Chapter Two

Intensity and Flux

2.1 INTRODUCTION

The main goal of this book is to understand the origin of and physical processes affecting a planetary spectrum. We shall see just how much information can be derived from a planetary spectrum: the kinds of gases and solid particles present, the planetary albedo, and constraints on the vertical temperature structure. We begin with concepts and variables used to quantify radiation traveling through a planetary atmosphere.

As a foundational language for exoplanet atmospheres, these radiation terms are so important that we spend a chapter defining them carefully. This is especially important because the definitions of intensity, surface flux, and flux at Earth are used differently in other books and in the literature. In this chapter we will use the term "surface" to describe either the solid surface of a planet or the layers of the atmosphere from which radiation emerges.

2.2 INTENSITY

To begin with we need a description of radiation in the exoplanet atmosphere. We may think of radiation as energy in the form of photons traveling through the planet atmosphere. This radiation interacts with different matter particles. The conventional description of radiation considers the energy of a number of identical photons in a single beam of radiation, called the intensity I.

As the beam of radiation travels through the planet atmosphere, photons will be absorbed into and emitted out of the beam. Different parts of the planet, therefore, have different intensities, and the intensity varies with frequency. We cannot measure the intensity coming from a specific part of the interior or exterior of an exoplanet. This is because the exoplanet is so distant that the planet atmosphere cannot be spatially resolved. Nevertheless, I is a macroscopic parameter describing the sum of all microscopic processes going on in the beam of radiation. We therefore need to compute I in detail and carry it along through calculations of radiative transfer until we are ready to compute the final quantities of radiation that we are interested in.

The intensity I is formally defined as the amount of energy passing through a surface area dA, within a differential solid angle $d\Omega$ centered about $\hat{\mathbf{n}}$, per frequency

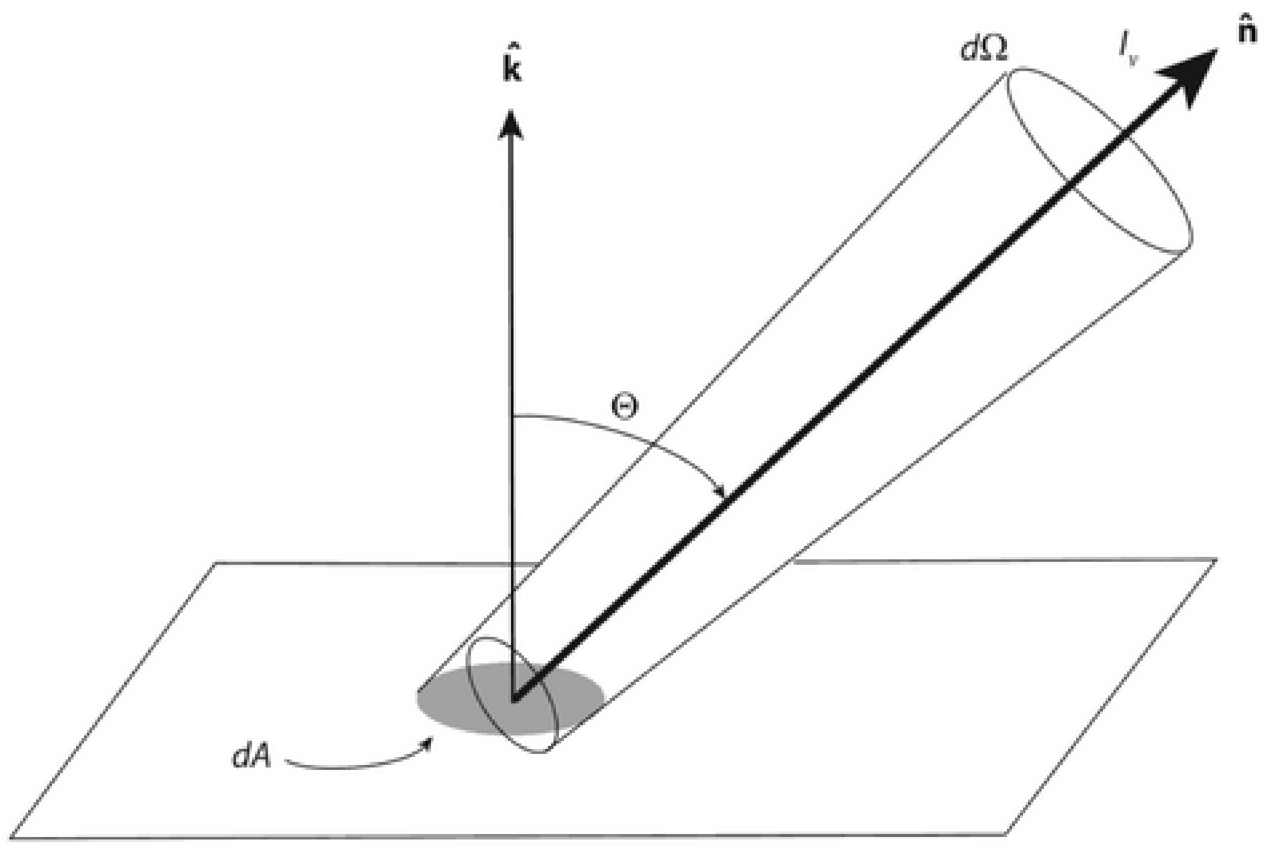

Figure 2.1 Definition of the specific intensity $I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)$.

interval, per unit time (Figure 2.1),

$$\boxed{dE(\nu, t) = I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} d\Omega dA d\nu dt.} \tag{2.1}$$

The SI units of I are J m^{-2} sr^{-1} s^{-1} Hz^{-1}. I is the intensity at location $\mathbf{x}$, going into direction $\hat{\mathbf{n}}$; directionality is implied despite the point that I is a scalar quantity.

We denote vectors in boldface, using italic for 1D and 2D vectors and roman for 3D vectors. When necessary to specify the direction of I or other scalars, we will denote a direction with a subscript.

The mean intensity is the intensity averaged over a solid angle,

$$\boxed{J(\mathbf{x}, \nu, t) = \frac{1}{4\pi} \int_\Omega I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t) d\Omega.} \tag{2.2}$$

See Figure 2.2 for a definition of solid angle.

2.3 FLUX AND OTHER INTENSITY MOMENTS

The quantity of radiation that we do measure from exoplanets is related to the flux. The flux is the net flow of energy through an arbitrarily oriented surface area dA with normal $\hat{\mathbf{n}}$, per frequency interval, per unit time. The flux $\mathbf{F}$ is derived from the intensity in direction $\hat{\mathbf{n}}$ integrated over solid angle Ω ,

$$\boxed{\mathbf{F}(\mathbf{x}, \nu, t) = \int_\Omega I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)\hat{\mathbf{n}} d\Omega.} \tag{2.3}$$

$\mathbf{F}(\mathbf{x}, \nu, t)$ has units of J m^{-2} s^{-1} Hz^{-1}. $\mathbf{F}(\mathbf{x}, \nu, t)$ is defined at each location $\mathbf{x}$ in the planetary atmosphere.

