UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 3: Interiors
Due: Tuesday, March 25, 2025, in class
This problem set is worth **60 points**.

1. **Overcoming the Coulomb barrier [15 pts]**

   In this problem you will show that classical mechanics predicts that hydrogen fusion cannot happen in the Sun.

   (a) Suppose two protons approach each other with equal speeds. What is the minimum speed needed to overcome the Coulomb barrier and collide, neglecting quantum effects? Take the radius of a proton to be $\approx$1 fermi $= 10^{-13}$ cm. [5 pts]

   Solution: Here we take the interaction potential to be the Coulomb potential $V(r) = z_{\rm surf} Z_2 e^2/r$ down to the point of contact at about 2 fm. Take a pair of interacting particles to both have velocity v at infinity – then their combined energy is $E = m_p v^2$. To overcome the Coulomb barrier, this energy must at least be equal to the potential at the radius of 2 fm. That is,

   $$V(2\ {\rm fm}) = m_p v^2 \quad \Longrightarrow \quad v_{\rm cl} = \sqrt{\frac{1}{m_p} V(2\ {\rm fm})} = \sqrt{\frac{z_{\rm surf} Z_2 e^2}{m_p\ 2\ {\rm fm}}}\,.$$

   For two protons, $z_{\rm surf} = Z_2 = +1$. Plugging in numbers, we find, classically, the individual velocities must be $\boxed{v_{\rm cl} = 8 \times 10^6\ {\rm m\ s^{-1}}}$ or 4% of the speed of light.

   (b) Assuming the proton speeds obey a Maxwell-Boltzmann distribution

   $$p(v) = \sqrt{\frac{2}{\pi} \left(\frac{m_p}{kT}\right)^3} v^2 \exp(-m_p v^2/2kT)$$

   with $T = 15.7 \times 10^6$ K (the central temperature of the Sun), what is the most probable speed (i.e., the speed at the peak of the distribution function)? [5 pts]

   Solution: First we give a reminder of how the Maxwell-Boltzmann distribution can be derived. In thermal equilibrium, the density of a given microstate is proportional to the Boltzmann factor of that state, which depends on the energy. Here, the energy is $E = \frac{1}{2} m_p \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$, with $\vec{\mathbf{v}}$ the velocity. The phase-space density is therefore proportional to

   $$p(\vec{\mathbf{v}}) d\vec{\mathbf{v}} \propto \exp\left(-m_p \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}/2kT\right) d\vec{\mathbf{v}} \quad \Longrightarrow \quad p(v)dv \propto v^2 \exp\left(-m_p v^2/2kT\right) dv,$$

   where $v = |\vec{\mathbf{v}}|$ and where the second equality comes from the assumption of isotropy and the volume element in 3-dimensional velocity space. One can integrate over v from 0 to ∞ to find the normalization constant; one finds

   $$p(v)dv = \sqrt{\frac{2}{\pi}} \left(\frac{m_p}{kT}\right)^{3/2} v^2 \exp\left(-m_p v^2/2kT\right) dv\,.$$

   The most probable velocity, defined by the maximum value of $p(v)$, is found by solving $p'(v_{\rm peak}) = 0$. This gives $v_{\rm peak} = \sqrt{2kT/m_p}$.

   For a temperature of 15.7×10^6 K, the most probable velocity is $\boxed{v_{\rm peak} \approx 5 \times 10^5\ {\rm m\ s^{-1}}}$. Comparing this to the classical velocity from Prob. **??**, we see that $v_{\rm cl}/v_{\rm peak} \approx 16$, which is not very close.

   (c) You might wonder whether a small minority of protons in the tail of the M-B distribution could fuse. Give an order of magnitude estimate for the number of protons in the Sun, and for the number of those protons that are energetic enough to fuse. You may find it useful to know that for large u_0,

   $$\frac{4}{\sqrt{\pi}} \int_{u_0}^{\infty} u^2 e^{-u^2} du \approx \frac{2}{\sqrt{\pi}}\, u_0 e^{-u_0^2}. \quad [5\text{pts}]$$

Solution: If we treat the sun as being composed completely of hydrogen, then the number of protons is simply $\boxed{N_p = M_\odot/m_p \approx 10^{57}}$.

