**UNIVERSITY OF KANSAS**
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

**Problem Set 2: Transport, Distributions, Eqns of State**
**Due**: Thursday, March 6, 2025, in class
This problem set is worth **70 points**.

1. **Saha equation and pure hydrogen [15 pts].** Consider a gas of pure hydrogen at fixed density and temperature. The ionization energy of hydrogen is $\chi_0 = 13.6$ eV. You may assume that all the hydrogen atoms (whether neutral or ionized) are in their ground energy state.

   (a) Write down the Saha equation relating the number densities of neutral and ionized hydrogen ($n_0$ and $n_1$, respectively). Make reasonable approximations to use numerical values for the partition functions.

   ***Solution:*** It's easy enough to write down the Saha equation:

$$\frac{n_1}{n_0} = \frac{2Z_1}{n_e Z_0}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} e^{-\chi_0/kT}.$$

   The partition function for neutral hydrogen is

$$Z_0 = 2(1 + 2^2 e^{-\chi_0(1-1/2^2)/kT} + ...) \approx 2 \; for \; kT << \chi_0.$$

   The partition function for ionized hydrogen is 1 since there are two possible orientations of the free electron's spin relative to the spin of the proton, and we've already written the factor of 2 in the Saha equation. Thus we have

$$\frac{n_1}{n_0} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}.$$

   (b) To find the individual densities, further constraints are required. Reasonable constraints are charge neutrality ($n_e = n_1$) and conservation of nucleon number ($n_1 + n_0 = n$), where the total hydrogen number density $n$ is a constant if the density $\rho$ is fixed. Rewrite the Saha equation in terms of the hydrogen ionization fraction $x = n_1/n$, eliminating $n_1$, $n_0$, and $n_e$. Does this equation have the expected limiting behavior for $T \to 0$ and $T \to \infty$?

   ***Solution:*** The two constraints are the conservation of charge and nucleon number, which can be written: $n_e = n_1$ and $n = n_1 + n_0$. Writing $x = n_1/n$, the Saha equation becomes

$$\begin{aligned}\frac{nx}{n-nx} &= \frac{1}{nx}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT} \\ \frac{x^2}{1-x} &= \frac{1}{n}\left(\frac{2\pi m_2 kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}\end{aligned}$$

   From this equation we see that as $T \to 0$, $x \to 0$; i.e., no ionization occurs. And as $T \to \infty$, $x \to 1$, indicating full ionization. These are the proper limiting behaviors.

   (c) Use the relation $\rho = m_{\rm H} n$ (where $m_{\rm H} = 1$ gm/$N_{\rm A}$, where $N_{\rm A} = 6.023 \times 10^{23}$ is Avogadro's number) to replace $n$ with $\rho$. Find an expression for the half-ionized ($x = 0.5$) path in the $\rho$-$T$ plane. Plot this path on a log-log plot for densities in the interesting range from $10^{-10}$–$10^{-2}$ g cm$^{-3}$

