UNIVERSITY OF KANSAS
Department of Physics
ASTR 794 — Prof. Crossfield — Spring 2025

Problem Set 5: Temperatures and Albedos
Due: Thursday, April XXXXX, 2025, in class
This problem set is worth **XXXX points**.

1. **Equilibrium Temperature [10 pts]** Assume a planet with radius R_P and Bond albedo A_B on a circular orbit with separation a around a star with radius R_* and temperature $T_{\rm eff}$. Assume that the planet's day and night sides emit as blackbodies with temperatures T_D and T_N, respectively, and that the planet's luminosity is driven entirely by absorbed incident radiation.

   (a) Show that

   $$T_D^4 + T_N^4 = \frac{1}{2}\left(\frac{R_*}{a}\right) T_{\rm eff}^4 \left(1 - A_B\right). \quad (1)$$

   Solution: The planet must be in energy balance, so $P_{abs} = P_{emit}$.

   $$P_{\rm abs} = \pi R_P^2 \left(1 - A_B\right) P_{\rm inc} \quad (2)$$

   and

   $$P_{\rm inc} = \frac{L_*}{4\pi a^2} = \frac{R_*^2}{a^2}\sigma_{SB} T_{\rm eff}^4. \quad (3)$$

   Meanwhile,

   $$P_{\rm emit} = 2\pi R_P^2 \sigma_{SB}\left(T_D^4 + T_N^4\right). \quad (4)$$

   Combining these relations gives the desired result.

   (b) For the Earth, plot T_N vs. T_D under this simple model.
   Solution:
   See Fig. 1 for the solution. Note that this is somewhat unrealistic, since it suggests that if one hemisphere is above freezing the other must be below freezing; and we know it doesn't freeze every time the sun goes down!

2. **Bond Albedos Depend on the Star [20 pts]**

   In this problem, you will investigate how a planet's Bond albedo A_B depends not just on its albedo spectrum A_λ but also on the incident spectrum of the host star (e.g., the spectral type).

   (a) Assume a planet orbiting its star at the "Earth-equivalent insolation distance" (EEID) — this is a crude approximation to the habitable zone that assumes a planet receives the same total incident flux $F_{\rm inc}$ from its star as does the Earth (call this F_{SC}, the Solar Constant flux).
   Derive the EEID in terms of $T_{\rm eff}$, R_*, and F_{SC} [6 pts].
   Solution: We must have $F_{\rm inc} = F_{SC}$, or

   $$F_{\rm inc} = \frac{R_*^2}{a_{\rm EEID}^2}\sigma T_{\rm eff}^4 = F_{SC} \quad (5)$$

   which then yields

   $$\boxed{a_{\rm EEID} = \sqrt{\frac{\sigma}{F_{SC}}} R_* T_{\rm eff}^2} \quad (6)$$

   This gives 1 AU for Earth/sun quantities, so the answer checks out.

(b) For main-sequence stars with $M_* \lesssim M_\odot$, a useful (but very approximate!) rule of thumb is that roughly speaking

$$\frac{R_*}{R_\odot} \approx \frac{T_*}{T_\odot} \approx \frac{M_*}{M_\odot}. \tag{7}$$

Under this approximation, determine how the EEID scales with stellar radius (or temperature, or mass) [6 pts].

Solution: Under this approximation, it must be true that

$$a_{\mathrm{EEID}} \propto R_* \times R_*^2 = R_*^3. \tag{8}$$

So,

$$\boxed{a_{\mathrm{EEID}} = 1 \text{ AU} \left(\frac{R_*}{R_\odot}\right)^3}. \tag{9}$$

(c) Assume a trivial but interesting planetary albedo spectrum that is zero for $\lambda \geq 1\,\mu m$ and 0.4 at shorter wavelengths. Assume that all stars radiate as blackbodies, and calculate what fraction of incident starlight is reflected away from this planet if it orbits a star of (i) 1.25 $M_\odot$, (i) 1.0 $M_\odot$, (i) 0.4 $M_\odot$. What is the implied Bond albedo A_B in each case? [8 pts]

Solution: The total (bolometric) reflected fraction of incident starlight is of course identical to the Bond albedo. To get this we must integrate the relevant spectrum over all wavelengths. Specifically, with a stellar spectrum $B_\lambda(T_*)$ we have a total reflectance of

$$A_B = \frac{\int_0^\infty A_\lambda B_\lambda(T_*)}{\int_0^\infty B_\lambda(T_*)} = \frac{0.4 \int_0^{1\ \mu\mathrm{m}} B_\lambda(T_*)\, d\lambda}{\int_0^\infty B_\lambda(T_*)\, d\lambda} \tag{10}$$