The number of protons faster than the classical velocity needed to fuse is the total number of protons times the probability that a proton has the requisite velocity,

$$N_{\text{cl. fuse}} = N_p\, P(v > v_{\text{cl}}) = N_p \int_{v_{\text{cl}}}^{\infty} p(v) dv$$

Plug in Eq. (1b) and nondimensionalize, setting $u = v/v_{\text{peak}} = v\sqrt{m_p/2kT}$.

$$P(v > v_{\text{cl}}) = \frac{4}{\sqrt{\pi}} \int_{u_{\text{cl}}}^{\infty} u^2 e^{-u^2} du\,,$$

where $u_{\text{cl}} \equiv v_{\text{cl}}/v_{\text{peak}}$. We must turn this integral into one which we know how to do in terms of special functions. One trick is to write it as a derivative of a Gaussian integral,

$$P(v > v_{\text{cl}}) = -\frac{d}{d\lambda}\left(\frac{2}{\sqrt{\pi}} \int_{u_{\text{cl}}}^{\infty} e^{-\lambda^2 u^2} du \right)_{\lambda=1}.$$

One can now look up the definition of the complementary error function

$$\text{erfc}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt\,.$$

Change variables again to $t = \lambda u$ to get

$$\begin{aligned} P(v > v_{\text{cl}}) &= -\frac{d}{d\lambda}\left(\frac{2}{\lambda\sqrt{\pi}} \int_{\lambda u_{\text{cl}}}^{\infty} e^{-t^2} dt \right)_{\lambda=1} = -\frac{d}{d\lambda}\left(\frac{1}{\lambda}\text{erfc}(\lambda u_{\text{cl}}) \right)_{\lambda=1} \\ &= \frac{1}{\lambda^2}\text{erfc}(\lambda u_{\text{cl}}) + \frac{2}{\lambda\sqrt{\pi}} e^{-\lambda^2 u_{\text{cl}}^2} u_{\text{cl}} \Big|_{\lambda=1} = \text{erfc}(u_{\text{cl}}) + \frac{2}{\sqrt{\pi}} e^{-u_{\text{cl}}^2} u_{\text{cl}}\,. \end{aligned}$$

One must now numerically evaluate this with $u_{\text{cl}} = v_{\text{cl}}/v_{\text{peak}} \approx 16$, which is problematic since $u_{\text{cl}} \gg 1$, so $P(v > v_{\text{cl}})$ is going to be very, very small. This can be made simpler by using an aysmptotic expansion for erfc (see e.g. Abromowitz and Stegun)

$$\text{erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \left(1 + \mathcal{O}(z^{-2})\right)\,.$$

We see that the second term in $P(v > v_{\text{cl}})$ is more important than the erfc term. Taking the log,

$$\ln P(v > v_{\text{cl}}) \approx \ln \frac{2}{\sqrt{\pi}} + \ln u_{\text{cl}} - u_{\text{cl}}^2 \approx -263\,.$$

or $P(v > v_{\text{cl}}) \approx 10^{-114}$.

Comparing this with the number of protons in the sun,

$$N_{\text{cl. fuse}} = N_p\, P(v > v_{\text{cl}}) \approx 10^{57}\, 10^{-114} \approx 10^{-57}\,.$$

There are no protons in the sun moving fast enough to classically overcome the Coulomb barrier in a collision with an equally energetic proton.

2. **Polytropes [35 pts].** They're old-fashioned, but polytropic interior models (where $P = K\rho^{1+1/n}$) can provide some insights that modern numerical models struggle to provide.

   (a) For generic index n, derive the Lane-Emden equation from the equation of hydrostatic equilibrium. [5 pts]

(b) Show that the total mass of a polytropic star is

$$M = 4\pi \rho_c \lambda_n^3 z_{\rm surf}^2 \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}}.$$

The factor λ_n is defined as

$$\lambda_n \equiv \left[(n+1) \frac{K\rho_c^{(1-n)/n}}{4\pi G} \right]^{1/2}$$

(you may assume this form), and $z_{\rm surf}$ specifies the outer radius of the star: $\phi_n(z_{\rm surf}) = 0$. [4 pts]

Solution:

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho dr \\ &= \int_0^{z_{\rm surf}} 4\pi \left(\lambda_n z\right)^2 \left(\rho_c \phi^n\right) \left(\lambda_n dz\right) \\ &= 4\pi \lambda_n^3 \rho_c \int_0^{z_{\rm surf}} z^2 \phi^n dz \end{aligned}$$

Using the Lane-Emden equation to replace ϕ^n,

$$\begin{aligned} M &= 4\pi \lambda_n^3 \rho_c \int_0^{z_{\rm surf}} z^2 \left(-\frac{1}{z^2} \frac{d}{dz} \left[z^2 \frac{d\phi_n}{dz} \right] \right) dz \\ &= -4\pi \lambda_n^3 \rho_c \int_0^{z_{\rm surf}} \frac{d}{dz} \left[z^2 \frac{d\phi_n}{dz} \right] dz \\ &= -4\pi \lambda_n^3 \rho_c \left[z^2 \frac{d\phi_n}{dz} \right]_{z=0}^{z=z_{\rm surf}} \\ &= -4\pi \lambda_n^3 \rho_c z_{\rm surf}^2 \left. \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}} \\ M &= \boxed{4\pi \lambda_n^3 \rho_c z_{\rm surf}^2 \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}}} \end{aligned}$$

The last line obtains since $\left. \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}} < 0$.