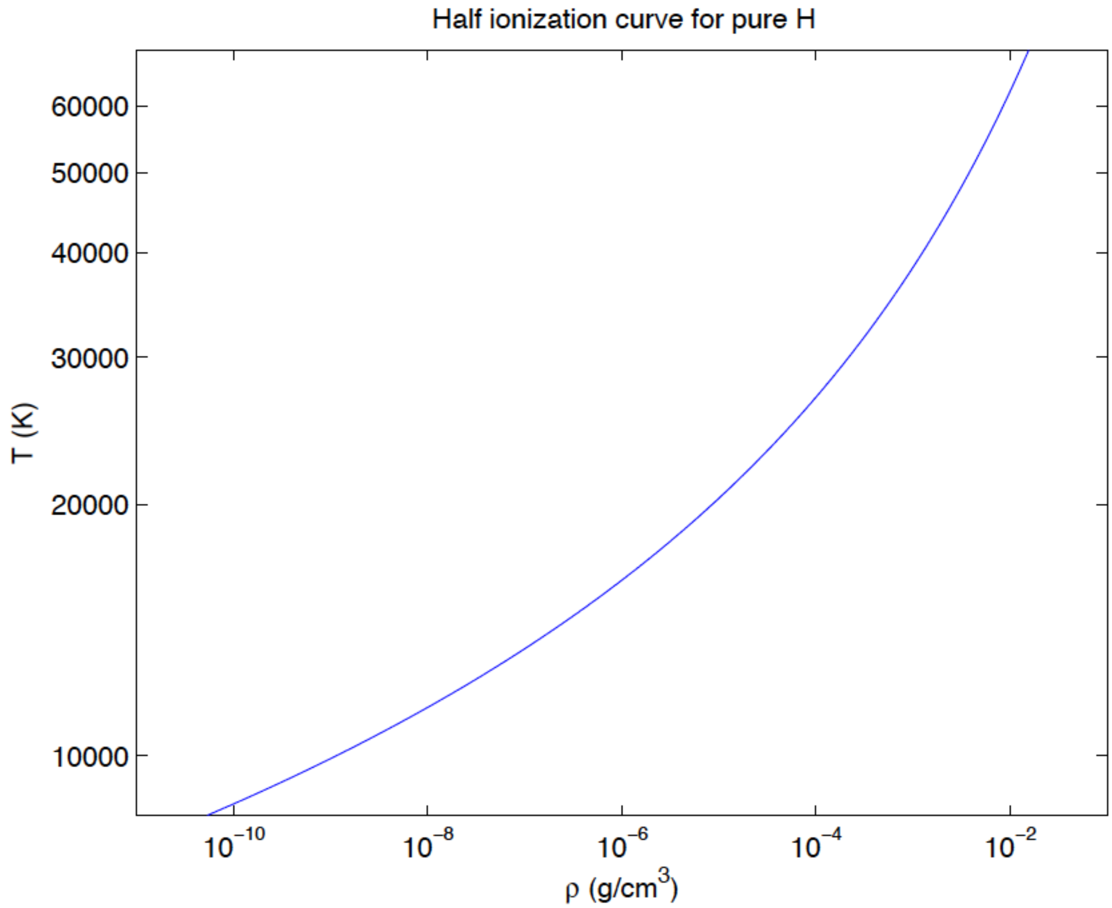


Figure 1: Half ionization curve for pure hydrogen.

***Solution:*** The mass density is given by $\rho = m_H n$, where $m_H$ is the mass of hydrogen, $1/N_A$. To get the half-ionization curve, set $x = 0.5$ in the Saha equation to obtain

$$\rho(T) = \frac{2}{N_A}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}.$$

This is the half-ionization curve shown in Figure 1.

2. **Saha equation and pure helium [20 pts].**

   Consider a gas of pure helium at fixed density and temperature. The ionization energies for helium are $\chi_0 = 24.6$ eV (from neutral to singly ionized) and $\chi_1 = 54.4$ eV (from singly to doubly ionized). You may assume that all the helium atoms (whether neutral, singly ionized, or doubly ionized) are in their ground energy state. Let $n_e$, $n_0$, $n_1$, and $n_2$ be the number densities of, respectively, free electrons, neutral atoms, singly-ionized atoms, and doubly-ionized atoms. The total number density of neutral atoms and ions is denoted by $n$. Define $x_e$ as the ratio $n_e/n$, and let $x_i$ be $n_i/n$ where $i = 0, 1, 2$. You should assume that the gas is electrically neutral. The degeneracy factors you need for the atoms and ions are 2 for He, 4 for $He^+$, and 2 for $He^{2+}$.

   (a) Construct the ratios $n_1/n_0$ and $n_2/n_1$ using the Saha equation. In doing so, take care in establishing the zero points of energy for the various constituents.

   ***Solution:*** The Saha equations are

$$\begin{aligned}\frac{n_1}{n_0} &= \frac{4}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}\\ \frac{n_2}{n_1} &= \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_1/kT}\end{aligned}$$

(b) Apply charge neutrality and nucleon number conservation ($n = n_0 + n_1 + n_2$) and recast the above Saha equations so that only $x_1$ and $x_2$ appear as unknowns. The resulting two equations have $T$ and $n$ [or, equivalently, $\rho = nm_{\rm He} = n(4\,{\rm gm}/N_{\rm A})$] as parameters.

