UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 4: Atmospheres
Due: Thursday, April 10, 2025, in class
This problem set is worth **70 points**.

1. **Gray, Plane-Parallel, Eddington-Approximation Atmosphere [20 pts]**

   Show that in a plane-parallel, gray atmosphere under the Eddington Approximation:

   (a) $S = \langle I \rangle$,

   (b) $P_{\rm rad} = \frac{F}{c}(\tau + Q)$ (where Q is a constant of integration),

   (c) $S = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right)$, and

   (d) $T(\tau) = T_{\rm eff}\left(\frac{3\tau}{4} + \frac{1}{2}\right)^{1/4}$.

2. **Limb darkening [20 pts].**

   In this problem you will derive a relation between the measured limb darkening of a star, and the source function of its photosphere. Let the intensity of the stellar disk be $I_\nu(r)$, where r is the distance from the center of the stellar disk in units of the stellar radius (i.e. $r = 0$ at the center, and $r = 1$ at the limb).

   (a) Instead of r it is traditional to express I_ν as a function of $\mu \equiv \sqrt{1 - r^2}$. Show that $\mu = \cos\theta$, where θ is the angle between the line of sight and the normal to the stellar surface.

   Solution: Refer to Fig. 1 for the geometry of the problem. The two rays toward Earth are parallel. Take the normal to the star's surface at some reduced radius r and continue it through to the center of the circle. This line intersects the two parallel rays, which is why the two angles labeled θ in the figure are the same angle.

   Construct the right triangle shown in the figure. The hypotenuse is 1 and the height is r. Therefore the base is $\sqrt{1 - r^2}$, from the Pythagorean theorem. From the definition of the cosine, $\boxed{\cos\theta = \sqrt{1 - r^2} = \mu}$.

   (b) We want an expression for the intensity at the stellar surface in terms of the source function. Start from the the radiative transfer equation for a plane-parallel atmosphere. Show that for an upward-propagating ray coming from far below to the top surface, the formal solution is

$$I_\nu(\mu) = \int_0^\infty d\tau_\nu \, \frac{S_\nu(\tau_\nu)}{\mu} e^{-\tau_\nu/\mu}, \tag{1}$$

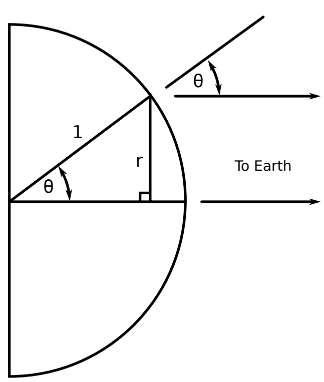

Figure 1: The geometry of Prob. 2.

where τ_ν is the vertical optical depth.

Solution: Let us take this opportunity to remind ourselves of the distinction between optical depth and vertical optical depth. Start with the form of the radiative transfer equation in terms of optical depth,

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu\,,$$

where $S_\nu = j_\nu/\alpha_\nu$ is the source function, and τ_ν is the actual optical depth. Define the new variable $d\xi_\nu \equiv -dz = -d\tau_\nu \cos\theta$ which is the vertical component along the ray, and the sign is chosen so that the "vertical optical depth" starts at 0 at the top (larger z) and increases as one goes down (smaller z). In terms of ξ_ν, the RTE becomes

$$I_\nu - S_\nu = \cos\theta \frac{dI_\nu}{d\xi_\nu} = \mu \frac{dI_\nu}{d\xi_\nu}\,.$$

Multiply by the integrating factor $e^{-\xi_\nu/\mu}$. Collect all the terms which have I_ν,

$$e^{-\xi_\nu/\mu} S_\nu = e^{-\xi_\nu/\mu}\left(I_\nu - \mu\frac{dI_\nu}{d\xi_\nu}\right) = -\mu \frac{d}{d\xi_\nu}\left(e^{-\xi_\nu/\mu} I_\nu\right)\,.$$

Divide through by μ and now the right hand side is a total derivative. Integrate over $d\xi_\nu$ from 0 to ∞,

$$\int_0^\infty d\xi_\nu e^{-\xi_\nu/\mu}\frac{S_\nu}{\mu} = -\int_0^\infty d\xi_\nu \frac{d}{d\xi_\nu}\left(e^{-\xi_\nu/\mu} I_\nu\right) = -e^{-\xi_\nu/\mu} I_\nu \Big|_{\xi_\nu=0}^{\xi_\nu=\infty} = I_\nu(\xi_\nu = 0, \mu)\,.$$

By an unfortunate convention, the symbol τ_ν is used instead of ξ_ν, but please be aware that the meaning is the vertical optical depth.

