

3. **Protons or photons? [8 pts]**

   At the center of the Sun, the density is approximately 150 g cm^{-3} and the temperature is about 15×10^6 K. Which is larger: the number density of protons, or the number density of photons? Give an order of magnitude estimate of each.

4. **The Eddington limit [20 pts]**

   A star with sufficiently high radiation pressure will spontaneously eject material from its surface. This sets a practical limit on the maximum luminosity of a star of a given mass.

   (a) [14 pts] Start with the radiative diffusion equation and the equation for hydrostatic equilibrium. Assume the opacity to be frequency-independent, and show that the luminosity at which the radiation pressure gradient equals the hydrostatic pressure gradient is given by

   $$L_{\text{Edd}} = \frac{4\pi G M c}{\kappa}, \tag{1}$$

   where M is the stellar mass. This is the "Eddington luminosity."

   (b) [6 pts] For ionized hydrogen, a minimum value for κ arises from Thomson scattering, which has cross-section $\sigma_T = 6.65 \times 10^{-25}$ cm^2. Show that for this case

   $$L_{\text{Edd}} \approx 3 \times 10^4 \, L_\odot \left(\frac{M}{M_\odot}\right), \tag{2}$$

   where $L_\odot = 3.839 \times 10^{33}$ erg s^{-1} and $M_\odot = 1.989 \times 10^{33}$ g.

5. **A fictional star [20 pts]**

   Consider a star of luminosity L with density distribution $\rho = \rho_0 \times (R/r)$, where R is the star's outer radius. Please don't ask how it manages to have such a simple density profile; this star exists only in the homework universe.

   All of the star's energy is generated from a very small region near $r = 0$, and is transported entirely by radiation (not convection). The opacity is dominated by electron (Thomson) scattering, with opacity κ_T (in cm$^-2$/g).

   (a) [4 pts] What is the star's effective temperature T_{eff}, in terms of the given quantities and fundamental constants?

   (b) [16 pts] Solve for the temperature as a function of r, in terms of ρ_0, T_{eff}, R, κ_T, and fundamental constants. For the outer boundary condition, assume that when $r = R$ then $T(R) = T_{\text{eff}}$ (this is just the gray-atmosphere result that $T(\tau = 2/3) = T_{\text{eff}}$).