$M_*/M_\odot$	T_{eff}/K	A_B
1.25	7220	0.33
1.0	5777	0.29
0.4	2310	0.049

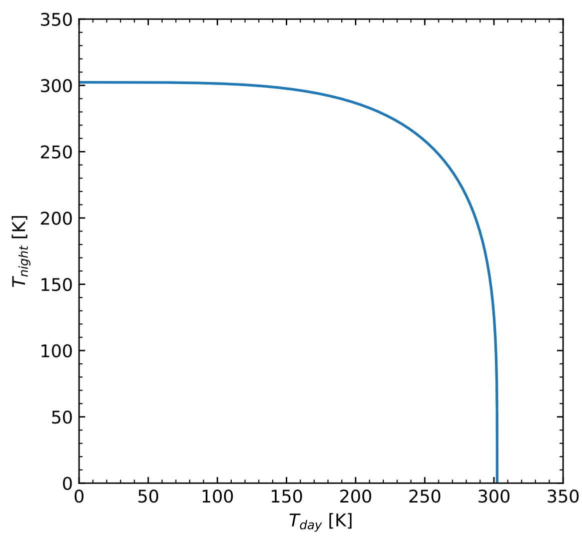

Figure 1: T_D vs. T_N for the Earth, in a simple two-hemisphere model.

So the Bond albedo depends on where the incoming radiation is strongest. The same albedo spectrum gives a different A_B depending on whether the planet is more or less reflective at the particular wavelengths where most of the incident starlight comes in.

3. **Brightness Temperatures [15 pts]**

(a) (5 pts) Calculate the surface flux density of an object emitting as a blackbody, (i.e., $F_{\lambda,\mathrm{surf}} = \pi B_\lambda$), at a wavelength of 4.5 μm and with temperatures of 300 K, 1000 K, and 3000 K. Give your answers in units of W/m^2/μm.

Solution: We just need to plug in $\lambda = 4.5\mu$m into the Planck function, $B_\lambda(T)$, for the three temperatures given – remembering to keep track of units and to multiply by π. When we do that we should get:

$$\boxed{F_{\lambda,\mathrm{surf}}(300\text{ K}) = 4.8\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

$$\boxed{F_{\lambda,\mathrm{surf}}(1000\text{ K}) = 8600\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

$$\boxed{F_{\lambda,\mathrm{surf}}(3000\text{ K}) = 110000\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

(b) (3 pts) Assume that the objects above are Brown Dwarfs: the size of Jupiter, and 10 pc away. Calculate the observed F_λ for each of the three temperatures above.

Solution: In this case we just need to remember that for a spherical object,

$$F_{\lambda,\mathrm{obs}} = \left(\frac{R}{d}\right)^2 F_{\lambda,\mathrm{surf}}. \tag{11}$$

We then plug in the appropriate values – the scale factor is 5.4×10^{-20} – and obtain:

$$\boxed{F_{\lambda,\mathrm{obs}}(300\text{ K}) = 2.6 \times 10^{-19}\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

$$\boxed{F_{\lambda,\mathrm{obs}}(1000\text{ K}) = 4.6 \times 10^{-16}\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

$$\boxed{F_{\lambda,\mathrm{obs}}(3000\text{ K}) = 5.7 \times 10^{-15}\text{ W m}^{-2}\,\mu\text{m}^{-1}}$$

(c) (3 pts) Using JWST, you observe a brown dwarf 10 pc away and measure a flux density of 3×10^{-15} W/m^2/μm. Assuming it is the size of Jupiter, what is its surface flux density?

Solution: We just use the same relations as presented in part (b) above. Multiplying the observed flux density by $(d/R)^2$ gives

$$\boxed{F_{\lambda,\mathrm{surf}} = 56000\text{ W m}^{-2}\,\mu\text{m}^{-1}} \tag{12}$$

(d) (4 pts) What must the surface temperature of this brown dwarf be, in order to explain your observed flux density?

Solution: From our answers to the previous part, we can see that the temperature needs to be between 1000 K and 3000 K. We could just keep guessing different temperatures, or we could just calculate F_λ for a range of temperatures and see where this observed value falls. Fig. 2 shows the result of that, which implies a temperature of $\boxed{T_\mathrm{surf} \approx 2100\text{ K}}$.