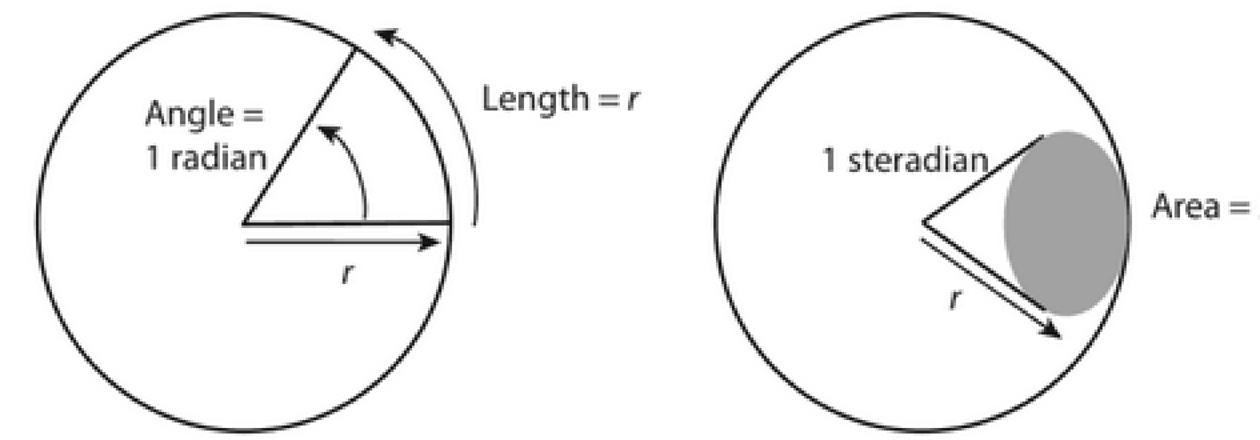

Figure 2.2 Definition of a radian (left panel) and a steradian (right panel). A steradian is related to the surface area of a sphere as a radian is related to the circumference of a circle. In 2D 1 radian is the angle subtended at the center of a circle by an arc length equal to the radius of a circle. In 3D 1 steradian is a measure of solid angle and is the solid angle subtended at the center of a sphere of radius r having an area r^2.

Flux is a vector and so we may write it in terms of its vector components. In a rectangular Cartesian coordinate system,

$$\mathbf{F}(\mathbf{x}, \nu, t) = F_i(\mathbf{x}, \nu, t)\hat{\mathbf{i}} + F_j(\mathbf{x}, \nu, t)\hat{\mathbf{j}} + F_k(\mathbf{x}, \nu, t)\hat{\mathbf{k}}. \tag{2.4}$$

In describing planetary atmospheres we are usually interested in the flux in one direction, the direction toward us, the observer. It is customary to take one component of the flux vector $F_k(\mathbf{x}, \nu, t)\hat{\mathbf{k}}$, writing only the magnitude $F_k(\mathbf{x}, \nu, t)$, a scalar quantity. We are essentially describing the energy flow in one direction

$$\boxed{F_k(\mathbf{x}, \nu, t) = \frac{dE(\nu, t)}{dA d\nu dt} = \int_\Omega I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} d\Omega.} \tag{2.5}$$

To further complicate the issue, the directional subscript is usually dropped so as to just write the flux in the direction of the observer as $F(\mathbf{x}, \nu, t)$.

The flux is also called the first moment of intensity. The zeroth moment of the intensity is the mean intensity, defined above in equation [2.2]. The second moment of intensity is

$$\boxed{\mathbf{K}(\mathbf{x}, \nu, t) = \frac{1}{4\pi} \int_\Omega I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)\hat{\mathbf{n}}\hat{\mathbf{n}} d\Omega.} \tag{2.6}$$

$\mathbf{K}(\mathbf{x}, \nu, t)$ is a tensor quantity related to the radiation pressure tensor $\mathbf{P}(\mathbf{x}, \nu, t)$, by $\mathbf{P} = \frac{1}{c}\mathbf{K}$, where c is the speed of light. When we come to use K in Chapter 6, we will take the magnitude of the tensor component of interest, a scalar quantity (as we have described for flux in equations [2.4] and [2.5]).

2.4 SURFACE FLUX

For exoplanet atmospheres we are most interested in the flux emerging from the planet surface. We call this kind of flux the surface flux. "Surface" refers to the layers of the planet atmosphere where most of the radiation originates without further interaction with gas, liquid, or solid particles. Surface flux is the outgoing

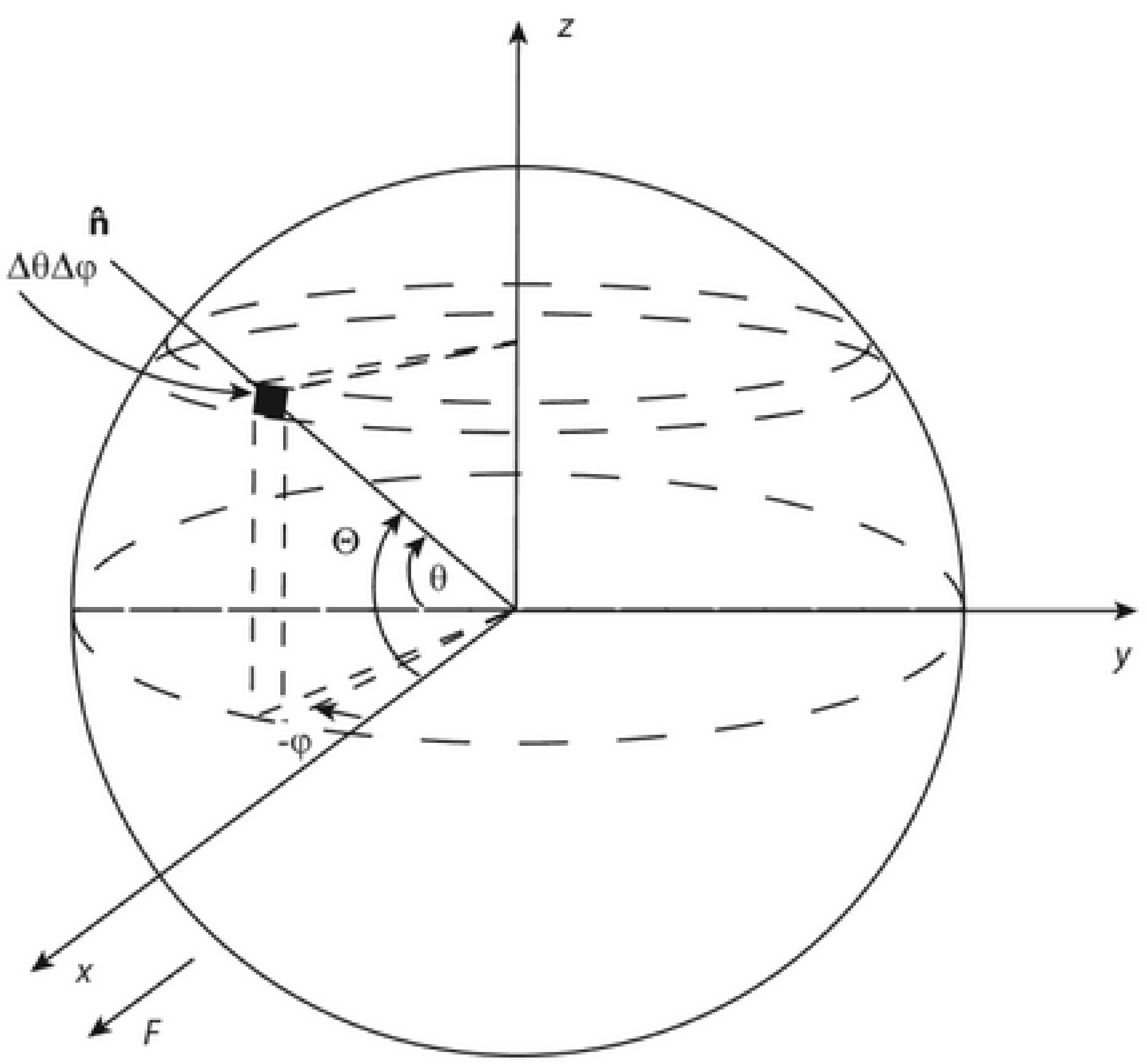

Figure 2.3 The spherical polar coordinate system using latitude θ and longitude ϕ. For flux computed along the x-axis, i.e., where the x-axis is $\hat{\mathbf{k}}$, and $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$ is $\cos\Theta$.