(c) Show that the ratio of the mean density to the central density is

$$\frac{\langle \rho \rangle}{\rho_c} = \frac{3}{z_{\rm surf}} \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}}. \quad \text{[4pts]}$$

Solution: Using our result from part a), the mean density of the star is,

$$\begin{aligned} \langle \rho \rangle &= \frac{M}{\frac{4}{3}\pi R^3} \\ &= \frac{4\pi \lambda_n^3 \rho_c z_{\rm surf}^2 \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}}}{\frac{4}{3}\pi \left(\lambda_n z_{\rm surf}\right)^3} \\ &= \frac{3\rho_c}{z_{\rm surf}} \left| \frac{d\phi_n}{dz} \right|_{z=z_{\rm surf}}. \end{aligned}$$

The desired result follows directly.

(d) Show that the central pressure is

$$P_c = \frac{GM^2}{R^4}\left[4\pi(n+1)\left|\frac{d\phi_n}{dz}\right|^2_{z=z_{\rm surf}}\right]^{-1}. \quad [4pts]$$

Solution: Starting from the polytropic equation of state, we have $P_c = K\rho_c^{1+1/n}$. Solving for K using equation 2b we find,

$$K = \frac{4\pi G\lambda_n^2}{(n+1)\rho_c^{(1-n)/n}}.$$

Thus,

$$\begin{aligned}
P_c &= K\rho_c^{1+1/n} \\
&= \frac{4\pi G\lambda_n^2\rho_c^2}{(n+1)} \\
&= \frac{4\pi G\lambda_n^2\langle\rho\rangle^2}{(n+1)\left(\frac{\langle\rho\rangle}{\rho_c}\right)^2} \\
&= \frac{4\pi G\lambda_n^2\left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2}{(n+1)\left(\frac{3}{z_{\rm surf}}\left|\frac{d\phi_n}{dz}\right|_{z=z_{\rm surf}}\right)^2} \\
&= \frac{GM^2\left(\lambda_n z_{\rm surf}\right)^2}{4\pi(n+1)R^6\left(\left|\frac{d\phi_n}{dz}\right|_{z=z_{\rm surf}}\right)^2} \\
&= \frac{GM^2}{R^4}\left[4\pi(n+1)\left|\frac{d\phi_n}{dz}\right|^2_{z=z_{\rm surf}}\right]^{-1}.
\end{aligned}$$

(e) Write a quick program that numerically solves the Lane-Emden equation for $\phi(z)$ and $z_{\rm surf}$ given arbitrary n. Provide your code (it doesn't have to be pretty). Use it to solve for $\phi(z)$ and plot it for n=1, 2, 3, and 4. Show that your result agrees well with the analytic solution of $\phi(z)$ for n=1. [8 pts]

Solution:

Fig. 1 shows the resulting plot. The code used is shown in the see solution below, in the next part.

(f) Use your code to model and plot $\phi(r)$, $P(r)$, $\rho(r)$, and the enclosed mass $M(r)$ for the Sun, assuming n=3. [4 pts]

Solution:

Fig. 2 shows the resulting profiles. The code used is shown below:

```
from pylab import *
import pandas as pd

G = 6.673e-11
rsun = 695508000.0
msun = 1.9891e+30

def solve_lane_emden(rstar, mstar, n, npts=10000, verbose=False,
    retparams=False):
   """
   :INPUTS:
    rstar : float
      Radius of star to model, in meters
```