Python code to reproduce these calculations and generate this plot is listed below.

```python
# Import necessary modules
import numpy as np
from pylab import *

# Define a function:
def bnu(T, lam):
    """Planck function in frequency.

    :INPUTS:
      T : scalar or array
        temperature in Kelvin

      lam : scalar or array
        wavelength in microns [but intensity will be per Hz]

    Value returned is in SI units: W/m^2/Hz/sr
    """
    from numpy import exp

    c  = 299792458 # speed of light, m/s
    h = 6.626068e-34 # SI units: Planck's constant
    k = 1.3806503e-23 # SI units: Boltzmann constant, J/K
    nu = c/(lam/1e6)
    expo = h*nu/(k*T)
    nuoverc = 1./ (lam/1e6)
    return ((2*h*nuoverc**2 * nu)) / (exp(expo)-1)
```

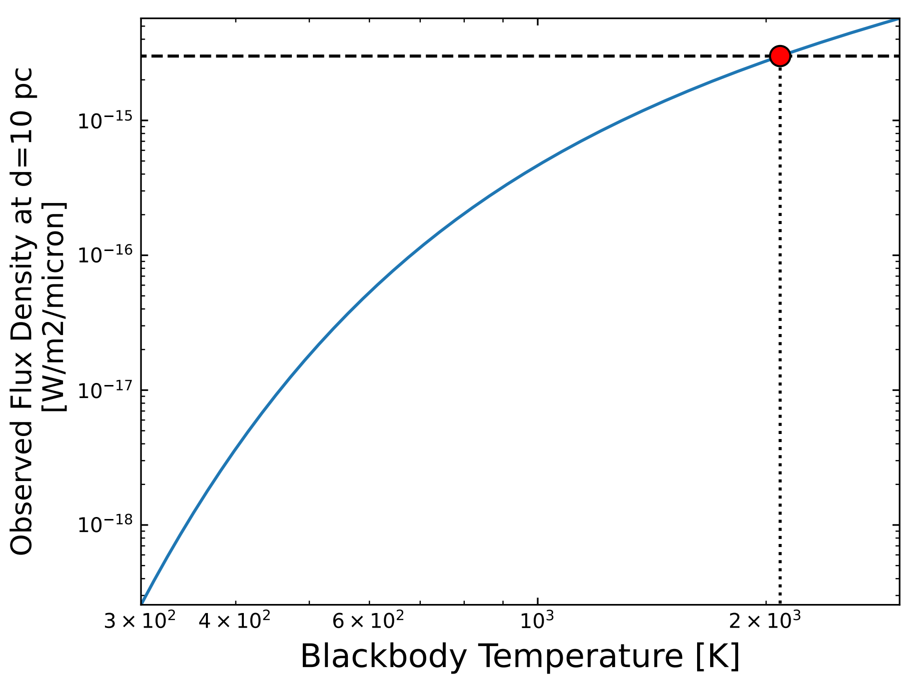

Figure 2: Flux density observed for an object emitting as a blackbody at a distance of 10 pc. The horizontal dashed line shows the observed flux density; the red circle and vertical dotted line indicate the brightness temperature required to match the observation.

```python
def blam(T, lam):
    """ Same as bnu, for returns W/m^2/um/sr """
    c = 299792458. # m/s
    nu = c/(lam/1e6) # Hz
    # Take advantage of the fact that (lam F_lam) = (nu F_nu):
    return bnu(T, lam) * (nu / lam)

parsec = 3.086e16 # Meters per parsec
rjupiter = 7.149e7 # Meters; Jupiter's radius

# Set up wavelengths and temperatures:
wavelength = 4.5 # microns
temperatures = np.array([300, 1000, 3000])
distance = 10 * parsec

flam_surf = np.pi * blam(temperatures, wavelength)
print('Surface flux density:')
for t,f in zip(temperatures, flam_surf):
    print('%i K: %1.2g W/m2/micron' % (t,f))


flam_obs = flam_surf * (rjupiter/distance)**2

print('Observed flux density:')
for t,f in zip(temperatures, flam_obs):
    print('%i K: %1.2g W/m2/micron' % (t,f))


flam_observed = 3e-15
flam_surf_inferred = flam_observed * (distance/rjupiter)**2

temps = np.arange(300, 3000)
flam_obs_hires = np.pi * blam(temps, wavelength) * (rjupiter/distance)**2

temp_inferred = np.interp(flam_observed, flam_obs_hires, temps)

figure()
loglog(temps, flam_obs_hires)
plot(xlim(), [flam_observed]*2, '--k')
plot([temp_inferred]*2, [0,flam_observed], ':k')
plot(temp_inferred, flam_observed, 'or', mew=1, mec='k', ms=10)
xlabel('Blackbody Temperature [K]', fontsize=16)
ylabel('Observed Flux Density at d=10 pc\n [W/m2/micron]', fontsize=14)
xlim(temps.min(), temps.max())
ylim(flam_obs_hires.min(), flam_obs_hires.max())
tight_layout()
```