flux at a particular point on the planet's surface. We will use $F_{\rm S}$ to distinguish the surface flux from the flux F which may be defined in a small volume anywhere in the planet atmosphere. Here we emphasize a subtle but important point. That is, as described in Section 2.3, we take only the magnitude of the flux vector in the outgoing direction of interest, a scalar quantity. So we will use the symbol $F_{\rm S}$ to denote surface flux in the outgoing direction of interest.

To describe the surface flux we first choose the spherical polar coordinate system (r, θ, ϕ) shown in Figure 2.3. In this coordinate system θ is the latitude and ϕ the longitude. This coordinate system is useful when considering a surface in a planetary latitude and longitude system. For example, $\theta = 0$ is defined according to the planet-star ecliptic plane, so that ($\theta = 0, \phi = 0$) corresponds to the planetary substellar point, the point that receives the most stellar radiation. On Earth, the equator ($\theta = 0$) is defined by the rotational axis of Earth.

We now go through the terms on the right-hand side of the flux definition equation [2.5] in order to derive an expression for the surface flux. In the spherical polar coordinate system we replace the vector $\mathbf{x}$ with (r, θ, ϕ), where again θ and ϕ refer to coordinates on the planetary surface. Furthermore, because we are only interested in the surface intensity, not the intensity at different altitudes in the planet atmosphere, we drop the reference to r, implicitly assuming $r = R_{\rm p}$. Now, the intensity $I(\mathbf{x}, \hat{\mathbf{n}}, \nu, t)$ has two vectors, one of which we have replaced by (θ, ϕ). For the second vector, the direction of $\hat{\mathbf{n}}$, we use two more angles θ_n and ϕ_n, with origin at $\hat{\mathbf{n}}$. We now write the surface intensity as $I_{\rm S}(\theta, \phi, \theta_n, \phi_n, \nu, t)$

We now move to the second term in equation [2.5] and note that $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta_n$.

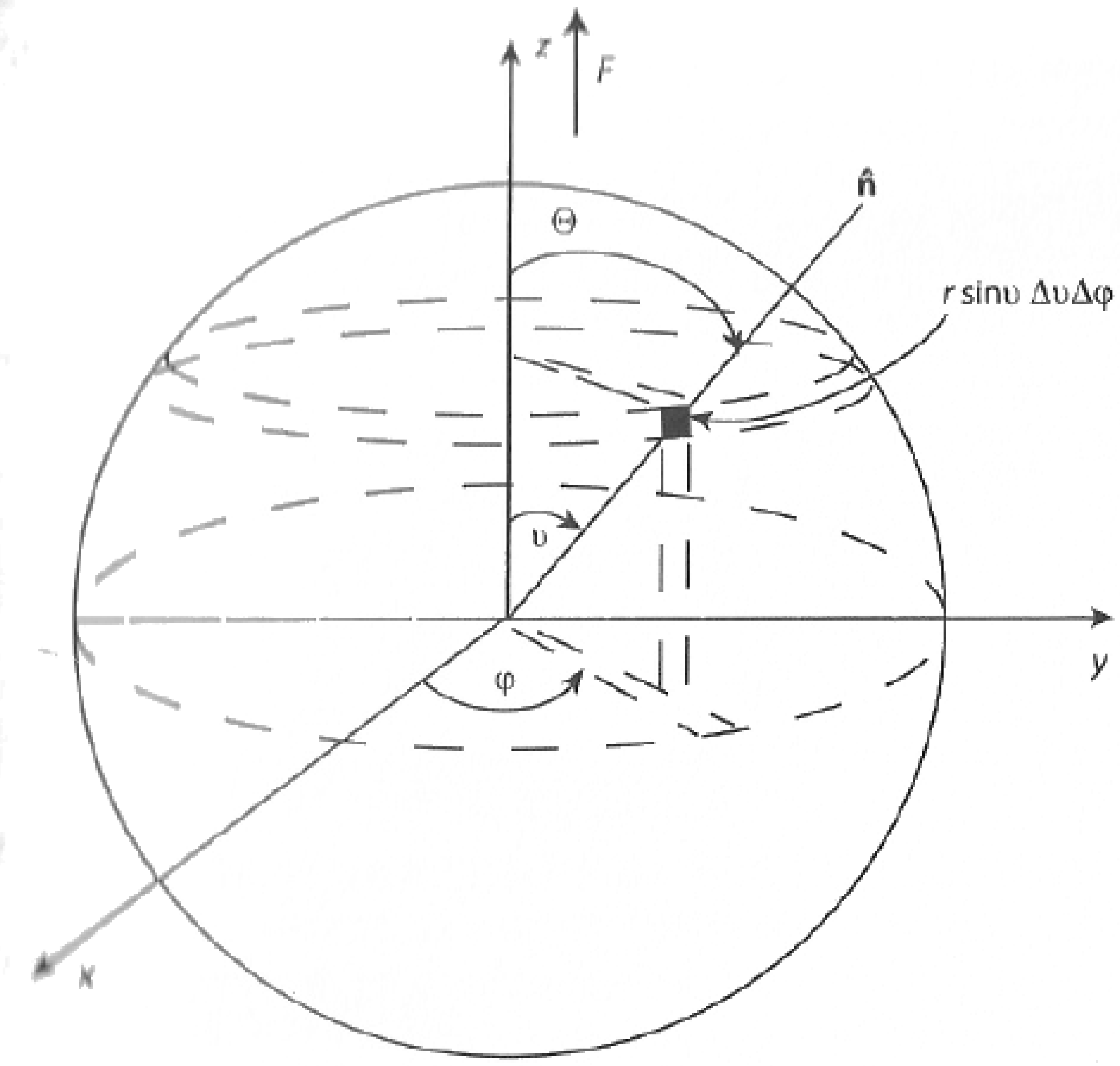

Figure 2.4 The spherical polar coordinate system using colatitude ϑ and longitude ϕ. For flux computed along the z-axis (the $\hat{\mathbf{k}}$ direction), $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\vartheta$.

From Figure 2.3, and using the spherical cosine law, we have $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta_n = \cos\theta_n \cos\phi_n$. Again, use of θ_n and ϕ_n means we no longer need $\hat{\mathbf{n}}$ or $\hat{\mathbf{k}}$ in our description of intensity.