***Solution:*** Charge conservation and nucleon number conservation can be written: $n = n_0 + n_1 + n_2$ and $n_e = n_1 + 2n_2$, so that the Saha equations become

$$\begin{aligned}\frac{x_1(x_1+2x_2)}{1-x_1-x_2} &= \frac{4}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}\\ \frac{x_2(x_1+2x_2)}{x_1} &= \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_1/kT}\end{aligned}$$

(c) Simultaneously solve the two Saha equations for $x_1$ and $x_2$ for temperatures in the range $4\times10^4 \le T \le 2\times10^5$ K. Do this for a fixed density with the three values $\rho = 10^{-4}$, $10^{-6}$, or $10^{-8}$ g cm$^{-3}$. You may find it more convenient to use the logarithm of your equations. Choose a dense grid in temperature because you will soon plot the results. Once you have found $x_1$ and $x_2$, also find $x_e$ and $x_0$ for the same range of temperature. Note that this is a numerical exercise; you will want to use a tool like Mathematica or Matlab for this.

***Solution:*** The mass density is given by $\rho = m_{He} n = 4n/N_A$, where $m_{He} = 4m_H = 4/N_A$ is the mass of the helium. The two Saha equations can then be written

$$\begin{aligned}f(x_1,x_2) &= x_1^2 + 2x_1x_2 + \frac{16}{\rho N_A}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_0/kT}(x_1+x_2-1) = 0\\ g(x_1,x_2) &= x_1x_2 + 2x_2^2 - \frac{4}{\rho N_A}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_1/kT}x_1 = 0\end{aligned}$$

This set of coupled, nonlinearn equations can be solved using nearly any multidimensional root-finding technique. I used a simple Newton-Raphson method, which works similarly to the Newton-Raphson method for solving a single equation. In this method, a guess is made for $\mathbf{x} = \binom{x_1}{x_2}$ and then the guess is refined using

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \delta\mathbf{x},$$

where

$$\delta\mathbf{x} = \begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = \begin{bmatrix}\partial f/\partial x_1 & \partial f/\partial x_2\\ \partial g/\partial x_1 & \partial g/\partial x_2\end{bmatrix}\begin{bmatrix}-f\\ -g\end{bmatrix}$$

Here is the MATLAB code that implements this procedure:

```
function heionization (rho, Tstart, Tspace, Tfinal)
% Plots the abundance of neutral, slightly ionized, and doubly ionized
% helium, as well as the electrons.
%
```

```
% x0 = n0/n, x1 = n1/n, x2 = n2/n, xe = ne/n, where n = rho * Na/4
%
% The temperature range is Tstart:Tspace:Tfinal % Constants

Na = 6.02214e23;
me = 9.1094e-28;
kB = 1.3807e-16;
h = 6.6261e-27;
kBeV = 8.617e-5;
chi0 = 24.6 ;
chi1 = 54.4 ;

x1 = [];
x2 = [];

%Loop through temperature for T = Tstart:Tspace:Tfinal

    % Define A and B:
    A = 16/Na/rho*(2*pi*me*kB*T/h^ 2).  ^ (3/2).  *exp(-chi0/kBeV./T);
    B = 4/Na/rho*(2*pi*me*kB*T/h^ 2).  ^ (3/2).  *exp(-chi1/kBeV./T);

    % Dumb initial guesses:
    x1guess = 0.5;
    x2guess = 0.5;
% Calculate f and g; correct until within tolerance 0.0001;

f = x1guess ^ 2 + 2 * x1guess * x2guess + A * (x1guess + x2guess - 1);
g = x1guess * x2guess + 2 * x2guess ^ 2 - B * x1guess;
err = max( abs(f), abs(g));

while (err > 0.0001)

    M = [2*x1guess + 2*x2guess + A 2*x1guess + A; x2guess - B x1guess + 4
* x2guess];
    dx = inv(M) * [-f -g];
    x1guess = x1guess + dx(1);
    x2guess = x2guess + dx(2);
    f = x1guess ^ 2 + 2 *x1guess *x2guess + A*(x1guess+x2guess-1);
    g = x1guess * x2guess + 2 * x2guess ^ 2- B*x1guess;
    err = max(abs(f), abs(g));
end

% Add solutions to list;
x1 = [x1 x1guess];
x2 = [x2 x2guess];

end
```

```
% Calculate xe and x0;
xe = x1 + 2 * x2;
x0 = 1 - x1 - x2 ;

Tvals = Tstart:Tspace:Tfinal ;

plot(Tvals, xe);
hold on;
plot(Tvals, x0, ':');
plot(Tvals, x1, '--');
plot(Tvals, x2, '-.';

return;
```

