UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 2: Transport, Distributions, Eqns of State
Due: Thursday, March 6, 2025, in class
This problem set is worth **70 points**.

1. **Saha equation and pure hydrogen [15 pts].** Consider a gas of pure hydrogen at fixed density and temperature. The ionization energy of hydrogen is $\chi_0 = 13.6$ eV. You may assume that all the hydrogen atoms (whether neutral or ionized) are in their ground energy state.

   (a) Write down the Saha equation relating the number densities of neutral and ionized hydrogen (n_0 and n_1, respectively). Make reasonable approximations to use numerical values for the partition functions.

   (b) To find the individual densities, further constraints are required. Reasonable constraints are charge neutrality ($n_e = n_1$) and conservation of nucleon number ($n_1 + n_0 = n$), where the total hydrogen number density n is a constant if the density ρ is fixed. Rewrite the Saha equation in terms of the hydrogen ionization fraction $x = n_1/n$, eliminating n_1, n_0, and n_e. Does this equation have the expected limiting behavior for $T \to 0$ and $T \to \infty$?

   (c) Use the relation $\rho = m_{\rm H} n$ (where $m_{\rm H} = 1$ gm$/N_{\rm A}$, where $N_{\rm A} = 6.023 \times 10^{23}$ is Avogadro's number) to replace n with ρ. Find an expression for the half-ionized ($x = 0.5$) path in the ρ-T plane. Plot this path on a log-log plot for densities in the interesting range from 10^{-10}–10^{-2} g cm^{-3}

2. **Saha equation and pure helium [20 pts].**

   Consider a gas of pure helium at fixed density and temperature. The ionization energies for helium are $\chi_0 = 24.6$ eV (from neutral to singly ionized) and $\chi_1 = 54.4$ eV (from singly to doubly ionized). You may assume that all the helium atoms (whether neutral, singly ionized, or doubly ionized) are in their ground energy state. Let n_e, n_0, n_1, and n_2 be the number densities of, respectively, free electrons, neutral atoms, singly-ionized atoms, and doubly-ionized atoms. The total number density of neutral atoms and ions is denoted by n. Define x_e as the ratio n_e/n, and let x_i be n_i/n where $i = 0, 1, 2$. You should assume that the gas is electrically neutral. The degeneracy factors you need for the atoms and ions are 2 for He, 4 for He^+, and 2 for He^{2+}.

   (a) Construct the ratios n_1/n_0 and n_2/n_1 using the Saha equation. In doing so, take care in establishing the zero points of energy for the various constituents.

   (b) Apply charge neutrality and nucleon number conservation ($n = n_0 + n_1 + n_2$) and recast the above Saha equations so that only x_1 and x_2 appear as unknowns. The resulting two equations have T and n [or, equivalently, $\rho = n m_{\rm He} = n(4\,{\rm gm}/N_{\rm A})$] as parameters.

   (c) Simultaneously solve the two Saha equations for x_1 and x_2 for temperatures in the range $4 \times 10^4 \le T \le 2 \times 10^5$ K. Do this for a fixed density with the three values $\rho = 10^{-4}$, 10^{-6}, or 10^{-8} g cm^{-3}. You may find it more convenient to use the logarithm of your equations. Choose a dense grid in temperature because you will soon plot the results. Once you have found x_1 and x_2, also find x_e and x_0 for the same range of temperature. Note that this is a numerical exercise; you will want to use a tool like Mathematica or Matlab for this.

   (d) Plot all your xs as a function of temperature for your chosen value of ρ. (Plot x_0, x_1, and x_2 on the same graph.) Identify the transition temperatures (half-ionization) for the two ionization stages.

3. **Stability against convection [10 pts]**

   (a) In lecture, we derived the condition

$$\left| \frac{dT}{dr} \right| < \frac{T}{P} \left(1 - \frac{1}{\gamma_a} \right) \left| \frac{dP}{dr} \right|$$

for stability against convection. Using the appropriate equation(s) of stellar structure and noting the sign of the radial gradients, show that this can be recast as a condition on the luminosity profile:

$$L(r) < \left(1 - \frac{1}{\gamma_a}\right) \frac{64\pi\sigma_{\rm SB} T^4 GM(r)}{3\kappa_R P}$$

(b) Show that to avoid convection in a stellar region where the equation of state is that of an ideal monatomic gas, the luminosity at a given radius must be limited by

$$L(r) < 1.22 \times 10^{-18} \frac{\mu T^3}{\kappa_R \rho} M(r)$$

where μ is the mean molecular weight, $T(r)$, κ_R is the Rosseland mean opacity, and $M(r)$ is the mass enclosed at radius r. All quantities are measured in the appropriate cgs units.

4. **Protons or photons? [10 pts]**

At the center of the Sun, the density is approximately 150 g cm^{-3} and the temperature is about 15×10^6 K. Which is larger: the number density of protons, or the number density of photons? Give an order of magnitude estimate of each.

5. **The Eddington limit [15 pts]**

A star with sufficiently high radiation pressure will spontaneously eject material from its surface. This sets a practical limit on the maximum luminosity of a star of a given mass.

(a) [10 pts] Start with the radiative diffusion equation and the equation for hydrostatic equilibrium. Assume the opacity to be frequency-independent, and show that the luminosity at which the radiation pressure gradient equals the hydrostatic pressure gradient is given by

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa}, \tag{1}$$

where M is the stellar mass. This is the "Eddington luminosity."

(b) [5 pts] For ionized hydrogen, a minimum value for κ arises from Thomson scattering, which has cross-section $\sigma_T = 6.65 \times 10^{-25}$ cm^2. Show that for this case

$$L_{\rm Edd} \approx 3 \times 10^4 \; L_\odot \; \left(\frac{M}{M_\odot}\right), \tag{2}$$

where $L_\odot = 3.839 \times 10^{33}$ erg s^{-1} and $M_\odot = 1.989 \times 10^{33}$ g.