Lastly, the differential solid angle for surface flux at a given location on the planet surface is defined by

$$d\Omega_n = \cos\theta_n d\theta_n d\phi_n. \tag{2.7}$$

The surface flux is

$$\boxed{F_{\rm S}(\theta, \phi, \nu, t) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} I_{\rm S}(\theta, \phi, \theta_n, \phi_n, \nu, t) \cos\phi_n \cos^2\theta_n d\theta_n d\phi_n.} \tag{2.8}$$

Recall that here $F_{\rm S}(\theta, \phi, \nu, t)$ is the scalar of the vector component of flux traveling in a given direction. Into which direction is the flux traveling? In this example, the surface flux is traveling out from the planet along direction $\hat{\mathbf{k}}$. We have further specified that the flux is at location θ, ϕ.

In some situations it is easier to solve for the surface flux (or surface intensity) in a coordinate system with one of the angles originating at the z-axis, as shown in Figure 2.4. The difference from the longitude-latitude spherical polar coordinate system is that the colatitude $\vartheta = 90° - \theta$ is used instead of the latitude θ as one of the independent variables. In this coordinate system, the substellar point is at the pole, and the flux is specified along the z-axis. The benefit of this system is that

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \cos\Theta_n = \cos\vartheta_n. \tag{2.9}$$

Furthermore, for problems symmetric in ϕ equations can be further simplified. The surface flux in this colatitude coordinate system is

$$\boxed{F_S(\vartheta,\phi,\nu,t) = \int_0^{2\pi}\int_0^{\pi/2} I_S(\vartheta,\phi,\vartheta_n,\phi_n,\nu,t)\cos\vartheta_n \sin\vartheta_n d\vartheta d\phi.} \quad (2.10)$$

We pause here to emphasize that the surface flux at (θ,ϕ) or (ϑ,ϕ) is usually written without reference to the surface coordinates, for example, by

$$F_S(\nu,t) = \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} I_S(\theta,\phi,\nu,t)\cos\phi\cos^2\theta d\theta d\phi. \quad (2.11)$$

or

$$F_S(\nu,t) = \int_0^{2\pi}\int_0^{\pi/2} I_S(\vartheta,\phi,\nu,t)\cos\vartheta\sin\vartheta d\vartheta d\phi. \quad (2.12)$$

Here θ and ϕ or ϑ and ϕ refer to the solid angle integration of the intensity, and even though the surface flux is for a specific surface element it is not specified. Although less precise, for a planet with uniform surface intensity this surface flux description is adequate, and we will be guilty of adopting it.

As an example of surface flux, we will compute the surface flux for a planet with uniform intensity $I_S(\theta,\phi,\nu,t) = I_S(\nu,t)$. This uniform intensity may be pulled out of the integrand in either equation [2.8] or equation [2.10] and the integrals performed to yield

$$\boxed{F_S(\nu,t) = \pi I_S(\nu,t).} \quad (2.13)$$

Sometimes it is useful to integrate the flux over all wavelengths, and we denote this wavelength-independent flux without a ν-dependence

$$\boxed{F_S(t) = \int_0^\infty F_S(\nu,t)d\nu.} \quad (2.14)$$

2.5 OBSERVED FLUX

What form of radiation are we able to measure at Earth? At Earth we see the planet as a point source, that is, as spatially unresolved. All radiation from the exoplanet hemisphere is averaged into a single value of flux. Recall that the surface intensity depends on location on the planet surface described by θ and ϕ. We must integrate the surface intensity from each location on the planet into a quantity we can actually measure.

At Earth we are measuring the energy collected from a planet subtended by an angle Ω, by a detector of a given area, in a frequency interval, and during some interval of time. We will denote the flux measured at the detector by $\mathcal{F}_\oplus(\nu,t)$. To derive $\mathcal{F}_\oplus(\nu,t)$ for a distant planet, we must realize that the integration over solid angle is at the detector and not at the planet surface.

In order to derive the surface flux at Earth we return to the definition of flux, equation [2.5], and use Figure 2.5. In the derivation of the flux at Earth it is convenient to use the colatitude spherical polar coordinate system (Figure 2.4). In this

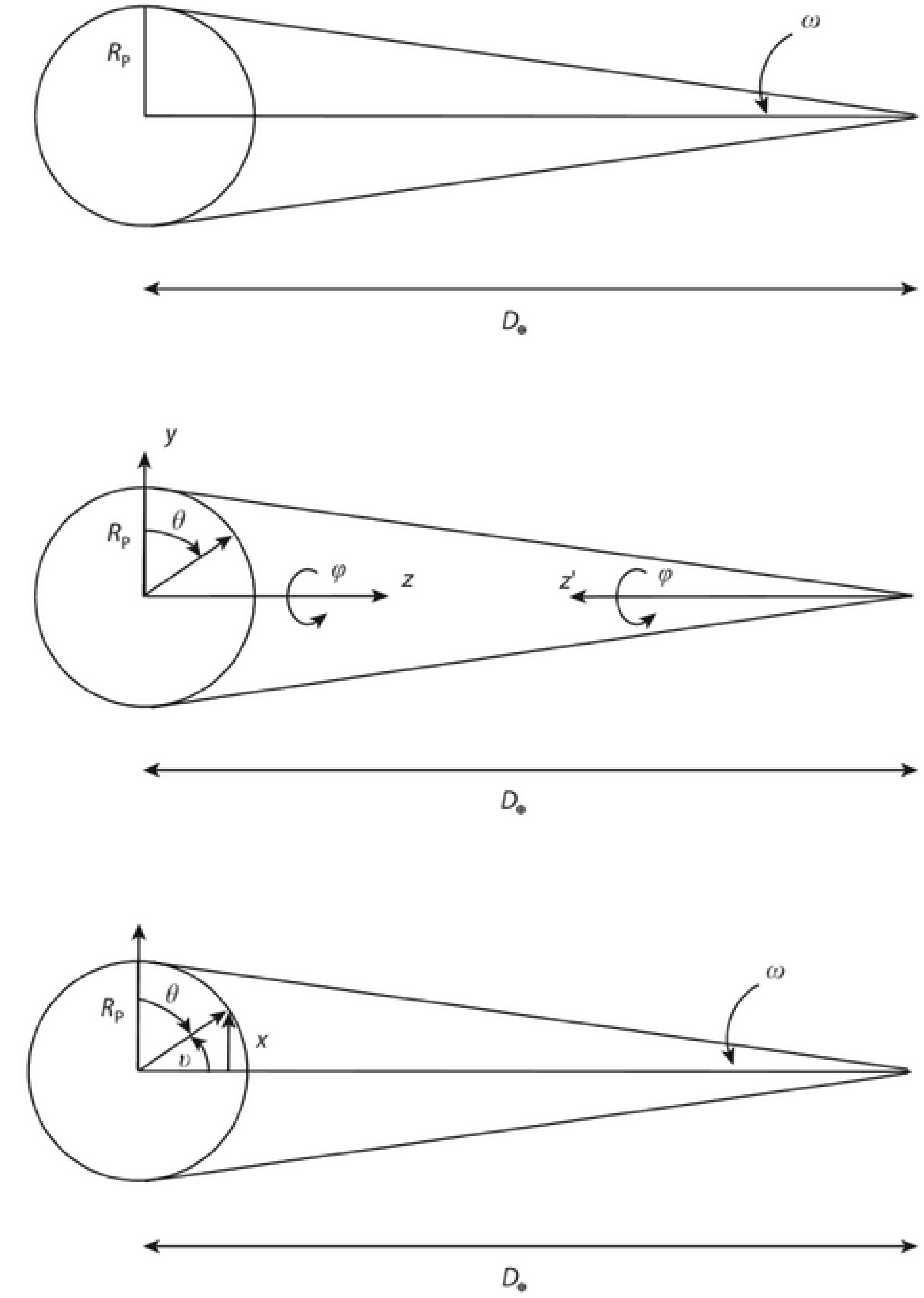

Figure 2.5 Derivation of the planet flux observed at Earth, $\mathcal{F}_\oplus(\nu,t)$.