(d) Plot all your $x$s as a function of temperature for your chosen value of $\rho$. (Plot $x_0$, $x_1$, and $x_2$ on the same graph.) Identify the transition temperatures (half-ionization) for the two ionization stages.

***Solution:*** Figures 2-4 show the ionization fraction for three densities. The half-ionization temperatures (defined as the lowest temperature at which the ionization fraction of a species is 0.5) are:

$\rho = 10^{-4}$ g/cm$^3$: T($x_1$ = 0.5) = $3.2 \times 10^4$ K, T($x_2$ = 0.5) = $8.1 \times 10^4$ K
$\rho = 10^{-6}$ g/cm$^3$: T($x_1$ = 0.5) = $2.2 \times 10^4$ K, T($x_2$ = 0.5) = $5.4 \times 10^4$ K
$\rho = 10^{-8}$ g/cm$^3$: T($x_1$ = 0.5) = $1.7 \times 10^4$ K, T($x_2$ = 0.5) = $4.0 \times 10^4$ K

3. **Stability against convection [10 pts]**

(a) In lecture, we derived the condition

$$\left|\frac{dT}{dr}\right| < \frac{T}{P}\left(1 - \frac{1}{\gamma_a}\right)\left|\frac{dP}{dr}\right|$$

for stability against convection. Using the appropriate equation(s) of stellar structure and noting the sign of the radial gradients, show that this can be recast as a condition on the luminosity profile:

$$L(r) < \left(1 - \frac{1}{\gamma_a}\right)\frac{64\pi\sigma_{\rm SB}T^4 GM(r)}{3\kappa_R P}$$

***Solution:*** We've derived the condition

$$\frac{\rho}{\gamma P}\frac{dP}{dr} - \frac{d\rho}{dr} > 0$$

for stability against convection. Using the ideal gas law $P = \rho kT/\mu m_p$, we can calculate $d\rho/dr$, and find

$$\frac{d\rho}{dr} = \frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{T}\frac{dT}{dr}.$$

Substituting into the condition for stability and simplifying, we obtain:

$$\frac{dT}{dr} > \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

Using the equation of radiative transport,

$$\frac{dT}{dr} = \frac{-3\kappa_R \rho L(r)}{16\pi a c T^3 r^2},$$

and solving the inequality for L(r):