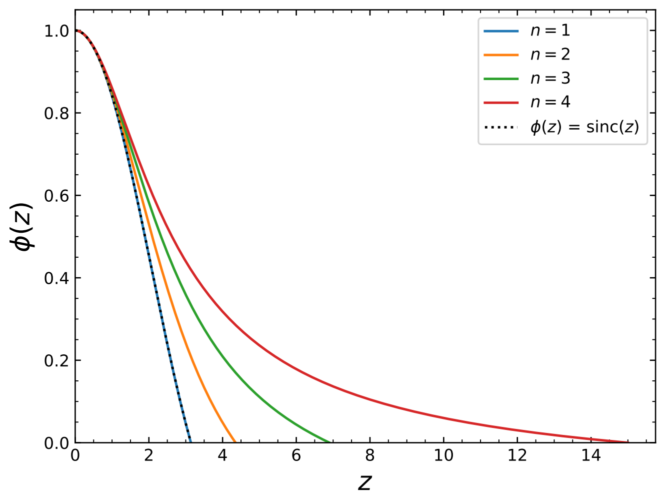

Figure 1: Polytropic model solutions for $n = 1, 2, 3, 4$. The numerical solution for $n = 1$ (solid blue line) agrees well with the analytic solution (dotted line).

```
   mstar : float
     Mass of star to model, in kg

   n : float (0<n<5)
     Polytropic index.

 :OUTPUTS:
   Pandas DataFrame with relevant internal profiles.



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 """
 z = np.linspace(0,15,npts+1)[1:]
 phi = np.ones(npts)
 dz = np.mean(np.diff(z))

 for ii in range(1,npts):
     dphidz = -1/z[ii-1]**2 * (phi[0:ii]**n * z[0:ii]**2 * dz).sum()
     phi[ii] = phi[ii-1] + dphidz * dz

 # My kludgey way to find the surface location:
 try:
     zsurf = z[(phi<=0).nonzero()[0]][0]
 except:
     zsurf = np.interp(0, phi[::-1], z[::-1])
 if verbose:
     print('For n=%1.3f, z_surf=%1.5f' % (n, zsurf))

 lambd = rstar / zsurf
 integral = ((z**2 * phi**n)[z<=zsurf] * dz).sum()
 rho_c = mstar / (4*pi*lambd**3 * integral)
 if verbose:
```

```
        print('For n=%1.3f, rho_c=%1.5f' % (n, rho_c))

    K = lambd**2 * (4*np.pi*G) / ((n+1) * (rho_c**(1./n - 1)))
    rho = rho_c * phi**n
    pressure = K * rho**(1 + 1./n)
    Menclosed = 4*np.pi*lambd**3 * rho_c * np.cumsum(((z**2 * phi**n) * dz))

    # Create an output table with all relevant quantities:
    out = pd.DataFrame(dict(z=z))
    out['r'] = z * lambd
    out['phi'] = phi
    out['rho'] = rho
    out['M'] = Menclosed
    out['P'] = pressure

    index = z<=zsurf
    if retparams:
        params = dict(n=n, zsurf=zsurf, rstar=rstar, mstar=mstar, npts_in=npts,
            npts_out=index.sum(), rho_c=rho_c, K=K, lambd=lambd)
        ret = (out[index], params)
    else:
        ret = out[index]
    return ret

rstar = rsun
mstar = msun
figure()
for n in [1,2,3,4]:
    out, params = solve_lane_emden(rstar, mstar, n, retparams=True)
    plot(out.z, out.phi, label='$n=%i$' % n)

plot(out.z[out.z<=np.pi], (np.sin(out.z)/out.z)[out.z<=np.pi], ':k',
     label='$\phi(z)$ = sinc$(z)$')
leg=legend()
xlim(0, xlim()[1])
ylim(0, ylim()[1])
xlabel('$z$', fontsize=16)
ylabel('$\phi(z)$', fontsize=16)


n = 3
rstar = rsun
mstar = msun
out, params = solve_lane_emden(rstar, mstar, n, retparams=True)


Pscale = int(np.log10(out.P[np.isfinite(out.P)].max()))

figure()
ax1=subplot(411)
plot(out.z/params['zsurf'], out.phi)
ylabel('$\phi(r)$')
ax2=subplot(412)
plot(out.z/params['zsurf'], out.rho)
ylabel('$\\rho(r)$ [kg/m$^3$]')
ax3=subplot(413)
plot(out.z/params['zsurf'], out.M/mstar)
ylabel('$M(r)/M_*$')
ax4=subplot(414)
```