coordinate system, the flux is defined as leaving the planet along the z-axis, and the z-axis is pointing toward the detector at Earth. Recall that the angle ϕ is defined about the z-axis. From the detector at Earth, the planet subtends the solid angle defined by

$$\Omega = \int_0^{2\pi}\int_0^{R_p/D_\oplus} \sin\omega d\omega d\phi. \quad (2.15)$$

A critical point in the derivation of measured flux is to recognize that the ϕ component of angle subtended by the planet at Earth is equivalent to the angle ϕ in the colatitude coordinate system on the planet (see the middle panel of Figure 2.5).

In the coordinate system of the detector at Earth we have for the flux

$$\mathcal{F}_{\oplus}(\nu,t) = \int_0^{2\pi}\int_0^{R_\mathrm{p}/D_\oplus} I_\mathrm{S}(\vartheta,\phi,\nu,t)\cos\omega\sin\omega d\omega d\phi. \tag{2.16}$$

Note that here we have retained the description of intensity in the coordinate system of the planet rather than change to the coordinate system at the detector.

We now convert the term $\cos\omega\sin\omega d\omega$ in equation [2.16] into the coordinate system of the planet. Using Figure 2.5, we see that

$$\omega = \frac{R_\mathrm{p}}{D_\oplus}\sin\vartheta. \tag{2.17}$$

Additionally, $d\omega = (R_\mathrm{p}/D_\oplus)\cos\vartheta d\vartheta$. In the limit that $R_\mathrm{p} \ll D_\oplus$, we have $\omega \ll 1$ and can make some approximations. Using a Taylor expansion, $\sin\omega \sim \omega$ and $\cos\omega \sim 1-\omega^2/2 \sim 1$. In this limit of $R_\mathrm{p} \ll D_\oplus$ we can also approximate $D_\oplus$ as the distance from the surface of the star to the observer at Earth. Equation [2.16] then becomes

$$\boxed{\mathcal{F}_{\oplus}(\nu,t) = \left(\frac{R_\mathrm{p}}{D_\oplus}\right)^2\int_0^{2\pi}\int_0^{\pi/2} I_\mathrm{S}(\vartheta,\phi,\nu,t)\cos\vartheta\sin\vartheta d\vartheta d\phi.} \tag{2.18}$$

We emphasize that $\mathcal{F}_{\oplus}(\nu,t)$ has the same dimensions as the surface flux in equation [2.10] and that in our limit that the planet is very distant from Earth the two fluxes are related by (see equation [2.12])

$$\boxed{\mathcal{F}_{\oplus}(\nu,t) = \left(\frac{R_\mathrm{p}}{D_\oplus}\right)^2 F_\mathrm{S}(\nu,t).} \tag{2.19}$$

2.6 LUMINOSITY AND OUTGOING ENERGY

Luminosity $L(t)$ is defined as the rate at which a planet radiates energy in all directions. Another way to think about luminosity is as the summation of flux passing through a closed surface encompassing the planet. For uniform flux $F_\mathrm{S}(\theta,\phi,t) = F_\mathrm{S}(t)$ and for a planet that radiates equally in all directions,

$$L(t) = \int_\mathrm{A} F_\mathrm{S}(t)dA, \tag{2.20}$$

in units of J s^{-1}. Using the surface element of a sphere

$$dA = R_\mathrm{p}^2\sin\theta d\theta d\phi, \tag{2.21}$$

we have

$$L(t) = R_\mathrm{p}^2\int_0^{2\pi}\int_0^{\pi} F_\mathrm{S}(t)\sin\theta d\theta d\phi, \tag{2.22}$$

and integrate to find

$$\boxed{L(t) = 4\pi R_\mathrm{p}^2 F_\mathrm{S}(t).} \tag{2.23}$$

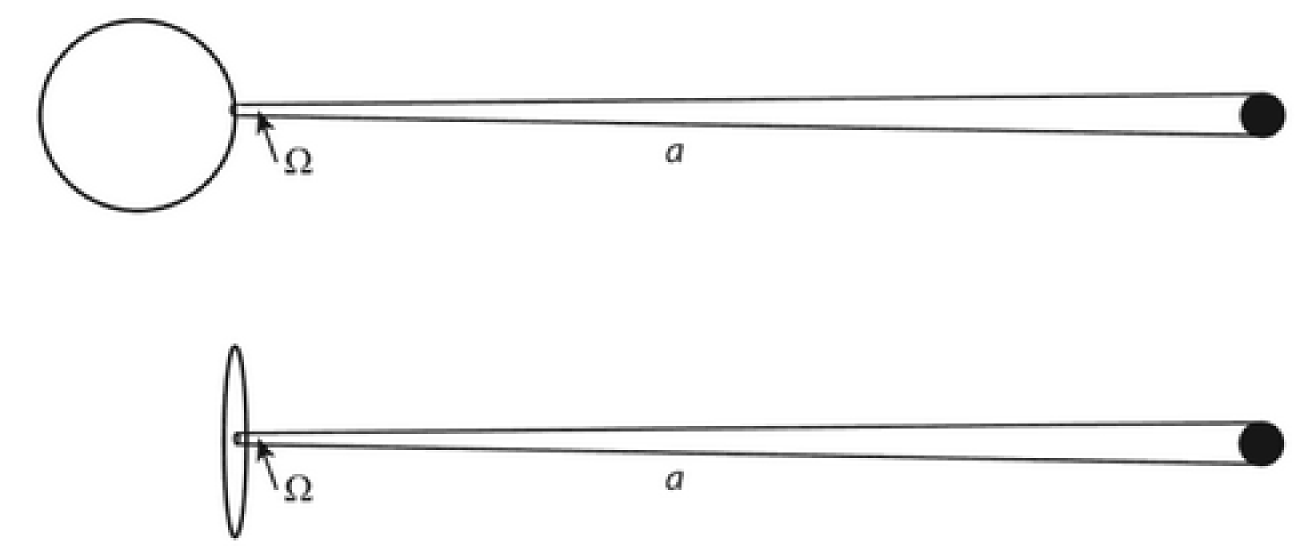

Figure 2.6 Incident flux on a sphere and a disk. The star subtends an angle Ω as seen from each location on the planet. To find the total energy per unit time incident on the planet, the flux at each location must be summed up over the planet hemisphere. Here a is semimajor axis.