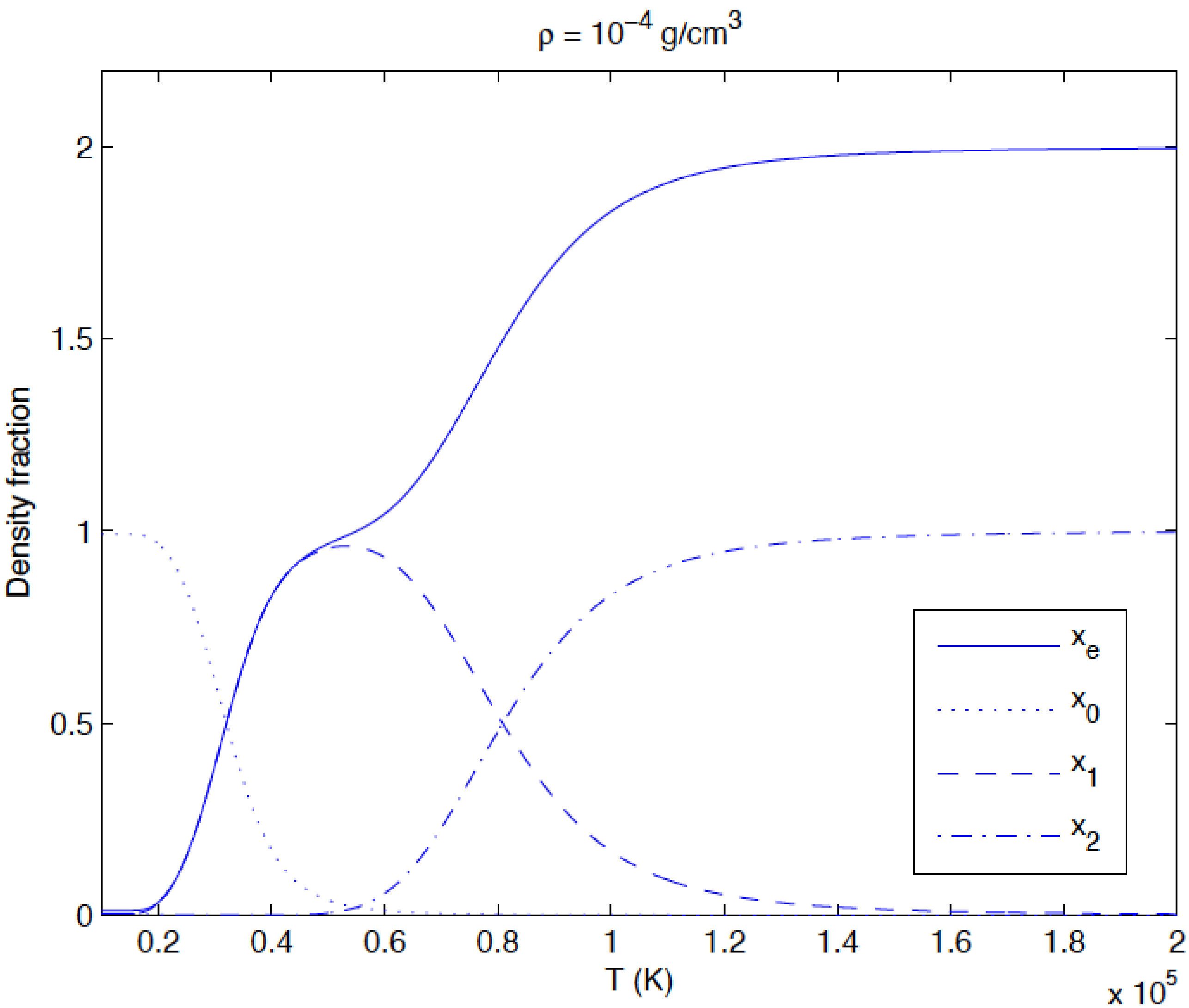


Figure 2: Ionization fractions for pure helium, $\rho = 10^{-4}$ g/cm$^{-3}$.