(c) Suppose the (unknown) source function can be represented by a polynomial,

$$S_\nu(\tau_\nu) = a_0 + a_1\tau_\nu + a_2\tau_\nu^2 + \cdots + a_n\tau_\nu^n. \tag{2}$$

Show that under this assumption the emergent intensity is given by

$$I_\nu(\mu) = a_0 + a_1\mu + 2a_2\mu^2 + \cdots + (n!)a_n\mu^n, \tag{3}$$

using the definite integral $\int_0^\infty x^n \exp(-x)dx = n!$. In this way the measured limb-darkening law can be used to determine the source function, and therefore the temperature stratification for an LTE atmosphere.

Solution: Substitute Eq. (2) into Eq. (1) and change variables to $x \equiv \tau_\nu/\mu$. This gives

$$I_\nu(\mu) = \int_0^\infty dx \sum_{i=0}^{n} a_i (x\mu)^i e^{-x}\,.$$

Each term in the sum may be integrated using the given definite integral, giving Eq. (3).

(d) Show that for a gray LTE atmosphere, the predicted limb darkening law for the wavelength-integrated intensity at the stellar surface is

$$\frac{I(\theta)}{I(0)} = \frac{2}{5} + \frac{3}{5}\cos\theta.$$

Solution: For a gray atmosphere ($\tau_\nu = \tau$), and in a plane parallel atmosphere with no energy generation ($dF/dz = 0$, flux is conserved from layer to layer), we found in class that

$$S = \langle I \rangle\,,$$

where $\langle f \rangle = \frac{1}{2}\int_{-1}^{1} f d\mu$ is an angular average. This yields the integro-differential equation

$$\frac{1}{2}\int_{-1}^{1} I d\mu = I - \mu\frac{dI}{d\tau}\,.$$

We simply state the solution,

$$S = \frac{3F}{4\pi}\left[\tau + q(\tau)\right] \approx \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right) .$$

This satisfies the assumption of Prob. 2c, with $a_1 = \frac{3F}{4\pi}$, $a_0 = \frac{2}{3}a_1$, and all other a_n's vanishing. Putting this into the result from Prob. 2c, find $I(\mu) = a_0 + a_1\mu$, where $\mu = \cos\theta$. Evaluating the ratio gives

$$\frac{I(\theta)}{I(0)} = \frac{a_0 + a_1 \cos\theta}{a_0 + a_1} = \frac{a_1(2/3 + \cos\theta)}{a_1(2/3 + 1)} = \frac{2}{5} + \frac{3}{5}\cos\theta\,.$$

3. **Radiative transfer in spherical coordinates [20 pts].**

After the past month's classes you should be familiar with the radiative diffusion equation for a plane-parallel atmosphere, an appropriate model for a thin photosphere. In this problem you will repeat those steps for a spherical atmosphere, as appropriate for the bulk of a star. We will assume the star is spherically symmetric and that consequently $I_\nu = I_\nu(r, \theta)$, where r is the radial coordinate and θ is the angle of a ray relative to the local radius vector (and *not* the polar angle referring to the position with respect to the stellar center). See Fig. 2.

(a) Use the chain rule,

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}, \tag{4}$$

to show that the radiative transfer equation (RTE) can be written

$$\cos\theta \frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\frac{\partial I_\nu}{\partial \theta} + \rho\kappa_\nu I_\nu - j_\nu = 0. \tag{5}$$

In this expression, κ_ν is the *opacity*, measured in units of cm^2 g^{-1}; and j_ν is the *emission coefficient*, measured in units of erg cm^{-3} s^{-1} sr^{-1} Hz^{-1} [both as defined by Rybicki & Lightman (p. 9-10)].