For giant exoplanets $L(t)$ is useful to describe the flux coming from the planet interior that is incident on the lower boundary of the atmosphere. This interior flux is considered to be uniform around the planet at the bottom of the atmosphere, and originates from gravitational potential energy left over from the planet's nascent contraction. $L(t)$ may also be used for the rate at which energy leaves the planet in all directions. Often, however, we want to compute the total flux passing through only one hemisphere of the planet, not the entire planet. This is because, when observing any exoplanet, flux from only one hemisphere is visible to us at any given time. For a planet that does not radiate uniformly in all directions from all locations, we have to take care not to use the luminosity $L(t)$, but instead to use energy per unit time,

$$E_\mathrm{S}(t) = R_\mathrm{p}^2\int_0^{\pi}\int_0^{\pi/2} F_\mathrm{S}(\theta,\phi,t)\sin\theta d\theta d\phi, \tag{2.24}$$

and, assuming uniform flux, integrate to find

$$E_\mathrm{S}(t) = 2\pi R_\mathrm{p}^2 F_\mathrm{S}(t). \tag{2.25}$$

We now turn from energy per unit time leaving the planet to energy per unit time incident on the planet.

2.7 INCIDENT FLUX AND INCIDENT ENERGY

For exoplanets, the amount of radiation from the star reaching the planet is critical. The radiation from the star heats the planet and ultimately governs the global energy balance. The stellar heating also drives mass motion in the planet atmosphere. We therefore want to know the amount of flux or energy per unit time from the parent star that falls on the planet surface. We call this incident radiation, incident intensity, incident flux, or incident energy, depending on the context. More generally, we may sometimes even call the incident radiation "irradiation." To derive an expression for incident flux we want to consider the solid angle subtended by the star on a surface element of the planet, as shown in Figure 2.6. We want to compute the flux from the star at a given location on the planet.

We use equation [2.18], an equation we previously derived for the planet's flux observed at Earth. We now want to know the flux of the star as observed from a location on the planet. In equation [2.18], we therefore replace R_p with R_* and $D_\oplus$ with the planet semimajor axis a. We also replace the planet surface intensity I_S with the stellar surface intensity $I_{\mathrm{S},*}$,

$$\mathcal{F}_\mathrm{inc}(\vartheta,\phi,\nu,t) = \left(\frac{R_*}{a}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} I_{\mathrm{S},*}(\nu,t)(\vartheta,\phi,\nu,t)\cos\vartheta\sin\vartheta d\vartheta d\phi. \tag{2.26}$$

To simplify this equation, we can fairly assume that the stellar intensity is uniform across the star's surface, yielding

$$\mathcal{F}_\mathrm{inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 \pi I_{\mathrm{S},*}(\nu,t), \tag{2.27}$$

where we recall that $\mathcal{F}(\nu,t) = \pi I(\nu,t)$ from equation [2.13], and then also have

$$\mathcal{F}_\mathrm{inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 F_{\mathrm{S},*}(\nu,t). \tag{2.28}$$

We have discussed the incident flux at one location of the planet (equations [2.27] and [2.28]). For many applications we will want to know the *total* incident energy per unit time, and we must integrate over the surface of the planet. We cannot simply multiply by $2\pi R_\mathrm{p}^2$, the surface area of the planet hemisphere. This is due to the fact that only the substellar point on the planet receives the full amount of stellar flux. The planet locations away from the substellar point receive an amount reduced by $\hat{\mathbf{n}}\cdot\hat{\mathbf{k}} = \cos\Theta$ (see Figure 2.3). We are familiar with this concept on Earth because the poles on Earth receive much less sunlight than the equatorial region. We proceed to integrate equation [2.28] over the surface area of one hemisphere

$$E_\mathrm{inc}(\nu,t) = \left(\frac{R_*}{a}\right)^2 F_{\mathrm{S},*}(\nu,t)\int_0^{2\pi}\int_0^{\pi/2} R_\mathrm{p}^2\cos\vartheta\sin\vartheta d\vartheta d\phi, \tag{2.29}$$

to find

$$E_\mathrm{inc}(\nu,t) = \pi R_\mathrm{p}^2\left(\frac{R_*}{a}\right)^2 F_{\mathrm{S},*}(\nu,t). \tag{2.30}$$

We also define a total incident energy over all wavelengths by

$$\boxed{E_\mathrm{inc}(t) = \pi R_\mathrm{p}^2\left(\frac{R_*}{a}\right)^2 F_{\mathrm{S},*}(t).} \tag{2.31}$$

We have just introduced the total flux and total energy incident on an exoplanet. For plane-parallel radiation from a distant point source, we will find use for a description of the incident stellar intensity as

$$I_*(\vartheta,\phi,\nu,t) = I_0\delta(\vartheta-\vartheta_0)\delta(\phi-\phi_0). \tag{2.32}$$

Here, the star is in the direction θ_0, ϕ_0 from the surface normal. We consider that the star is far enough away from the planet so that the only the rays in one direction

are incident on the planet. At each location on the planet, the star is in a different position on the sky, and hence θ_0 and ϕ_0 are different for each surface element. The incident flux at a given location (θ,ϕ) on the planet surface is

$$\begin{aligned}\mathcal{F}_\mathrm{inc}(\vartheta,\phi,\nu,t) &= \left(\frac{R_*}{a}\right)^2 \int_0^{2\pi}\int_0^{\pi/2} I_0\delta(\vartheta-\vartheta_0)\delta(\phi-\phi_0)\cos\vartheta\sin\vartheta d\vartheta d\phi \\ &= \left(\frac{R_*}{a}\right)^2 \cos\vartheta_0\sin\vartheta_0 I_0.\end{aligned} \tag{2.33}$$

2.8 BLACK BODY INTENSITY AND BLACK BODY FLUX

Black body flux can be used to estimate both the flux incoming to and outgoing from an exoplanet. We will assume that the reader is already familiar with a black body radiator and the derivation of the intensity black body radiation. Here we aim to describe the black body intensity and flux in our framework and its relevance to exoplanetary atmospheres. A black body is a "perfect" radiator that absorbs all radiation incident on it and reemits radiation in a frequency spectrum depending only on its temperature T. Black body radiation is furthermore isotropic and hence has no $\hat{\mathbf{n}}$-dependence. The black body radiation depends only on temperature and frequency and can be described by $B(T,\nu)$.

The Planck function describes the intensity of black body radiation,

$$B(T,\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{h\nu/kT}-1}. \tag{2.34}$$

Here h is Planck's constant, k is Boltzmann's constant, and c is the speed of light. The black body intensity by definition has the same dimensions as the intensity I, units of J m^{-2} sr^{-1} s^{-1} Hz^{-1}. Because the temperature varies with location in the planet atmosphere, and possibly with time, we can also write the black body intensity as

$$B(\mathbf{x},\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{h\nu/kT}-1}. \tag{2.35}$$

We emphasize that our description of black body radiation is radiation per frequency bin ($d\nu$). Black body radiation per wavelength bin $d\lambda$ must include the conversion factor $d\nu = -c/\lambda^2 d\lambda$, where the $-$ sign can be absorbed into $d\lambda$,

$$B(T,\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{\mathrm{e}^{hc/\lambda kT}-1}. \tag{2.36}$$

The black body flux can be computed from the black body intensity B using the flux definition in equation [2.5]. Black body radiation is isotropic, that is, uniform in all directions, so that outward from one hemisphere

$$F_\mathrm{S}(T,\nu) = \pi B(T,\nu). \tag{2.37}$$

We will always use B as black body intensity and πB as black body flux.

