UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 3: Interiors
Due: Tuesday, March 25, 2025, in class
This problem set is worth **60 points**.

1. **Overcoming the Coulomb barrier [15 pts]**

   In this problem you will show that classical mechanics predicts that hydrogen fusion cannot happen in the Sun.

   (a) Suppose two protons approach each other with equal speeds. What is the minimum speed needed to overcome the Coulomb barrier and collide, neglecting quantum effects? Take the radius of a proton to be $\approx$1 fermi $= 10^{-13}$ cm. [5 pts]

   (b) Assuming the proton speeds obey a Maxwell-Boltzmann distribution

   $$p(v) = \sqrt{\frac{2}{\pi}\left(\frac{m_p}{kT}\right)^3} v^2 \exp(-m_p v^2/2kT)$$

   with $T = 15.7 \times 10^6$ K (the central temperature of the Sun), what is the most probable speed (i.e., the speed at the peak of the distribution function)? [5 pts]

   (c) You might wonder whether a small minority of protons in the tail of the M-B distribution could fuse. Give an order of magnitude estimate for the number of protons in the Sun, and for the number of those protons that are energetic enough to fuse. You may find it useful to know that for large u_0,

   $$\frac{4}{\sqrt{\pi}} \int_{u_0}^{\infty} u^2 e^{-u^2} du \approx \frac{2}{\sqrt{\pi}} u_0 e^{-u_0^2}. \quad \text{[5pts]}$$

2. **Polytropes [35 pts].** They're old-fashioned, but polytropic interior models (where $P = K\rho^{1+1/n}$) can provide some insights that modern numerical models struggle to provide.

   (a) For generic index n, derive the Lane-Emden equation from the equation of hydrostatic equilibrium. [5 pts]

   (b) Show that the total mass of a polytropic star is

   $$M = 4\pi\rho_c \lambda_n^3 z_{\rm surf}^2 \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}} .$$

   The factor λ_n is defined as

   $$\lambda_n \equiv \left[(n+1) \frac{K\rho_c^{(1-n)/n}}{4\pi G} \right]^{1/2}$$

   (you may assume this form), and $z_{\rm surf}$ specifies the outer radius of the star: $\phi_n(z_{\rm surf}) = 0$. [4 pts]

   (c) Show that the ratio of the mean density to the central density is

   $$\frac{\langle \rho \rangle}{\rho_c} = \frac{3}{z_{\rm surf}} \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}} . \quad \text{[4pts]}$$

   (d) Show that the central pressure is

   $$P_c = \frac{GM^2}{R^4} \left[4\pi(n+1) \left| \frac{d\phi_n}{dz} \right|^2_{z=z_{\rm surf}} \right]^{-1} . \quad \text{[4pts]}$$

   (e) Write a quick program that numerically solves the Lane-Emden equation for $\phi(z)$ and $z_{\rm surf}$ given arbitrary n. Provide your code (it doesn't have to be pretty). Use it to solve for $\phi(z)$ and plot it for n=1, 2, 3, and 4. Show that your result agrees well with the analytic solution of $\phi(z)$ for n=1. [8 pts]

(f) Use your code to model and plot $\phi(r)$, $P(r)$, $\rho(r)$, and the enclosed mass $M(r)$ for the Sun, assuming n=3. [4 pts]

(g) Compute the implied nuclear luminosity of your polytropic Solar model. Take the nuclear energy generation rate per unit volume to be

$$\epsilon_V = (2.46 \times 10^6)\, \rho^2 X^2 T_6^{-2/3} \exp\left(-33.81\, T_6^{-1/3}\right) \;\; \mathrm{erg\ s^{-1}\ cm^{-3}},$$

where ρ is in g cm^{-3}, T_6 is the temperature in units of 10^6 K, and $X = 0.6$ is the hydrogen mass fraction. First, write the calculation as the product of a dimensioned constant and a dimensionless integral involving ϕ_n and z. (For the T_c inside the integral you can use 15.7×10^6 K.) Show the value of your constant, and the form of the dimensionless integral. Then, evaluate the nuclear luminosity in erg s^{-1}. Compare to the actual luminosity of 3.839×10^{33} erg s^{-1}. [6 pts]

3. **Opacity due to Thomson scattering [10 pts]**

Consider an atmosphere of completely ionized hydrogen having the same mass density as Earth's atmosphere at sea level ($\rho = 1.23$ kg m^{-3}). Calculate the path length over which a beam of light would be attenuated to half of its original intensity, due to Thomson scattering by free electrons.