UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 4: Atmospheres
Due: Thursday, April 10, 2025, in class
This problem set is worth **70 points**.

1. **Gray, Plane-Parallel, Eddington-Approximation Atmosphere [20 pts]**

   Show that in a plane-parallel, gray atmosphere under the Eddington Approximation:

   (a) $S = \langle I \rangle$,

   (b) $P_{\rm rad} = \frac{F}{c}\left(\tau + Q\right)$ (where Q is a constant of integration),

   (c) $S = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right)$, and

   (d) $T(\tau) = T_{\rm eff}\left(\frac{3\tau}{4} + \frac{1}{2}\right)^{1/4}$.

2. **Limb darkening [20 pts].**

   In this problem you will derive a relation between the measured limb darkening of a star, and the source function of its photosphere. Let the intensity of the stellar disk be $I_\nu(r)$, where r is the distance from the center of the stellar disk in units of the stellar radius (i.e. $r = 0$ at the center, and $r = 1$ at the limb).

   (a) Instead of r it is traditional to express I_ν as a function of $\mu \equiv \sqrt{1 - r^2}$. Show that $\mu = \cos\theta$, where θ is the angle between the line of sight and the normal to the stellar surface.

   (b) We want an expression for the intensity at the stellar surface in terms of the source function. Start from the the radiative transfer equation for a plane-parallel atmosphere. Show that for an upward-propagating ray coming from far below to the top surface, the formal solution is

   $$I_\nu(\mu) = \int_0^\infty d\tau_\nu \, \frac{S_\nu(\tau_\nu)}{\mu} e^{-\tau_\nu/\mu}, \tag{1}$$

   where τ_ν is the vertical optical depth.

   (c) Suppose the (unknown) source function can be represented by a polynomial,

   $$S_\nu(\tau_\nu) = a_0 + a_1\tau_\nu + a_2\tau_\nu^2 + \cdots + a_n\tau_\nu^n. \tag{2}$$

   Show that under this assumption the emergent intensity is given by

   $$I_\nu(\mu) = a_0 + a_1\mu + 2a_2\mu^2 + \cdots + (n!)a_n\mu^n, \tag{3}$$

   using the definite integral $\int_0^\infty x^n \exp(-x)dx = n!$. In this way the measured limb-darkening law can be used to determine the source function, and therefore the temperature stratification for an LTE atmosphere.

   (d) Show that for a gray LTE atmosphere, the predicted limb darkening law for the wavelength-integrated intensity at the stellar surface is

   $$\frac{I(\theta)}{I(0)} = \frac{2}{5} + \frac{3}{5}\cos\theta.$$

3. **Radiative transfer in spherical coordinates [20 pts].**

After the past month's classes you should be familiar with the radiative diffusion equation for a plane-parallel atmosphere, an appropriate model for a thin photosphere. In this problem you will repeat those steps for a spherical atmosphere, as appropriate for the bulk of a star. We will assume the star is spherically symmetric and that consequently $I_\nu = I_\nu(r, \theta)$, where r is the radial coordinate and θ is the angle of a ray relative to the local radius vector (and *not* the polar angle referring to the position with respect to the stellar center). See Fig. 1.

(a) Use the chain rule,

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}, \tag{4}$$

to show that the radiative transfer equation (RTE) can be written

$$\cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial I_\nu}{\partial\theta} + \rho\kappa_\nu I_\nu - j_\nu = 0. \tag{5}$$

In this expression, κ_ν is the *opacity*, measured in units of cm^2 g^{-1}; and j_ν is the *emission coefficient*, measured in units of erg cm^{-3} s^{-1} sr^{-1} Hz^{-1} [both as defined by Rybicki & Lightman (p. 9-10)].

(b) Integrate the RTE over all solid angles to show

$$\frac{dF_\nu}{dr} + \frac{2}{r}F_\nu + c\rho\kappa_\nu u_\nu - \rho\epsilon_\nu = 0, \tag{6}$$

where ϵ_ν is the (angle-averaged) *emissivity* as defined on p. 9 of Rybicki & Lightman.

(c) Multiply the RTE by $\cos\theta$ and integrate over all solid angles to show

$$c\frac{dp_\nu}{dr} + \rho\kappa_\nu F_\nu = 0, \tag{7}$$

where you have assumed j_ν to be isotropic, and I_ν to be nearly isotropic. Here, p_ν is the *specific radiation pressure* given by

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega. \tag{8}$$

(d) Use the preceding equation, as well as the blackbody formula for radiation pressure, the relation $F = L/4\pi r^2$ and the definition of the Rosseland mean opacity κ_R to show

$$\frac{dT}{dr} = -\frac{3\rho\kappa_R L}{64\pi\sigma r^2 T^3}. \tag{9}$$

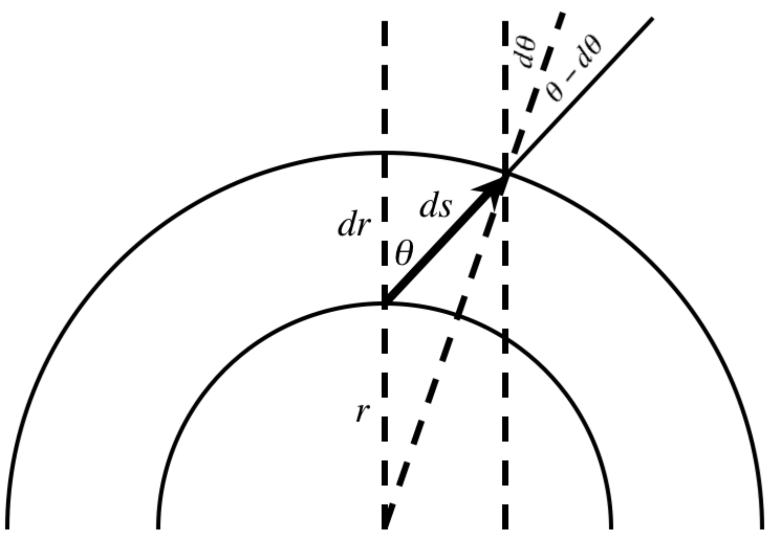

Figure 1: Geometry relevant to Prob. 3. A photon propagates a distance ds along a direction θ from the local radius vector. As a result its radial coordinate increases by dr and the angle to the local radius vector decreases by $d\theta$.

4. **Corona time [10 pts].**

   The solar corona may have a base electron density of 10^8 cm^{-3} at $T = 2 \times 10^6$ K. Assume that the corona has an inner radius equal to that of the Sun, the corona is isothermal and that it obeys the equation of hydrostatic equilibrium. Compute the X-ray free-free emission from this model corona and compare with the total luminosity of the Sun