Stellar and planet atmospheres can be approximated as black body radiators, even though a black body is a highly idealized construct. Figure 2.7 shows the black

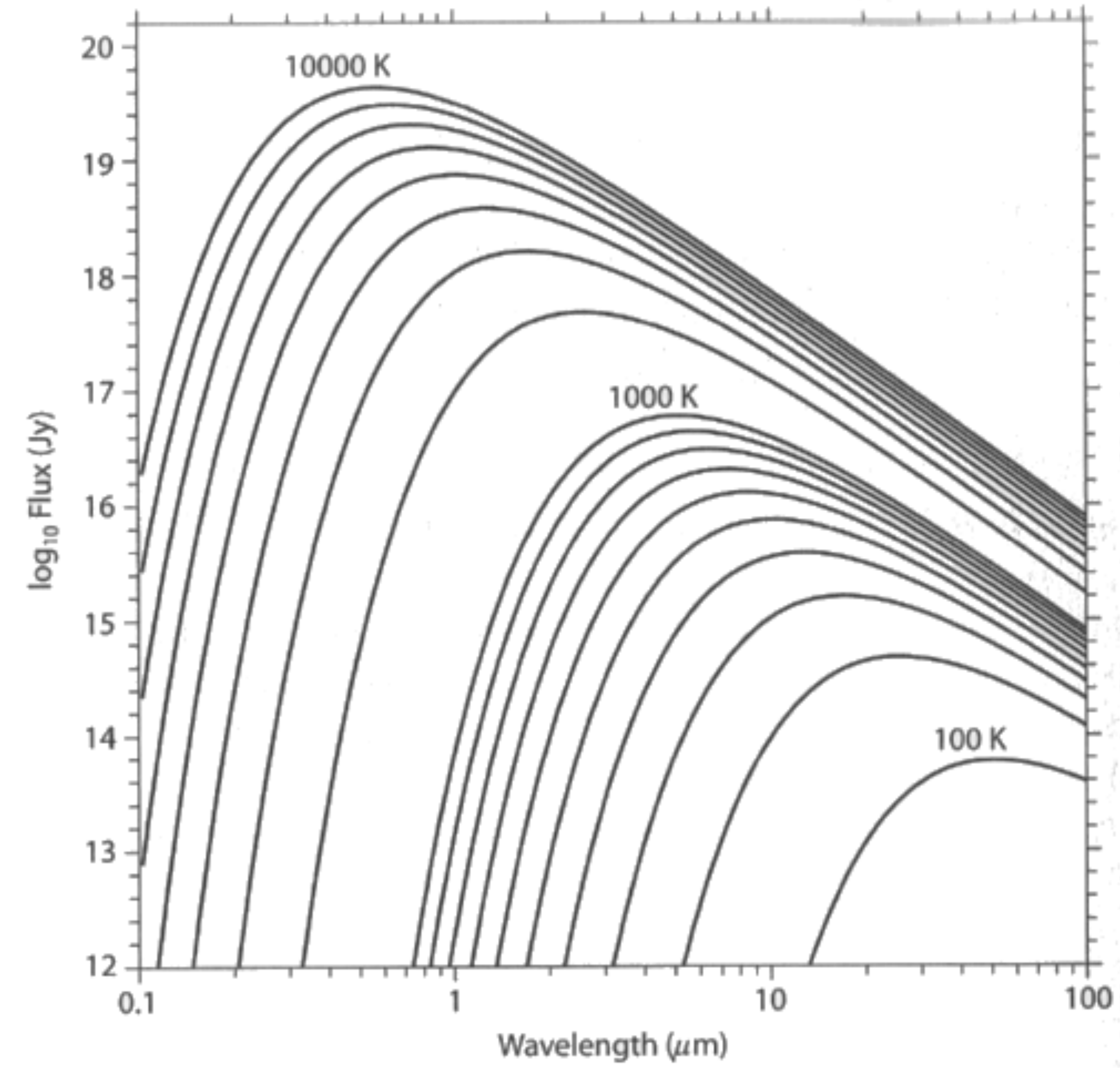

Figure 2.7 Black body surface fluxes (in units of Jy = 10^{-26} W m^{-2} Hz^{-1}) for a range of temperatures. The black body fluxes are spaced by 100 K in the range 100 to 1000 K and by 1000 K in the range 1000 to 10,000 K. The Sun may be represented by a 5750 K black body, while Earth can be approximated by a 300 K black body.

body surface flux for different temperatures. The effective temperature of stars with known planets ranges from 3000 to 6000 K, and known planets have effective temperatures ranging from 60 to over 2000 K. The magnitude and frequency peak of the black body flux increase with decreasing temperature.

2.9 LAMBERT SURFACE

A Lambert surface is often used to approximate the reflectivity of planetary bodies. Understanding a Lambert surface helps to understand the difference between two fundamental concepts: the intensity emanating from an object and the intensity measured by a distant observer. A Lambert surface is a surface that scatters intensity isotropically (i.e., equally in all directions).

What does it mean conceptually for a surface to scatter equal intensity in all directions? Equal intensity means that the apparent surface brightness of an area element is the same from any viewing angle. As an example, consider a sheet of white paper illuminated from above. A piece of white paper is an approximate plane Lambertian reflector. In other words, if you hold a sheet of paper up to

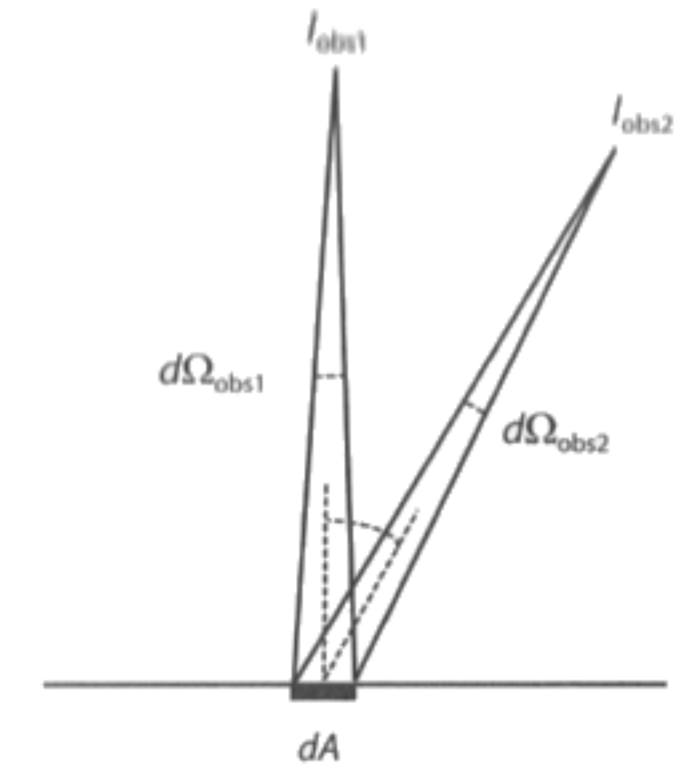

Figure 2.8 Intensity observed by a distant viewer. The angle subtended by dA is smaller at observer 2 than at observer 1.

the light and view it from different directions, the paper appears to be the same brightness. A second example is holding a lightbulb up to a white painted wall. The wall is brightest perpendicular to the light bulb, but the bright spot's location and extent does not change with changing viewing angle.