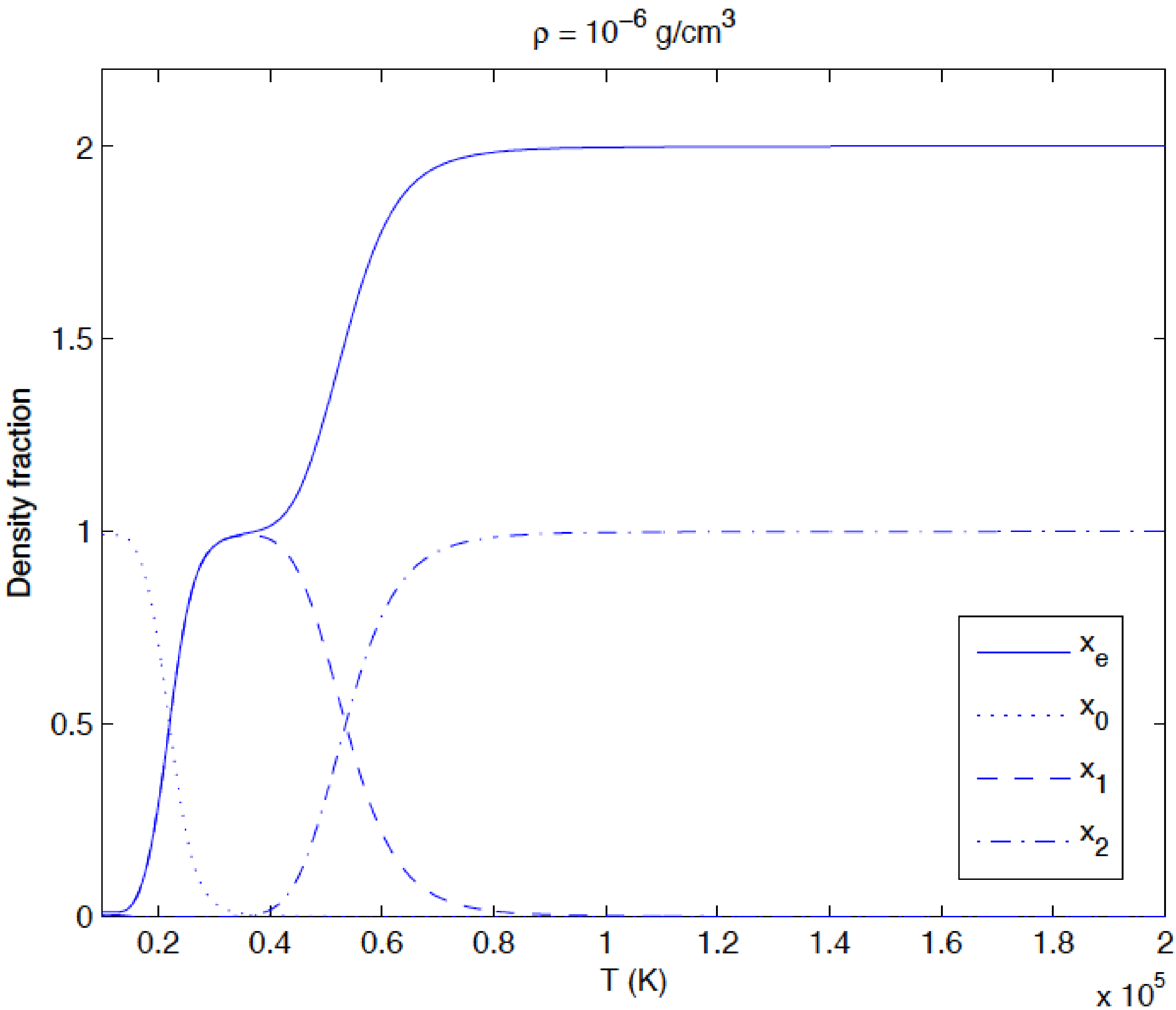


Figure 3: Ionization fractions for pure helium, $\rho = 10^{-6}$ g/cm$^{-3}$.