Solution: Consider a photon traveling a distance ds along a ray at an angle θ from the local radius vector (See Figure 2). Then, the radial distance the photon has traveled is $dr = ds\cos\theta$, while the incremental difference in angle between the ray and the local radial vector is $d\theta = -\frac{ds\sin\theta}{r}$. Thus, we find

$$\begin{aligned}\frac{dr}{ds} &= \cos\theta \\ \frac{d\theta}{ds} &= -\frac{\sin\theta}{r}\end{aligned}$$

Substitution into the equation of radiative transfer,

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds} = -\rho\kappa_\nu I_\nu + j_\nu,$$

with the chain rule, yields the desired result.

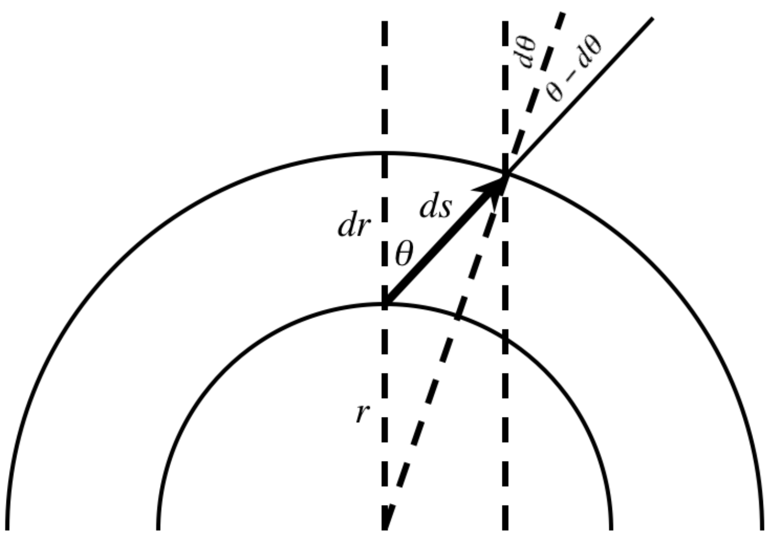

Figure 2: Geometry relevant to Prob. 3. A photon propagates a distance ds along a direction θ from the local radius vector. As a result its radial coordinate increases by dr and the angle to the local radius vector decreases by $d\theta$.

(b) Integrate the RTE over all solid angles to show

$$\frac{dF_\nu}{dr} + \frac{2}{r}F_\nu + c\rho\kappa_\nu u_\nu - \rho\epsilon_\nu = 0, \tag{6}$$

where ϵ_ν is the (angle-averaged) *emissivity* as defined on p. 9 of Rybicki & Lightman.

Solution:

$$\begin{aligned}
0 &= \int \left[\cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial I_\nu}{\partial\theta} + \rho\kappa_\nu I_\nu - j_\nu\right] d\Omega \\
&= \frac{\partial}{\partial r}\int \cos\theta I_\nu d\Omega - \frac{1}{r}\int \sin\theta\frac{\partial I_\nu}{\partial\theta}d\Omega + \rho\kappa_\nu\int I_\nu d\Omega - \int j_\nu d\Omega \\
&= \frac{\partial F_\nu}{\partial r} - \frac{2\pi}{r}\int \sin^2\theta\frac{\partial I_\nu}{\partial\theta}d\theta + \rho\kappa_\nu c u_\nu - 4\pi j_\nu \\
&= \frac{\partial F_\nu}{\partial r} - \frac{2\pi}{r}\left[\sin^2\theta I_\nu\,|_{\theta=0}^{\pi} - \int 2\cos\theta\sin\theta I_\nu d\theta\right] + \rho\kappa_\nu c u_\nu - \rho\epsilon_\nu \\
&= \frac{\partial F_\nu}{\partial r} + \frac{1}{r}\left[\int 2\cos\theta I_\nu d\Omega\right] + \rho\kappa_\nu c u_\nu - \rho\epsilon_\nu \\
&= \frac{\partial F_\nu}{\partial r} + \frac{2}{r}F_\nu + \rho\kappa_\nu c u_\nu - \rho\epsilon_\nu
\end{aligned}$$