We will now expand on the conceptual description of a Lambert surface, by showing that a plane Lambert surface, illuminated from one direction perpendicular to the surface normal, has the same apparent brightness as viewed from any direction. Figure 2.8 shows the intensity viewed by an observer looking perpendicular to the plane and an observer viewing at an angle ϑ from the surface normal. Both observers are viewing the same differential area dA on the Lambert surface. We note that the distance to the observer is much greater than the size of the surface element. From the definition of intensity (equation [2.1]) we have the intensity measured by observer 1,

$$I_{\mathrm{obs1}} = \frac{dE_{\mathrm{obs1}}}{dA d\Omega_{\mathrm{obs1}} \hat{\mathbf{n}}_{\mathrm{obs1}} \cdot \hat{\mathbf{k}}}, \tag{2.38}$$

and, similarly, the intensity measured by observer 2 is

$$I_{\mathrm{obs2}} = \frac{dE_{\mathrm{obs2}}}{dA d\Omega_{\mathrm{obs2}} \hat{\mathbf{n}}_{\mathrm{obs2}} \cdot \hat{\mathbf{k}}}. \tag{2.39}$$

We have by definition of the coordinate system (Figure 2.8) that $\hat{\mathbf{n}}_{\mathrm{obs1}} \cdot \hat{\mathbf{k}} = 1$ and $\hat{\mathbf{n}}_{\mathrm{obs2}} \cdot \hat{\mathbf{k}} = \cos\vartheta$. In other words, the solid angle subtended by the surface element dA at observer 2 is smaller than the solid angle subtended by dA at observer 1. This is more easily apparent for the extreme case of $\vartheta \sim 90°$ where the solid angle subtended approaches zero. We also have the relationship

$$dE_{\mathrm{obs2}} = \cos\vartheta dE_{\mathrm{obs1}}, \tag{2.40}$$

because for isotropic scattering the energy (or number of photons) drops off as $\cos\vartheta$ away from the normal (i.e., away from the direction of incident radiation).

Putting the above equations together, we have, from observer 2

$$I_{\mathrm{obs2}} = \frac{dE_{\mathrm{obs1}} \cos\vartheta}{dA d\Omega_{\mathrm{obs2}} \cos\vartheta}. \tag{2.41}$$

From comparison with equation [2.38] we see that

$$I_{\mathrm{obs1}} = I_{\mathrm{obs2}}. \tag{2.42}$$

For a plane Lambert surface the intensity—or brightness—appears the same from all directions: the smaller number of photons emerging by the slant direction is compensated by the smaller subtended angle of an area element.

2.10 SUMMARY

We have presented fundamental definitions and concepts needed to describe radiation traveling through a planetary atmosphere. We began with a precise definition of the intensity and flux to be used throughout this book, quantities that have a variety of definitions in the literature (see exercise 1). We made a careful investigation of the surface flux on a planet as compared to the observed flux at Earth, and showed that these are the same if the planet is far enough away and if the planet's intensity is uniform across the planet's surface. Here the word "surface" might refer to a solid planetary surface like Earth's, or, in the case of a giant planet, it might refer to the deep atmosphere layers that become optically thick (akin to the photosphere of the Sun). We continued to define the quantities of outgoing luminosity and incident stellar flux, as well as the challenging concept of the Lambertian surface. With a handle on the fundamental definitions and concepts we are ready to embark on the task of understanding the basic physical characteristics and observables of exoplanets, planetary temperatures, albedos, and flux ratios.

REFERENCES

For further reading

For further introduction to the definitions of intensity, flux, and black body radiation:

- Chapter 1 in Rybicki, G. B., and Lightman, A. P. 1986. *Radiative Processes in Astrophysics* (New York: J. Wiley and Sons).
- Mihalas D. 1978. *Stellar Atmospheres* (2nd ed.; San Francisco: W. H. Freeman).
- Shu, F. 1991. *The Physics of Astrophysics. Vol. I, Radiation* (Mill Valley: University Science Books).

Reference for this chapter

1. McCluney, W. R. 1994. *Introduction to Radiometry and Photometry* (Norwood: Artech House).

Table 2.1 SI radiometry units.

Quantity	SI unit (abbr.)	Notes
Radiant energy	J	Energy
Radiant flux	W	Also called radiant power
Radiant intensity	W sr^{-1}	Power per unit solid angle
Radiance	W sr^{-1} m^{-2}	Power per unit solid angle per unit area
Irradiance	W m^{-2}	Power incident on a surface
Radiant exitance or Radiant emitance	W m^{-2}	Power emitted from a surface
Radiosity	W m^{-2}	Emitted + reflected power from a surface
Spectral radiance	W sr^{-1} m^{-3} or W sr^{-1} m^{-2} Hz^{-1}	
Spectral irradiance	W m^{-3} or W m^{-2} Hz^{-1}	

Table adapted from [1].

EXERCISES

1. List the variables introduced in this chapter that describe radiation, including their dimensions. The SI system has standard definitions for radiation terms that differ from the ones conventionally used for exoplanets. Compare the radiation terms used in this chapter to the SI radiation terms in Table 2.1

2. Explain the meaning of isotropic radiation. Show that for isotropic radiation the flux integrated over a hemispheric solid angle is $F = \pi I$ but that flux integrated over a solid angle is $F = 0$. Show that for isotropic radiation $I = J$.

3. Show that the intensity I does not depend on distance in a medium with no extinction or emission. Show that the flux $\mathcal{F}$ follows the inverse square law $\mathcal{F} \sim 1/d^2$, where d is distance away from the source, and that this distance dependency is not in conflict with the constancy of I.

4. Show that the general expression for solid angle (in units of steradians) is

$$\Omega = \frac{1}{R^2}\int_0^{\theta} 2\pi R \sin\theta R d\theta = 2\pi(1 - \cos\theta). \tag{2.43}$$

   What is the large-angle limit as $\theta \to \pi$? What is the small-angle limit as $\theta \to 0$?

5. Show that $\cos\Theta = \cos\phi \cos\theta$ in the spherical polar coordinate system in Figure 2.3.

6. Frequency or wavelength of maximum flux and energy from a black body.
   a. Derive Wien's Law as a function of frequency. Wien's Law describes the relationship between the peak of emission of a black body radiator and frequency. Wien derived his law from thermodynamic arguments but you may use the derivative of the black body radiation (equation [2.34]).

   b. Repeat part a, beginning with equation [2.36], to derive Wien's Law as a function of wavelength.

   c. Use Wien's Law to estimate the wavelength and frequency at which the Sun's emitted energy peaks. We will assume that the Sun's emitted flux can be approximated by a black body of temperature 5750 K. Are the frequency and wavelength the same or different? Explain.

   d. Repeat part b for an M star with a temperature of 3500 K.

7. For incident radiation from a star onto a planet, the expression for flux is different for an extended source compared to a point source (Section 2.7.) At what star-planet separation can the star reasonably be approximated as a point source?