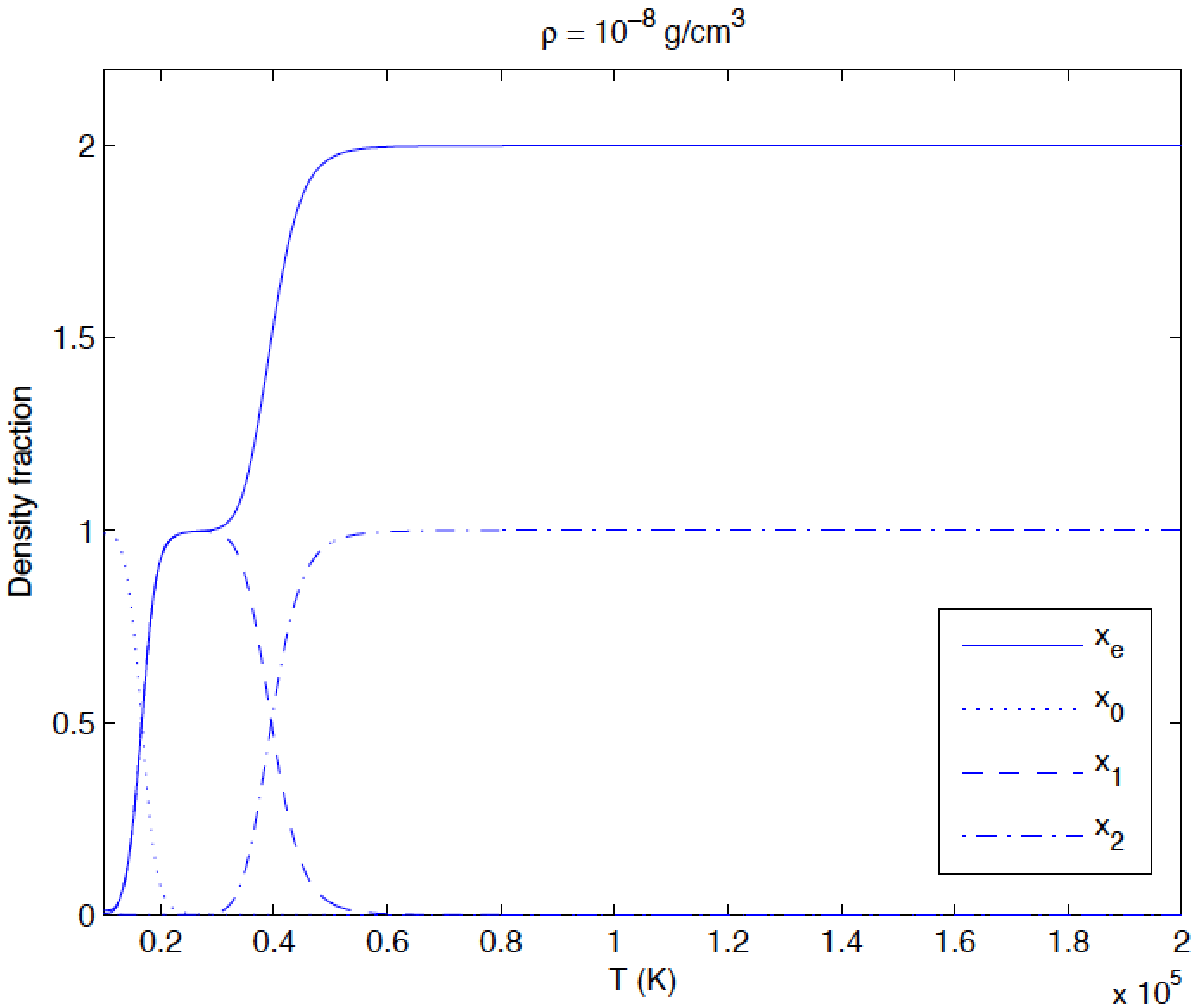


Figure 4: Ionization fractions for pure helium, $\rho = 10^{-8}$ g/cm$^{-3}$.

$$L(r) < \frac{-16\pi acT^4r^2}{3\kappa_R\rho P}\frac{dP}{dr}\left(1-\frac{1}{\gamma}\right)$$

And substituting in the equation for hydrostatic equilibrium, we get

$$\begin{aligned} L(r) &< \left(1-\frac{1}{\gamma}\right)\frac{16\pi acT^4GM(r)}{3\kappa_R P} \\ L(r) &< \left(1-\frac{1}{\gamma}\right)\frac{64\pi\sigma_{SB}T^4GM(r)}{3\kappa_R P} \end{aligned}$$

(b) Show that to avoid convection in a stellar region where the equation of state is that of an ideal monatomic gas, the luminosity at a given radius must be limited by

$$L(r) < 1.22\times10^{-18}\frac{\mu T^3}{\kappa_R\rho}M(r)$$

where $\mu$ is the mean molecular weight, $T(r)$, $\kappa_R$ is the Rosseland mean opacity, and $M(r)$ is the mass enclosed at radius $r$. All quantities are measured in the appropriate cgs units.