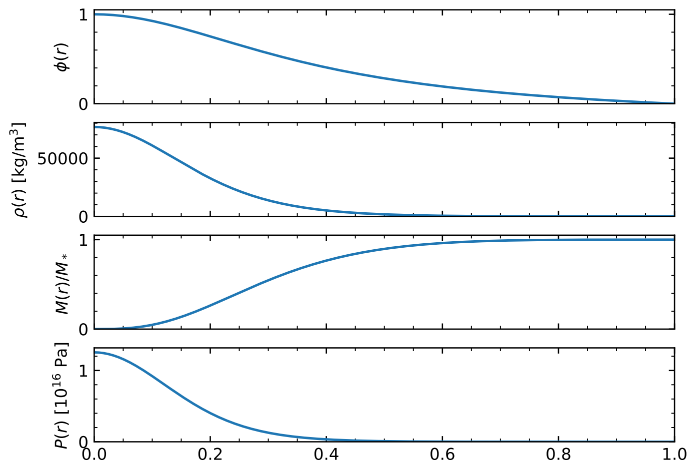

Figure 2: Solar polytropic model for $n = 3$.

```
plot(out.z/params['zsurf'], out.P/10**Pscale)
ylabel('$P(r)$ [$10^{%i}$ Pa]' % Pscale )


axs = [ax1, ax2, ax3, ax4]
[ax.set_ylim(0, ax.get_ylim()[1]) for ax in axs]
[ax.set_xlim(0, 1) for ax in axs]
[ax.set_xlabel('') for ax in axs[0:-1]]
[ax.set_xticklabels([]) for ax in axs[0:-1]]
```

(g) Compute the implied nuclear luminosity of your polytropic Solar model. Take the nuclear energy generation rate per unit volume to be

$$\epsilon_V = (2.46 \times 10^6)\, \rho^2 X^2 T_6^{-2/3} \exp\left(-33.81\, T_6^{-1/3}\right) \ \text{erg s}^{-1}\ \text{cm}^{-3},$$

where ρ is in g cm^{-3}, T_6 is the temperature in units of 10^6 K, and $X = 0.6$ is the hydrogen mass fraction. First, write the calculation as the product of a dimensioned constant and a dimensionless integral involving ϕ_n and z. (For the T_c inside the integral you can use 15.7×10^6 K.) Show the value of your constant, and the form of the dimensionless integral. Then, evaluate the nuclear luminosity in erg s^{-1}. Compare to the actual luminosity of 3.839×10^{33} erg s^{-1}. [6 pts]

Solution: Plugging ρ_c, T_c and X into ϵ_V gives

$$\epsilon = (3.53 \times 10^9 \text{ erg s}^{-1} \text{ cm}^{-3})\phi_3^{16/3} \exp(-13.5/\phi_3^{1/3})\,.$$

This may be integrated over the volume to calculate the total luminosity,

$$L = \int_0^R \epsilon 4\pi r^2 dr = 4\pi\lambda_3^3 \int_0^{z_{\text{surf}}} \epsilon z^2 dz\,.$$

Numerically integrating the luminosity and plugging in the value of $\lambda \approx 7.66 \times 10^9$ cm gives

$$L_{\odot,\text{poly}} \approx \left(2.25 \times 10^{40} \text{ erg s}^{-1}\right) \int_0^{z_{\text{surf}}} \phi_3^{16/3} e^{\left(-13.5/\phi_3^{1/3}\right)} z^2 dz \approx 6.5 \times 10^{33} \text{ erg s}^{-1}.$$

This is about a factor of two greater than the measured solar luminosity of $L_{\odot,\mathrm{meas}} \approx 3.84 \times 10^{33}$ erg s^{-1}.

3. **Opacity due to Thomson scattering [10 pts]**

   Consider an atmosphere of completely ionized hydrogen having the same mass density as Earth's atmosphere at sea level ($\rho = 1.23$ kg m^{-3}). Calculate the path length over which a beam of light would be attenuated to half of its original intensity, due to Thomson scattering by free electrons.

   Solution: Consider the radiative transfer equation along a path with no source term,

$$I(s) = I_0 e^{-\tau}, \qquad \tau \equiv \int_0^s \alpha ds\,.$$

   The optical depth at half-attenuation is found from

$$\frac{I(s)}{I_0} = \frac{1}{2} = e^{-\tau_{1/2}} \quad \Longrightarrow \quad \tau_{1/2} = \ln 2\,.$$

   For a homogenous medium, α is independent of the point along the path, so $\tau = \alpha s$.

   To find α due to Thomson scattering, rewrite it in terms of the scattering cross section, $\alpha = n_e \sigma_T$ where n_e is the number density of scatterers (electrons), and recall that

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 .$$

   Convert the number density n_e into the mass density of the atmosphere by using charge neutrality, so $n_e = n_p \approx \rho/m_p$ where ρ is the total mass density; most of the mass density comes from protons. Combining, find

$$\ln 2 = \alpha s_{1/2} = \frac{\rho}{m_p}\sigma_T s_{1/2} \quad \Longrightarrow \quad \boxed{s_{1/2} = \frac{m_p \ln 2}{\rho \sigma_T}}\,.$$

   The density of Earth's atmosphere at sea level depends on temperature, of course. However, the quantity is standardized at 1 atm and 15°C as $\rho = 1.23$ kg m^{-3}. Finally plugging in numbers yields $\boxed{s_{1/2} \approx 14 \text{ m}}$.