(c) Multiply the RTE by $\cos\theta$ and integrate over all solid angles to show

$$c\frac{dp_\nu}{dr} + \rho\kappa_\nu F_\nu = 0, \tag{7}$$

where you have assumed j_ν to be isotropic, and I_ν to be nearly isotropic. Here, p_ν is the *specific radiation pressure* given by

$$p_\nu = \frac{1}{c}\int I_\nu \cos^2\theta\, d\Omega. \tag{8}$$

Solution:

$$\begin{aligned}
0 &= \int \cos\theta\left[\cos\theta\,\frac{\partial I_\nu}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial I_\nu}{\partial\theta} + \rho\kappa_\nu I_\nu - j_\nu\right] d\Omega \\
&= \frac{d}{dr}\int \cos^2\theta I_\nu d\Omega - \frac{1}{r}\int \cos\theta\sin\theta\frac{\partial I_\nu}{\partial\theta}d\Omega + \rho\kappa_\nu\int\cos\theta I_\nu d\Omega - \int\cos\theta j_\nu d\Omega \\
&= c\frac{dp_\nu}{dr} - \frac{2\pi}{r}\int \sin^2\theta\cos\theta\frac{\partial I_\nu}{\partial\theta}d\theta + \rho\kappa_\nu F_\nu - 0 \\
&= c\frac{dp_\nu}{dr} - \frac{2\pi}{r}\left[\sin^2\theta\cos\theta I_\nu\,|_{\theta=0}^{\pi} - \int \sin\theta\left(3\cos^2\theta - 1\right) I_\nu d\theta\right] + \rho\kappa_\nu F_\nu \\
&= c\frac{dp_\nu}{dr} + \frac{1}{r}\left[\int I_\nu\left(3\cos^2\theta - 1\right)d\Omega\right] + \rho\kappa_\nu F_\nu \\
&= c\frac{dp_\nu}{dr} + \frac{1}{r}\left[3cp_\nu - cu_\nu\right] + \rho\kappa_\nu F_\nu \\
&\approx c\frac{dp_\nu}{dr} + \rho\kappa_\nu F_\nu
\end{aligned}$$

Where in the last line we have noted that $p_\nu = \frac{1}{3}u_\nu$ since I_ν is nearly isotropic.

(d) Use the preceding equation, as well as the blackbody formula for radiation pressure, the relation $F = L/4\pi r^2$ and the definition of the Rosseland mean opacity κ_R to show

$$\frac{dT}{dr} = -\frac{3\rho\kappa_R L}{64\pi\sigma r^2 T^3}. \tag{9}$$

Solution: The radiation pressure of a blackbody is given by

$$p_\nu = \frac{u_\nu}{3} = \frac{4\pi B_\nu}{3c}.$$

Substituting in our result from part b) we find:

$$\begin{aligned}
0 &= c\frac{dp_\nu}{dr} + \rho\kappa_\nu F_\nu \\
&= \frac{4\pi}{3\kappa_\nu}\frac{dB_\nu}{dr} + \rho F_\nu \\
&= \int \left[\frac{4\pi}{3}\frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}\frac{dT}{dr} + \rho F_\nu\right] d\nu \\
&= \frac{4\pi}{3}\frac{dT}{dr}\int \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu + \rho\int F_\nu d\nu \\
&= \frac{4\pi}{3}\frac{dT}{dr}\frac{1}{\kappa_R}\frac{d}{dT}\int B_\nu d\nu + \rho F \\
&= \frac{4\pi}{3\kappa_R}\frac{dT}{dr}\frac{d}{dT}\left(\frac{\sigma T^4}{\pi}\right) + \frac{\rho L}{4\pi r^2} \\
0 &= \frac{16\sigma T^3}{3\kappa_R}\frac{dT}{dr} + \frac{\rho L}{4\pi r^2} \\
\frac{dT}{dr} &= -\frac{3\rho\kappa_R L}{64\pi\sigma r^2 T^3}.
\end{aligned}$$

4. **Corona time [10 pts].**

   The solar corona may have a base electron density of 10^8 cm^{-3} at $T = 2 \times 10^6$ K. Assume that the corona has an inner radius equal to that of the Sun, the corona is isothermal and that it obeys the equation of hydrostatic equilibrium. Compute the X-ray free-free emission from this model corona and compare with the total luminosity of the Sun