***Solution:*** For an ideal monotomic gas, $\gamma = \frac{5}{3}$ and $P = \rho kT/\mu m_p$. Plugging in these expressions, we arrive at the desired result (in cgs units):

$$L(r) < 1.22\times10^{-18}\frac{\mu T^3}{\kappa_R\rho}M(r)$$

4. **Protons or photons? [10 pts]**

At the center of the Sun, the density is approximately 150 g cm$^{-3}$ and the temperature is about $15\times10^6$ K. Which is larger: the number density of protons, or the number density of photons? Give an order of magnitude estimate of each.

***Solution:*** The number density of protons is roughly

$$n_p \approx \frac{\rho_c}{m_p} = \boxed{9.03\times10^{25}\ \mathrm{cm}^{-3}},$$

where we have neglected the effect of He and considered a pure hydrogen composition. The number density of photons is can be related to the temperature

$$n_\gamma = \int\frac{u_\nu}{h\nu}d\nu = \int\frac{4\pi B_\nu(T)}{ch\nu} = \frac{16\pi k_B^3T^3}{c^3h^3}\zeta(3) \approx \boxed{6.9\times10^{22}\ \mathrm{cm}^{-3}},$$

where $\zeta(3)\approx1.20$ is the Reimann zeta function. The number density of protons is more than a thousand times higher than the number density of photons.

5. **The Eddington limit [15 pts]**

A star with sufficiently high radiation pressure will spontaneously eject material from its surface. This sets a practical limit on the maximum luminosity of a star of a given mass.

(a) [10 pts] Start with the radiative diffusion equation and the equation for hydrostatic equilibrium. Assume the opacity to be frequency-independent, and show that the luminosity at which the radiation pressure gradient equals the hydrostatic pressure gradient is given by

$$L_{\mathrm{Edd}} = \frac{4\pi GMc}{\kappa}, \quad (1)$$

where $M$ is the stellar mass. This is the "Eddington luminosity."

***Solution:*** The second moment of the radiative transfer equation in spherical coordinates is (see problem set 3, problem 8c)

$$c\frac{dP_\nu}{dr} = -\rho\kappa_\nu F_\nu \,.$$

Integrating over frequency, using the fact that $\kappa$ is independent of frequency, and using $F = L/4\pi r^2$, we find the radiation pressure gradient to be

$$\frac{dP_r}{dr} = -\frac{\rho\kappa L}{4\pi r^2 c} \,.$$

If radiation pressure is the dominant source of pressure, then, using the equation of hydrostatic equilibrium,

$$-\frac{\rho\kappa L}{4\pi r^2 c} = -\frac{GM}{r^2}\rho \,.$$

This holds when *all* of the pressure support is provided by radiation. In this case the luminosity no longer depends on density or radius; it is simply

$$L_{\rm Edd} = \frac{4\pi GMc}{\kappa} \,.$$

(b) [5 pts] For ionized hydrogen, a minimum value for $\kappa$ arises from Thomson scattering, which has cross-section $\sigma_T = 6.65 \times 10^{-25}$ cm$^2$. Show that for this case

$$L_{\rm Edd} \approx 3 \times 10^4 \, L_\odot \left(\frac{M}{M_\odot}\right) , \tag{2}$$

where $L_\odot = 3.839 \times 10^{33}$ erg s$^{-1}$ and $M_\odot = 1.989 \times 10^{33}$ g.

***Solution:*** Since $n\sigma = \kappa\rho$, we have $\kappa_T = n\sigma_T/\rho$. Here $n$ is the number density of scatterers (electrons), but $\rho$ is the mass density of the medium (mostly protons). From charge neutrality, $n_e = n_p$, giving $\kappa_T = \sigma_T/m_p$. Plugging this and the solar mass into $L_{\rm Edd}$ gives Eq. (2).